Webster, A. G.
The theory of
electricity and magnetism

THE THEORY

OF

ELECTRICITY AND MAGNETISM.

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BEING

LECTURES ON MATHEMATICAL PHYSICS

BY

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PREFACE.

SOME justification is perhaps necessary for the appearance of another treatise on Electricity and Magnetism, in view of the numerous ones already existing in English. This book is the result of a demand encountered in my own experience in teaching, and is based upon various courses of lectures that I have delivered at Clark University during the last six years. The classical treatise of Maxwell, which must always remain as a point of departure for the modern treatment of the subject, is ill adapted to the purpose of a text-book. To ask a student to attempt to assimilate the contents of the two volumes of Maxwell in a year, or even in two years, is only to expose him to the severest pangs of mental indigestion. Again, Maxwell's own views are there presented by him with not the greatest clearness, while severe demands are made upon the student's mathematical attainments. The excellent treatises of Mascart and Joubert and of Watson and Burbury follow Maxwell with considerable closeness. Professor Gray's admirable treatise, though containing much recent matter, suffers under the disadvantage of being in three volumes, while the very convenient little book of Mr Emtage is somewhat restricted in scope. Professor J. J. Thomson's altogether delightful Elements of the Mathematical Theory, which appeared when the present

book was nearly ready for the press, while extremely modern as well as clear, is addressed to a somewhat different class of students from that contemplated in writing the present book.

The theoretical writings of Hertz, Heaviside, Cohn and others have resulted in the systematization of Maxwell's theory and have made possible improvements in the mode of its presentation and nomenclature not contemplated by him. The extremely important and original contributions of Mr Oliver Heaviside are unfortunately but little adapted to the use of the student on account of their very voluminous character as a whole, as well as of an extreme conciseness of expression in individual parts. The few brilliant chapters on theoretical matters left by Hertz are hardly by way of exposition, but rather of a summing up of the conclusions of the theory.

It has been my aim in the preparation of this volume to present to the student the results of the theory as it stands to-day after the labors of Faraday, Maxwell, Helmholtz, Hertz and Heaviside. Here it may be convenient to state what I consider to be the essentials of Maxwell's theory as distinguished from the old theories. To this question may very well be made the answer of Hertz: "Maxwell's theory is Maxwell's system of equations." But to specify more fully the points of difference, they are in the opinion of the writer:

1°. The localization of the energy in the medium.

2°. The magnetic action of displacement currents.

While starting from the standpoint of Energy, I have not thought it advisable to abolish the usual terms repugnant to so many writers, who assuming the attitude of Maxwellians *par excellence,* deny the existence of Electricity. Maxwell himself was not one of these. Feeling that the consideration of the Newtonian Potential Function is indispensable, not only for the old theory of action at a distance, but for the modern theory, and in addition that it introduces the student to many of the methods that he will need in various branches of mathematical physics, I

have prefixed to the treatment of Electricity a rather complete treatment of the Potential by itself, including the properties of polarized distributions. It has been the custom of English writers to include chapters on the Potential in works on Analytical Statics, as in the cases of the admirable treatises of Routh and Minchin. It will probably be admitted, however, that the inclusion of this subject in a treatise mainly devoted to the consideration of rods, strings, and billiard balls is no more appropriate than in one devoted to Electricity and is less likely to attract the student of the latter.

It is unfortunately the case that graduates of our American colleges are as a rule insufficiently prepared in the departments of mathematics necessary in approaching the subject of mathematical physics. In fact, I know of but three text-books on the Calculus in English, those of Greenhill, Williamson and Byerly, that give a treatment of Green's Theorem. I have therefore considered it expedient to prefix a mathematical introduction giving a short treatment of the important subjects of Definite Integrals and of the Theory of Functions of a Complex Variable, indispensable to a study of the Potential Function. For the same reason, I have included a treatment of the fundamental principles of Mechanics *ab initio*, including the deduction of the Principle of Energy, Hamilton's Principle, and Lagrange's Equations of Motion. I have followed the example of Boltzmann in making the deduction of the equations of the Electromagnetic Field depend on Hamilton's Principle by means of the properties of Helmholtz's Cyclic Systems, the treatment of which is here added. These chapters are extracted from my lectures on Dynamics. In this manner it has come about that the book is nearly half finished before the word *electricity* is mentioned. This may be objectionable to some persons, but I consider it of great importance that the student should be well supplied with tools and practised in their use before he is called upon to use them on a new and unfamiliar subject. The physical difficulties connected with electricity are

great enough without being mixed up with mathematical ones. It is also a pity to have the student get the idea that certain theorems pertain to electricity, when they really are simply matters of geometry or analysis. I have whenever possible attempted to bring out the geometrical or physical nature of the processes involved before coming to the electrical application. Thus these introductory chapters may serve as a sort of general introduction to Mathematical Physics.

After a treatment of the problem of Electrostatics in a single medium by means of the Principle of Virtual Work the usual methods of attacking electrostatic problems are treated. These chapters pertain to either the new or the old theory. I have then inserted the chapter on Electrokinetics, somewhat out of its natural order, in order to bring out the geometrical ideas involved in the so-called Law of Ohm. Of these application is made in the treatment of Dielectrics and Magnetizable Bodies, which is carried out in such a manner as to show the close parallelism between the two classes of phenomena there treated, a point not always insisted on by Maxwell, but clearly brought out by Hertz and Heaviside. On account of this the symmetrical notation of Hertz is adopted in preference to that of Maxwell. I have however kept the term *induction* used by Maxwell for magnetism alone, instead of the term *polarization* used by Hertz, which I have used in the more usual sense of moment of unit volume. I regret not having been able to respond to the appeal made by Boltzmann to future writers to follow Maxwell's notation. I feel that it is more important to have a good notation than a familiar one, and that it is a first essential of a good notation that it should be symmetrical. The indiscriminate mixture of Greek, Roman and German letters used by Maxwell is as unfortunate as the dissymmetry with respect to electrical and magnetic phenomena.

It is hardly necessary to say that vector methods have been used throughout, although the abbreviated notation of Hamilton

and Heaviside has been little used. It is easy to lay so much stress on symbolism that the student loses sight of the real simplicity of the method. In order, however, to show its extreme utility, particularly in connection with the operator ∇, the essentials of the quaternion notation have been explained in the first chapter.

As the aim of the book has been to present an introduction to the mathematical theory of electricity, little or no reference has been made to experimental methods—in fact it seems that such subjects as standard cells, dynamos, or galvanometers should be treated in a separate work, and I have no desire to add to the large number of such already existing. At the same time it is hoped that the principles involved in the various modes of measurement are all herein contained.

The figures with which the book is illustrated, while but a few of them are new, have in no case been copied from existing figures, but have been, if necessary, recalculated, and in every case redrawn on a large scale and photographed down to the required size. For the amount of labor here involved I am under great obligations to Messrs W. P. Boynton and T. W. Edmondson, fellows of Clark University, who have undertaken the whole matter. As the proof has been read only by the author, it is probable that a certain number of errors have crept in, which it is hoped may be excused.

In conclusion my aim has been to present a brief, connected treatise embodying the essential points of the theory and suitable for assimilation by the student in a period of time not exceeding a year. To this end I have considered only the usual methods of treating the various subjects, and included enough examples to illustrate their working, and no more. If it be considered that unnecessary matter has been included it may be replied that this may easily be omitted, and that it is safer to include too much than to make unwarrantable assumptions regarding the knowledge possessed by the student. If the book shall succeed in

clearing up some of the difficulties generally encountered by the student and in inducing him to read the classical writings of Maxwell, Helmholtz, Hertz and Heaviside the object of its author will have been achieved.

A. G. WEBSTER.

WORCESTER, MASS.,
Dec. 23, 1896.

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ERRATUM.

Page 140, lines 12 and 13, *for* revolution *read* translation.

MATHEMATICAL INTRODUCTION.

CHAPTER I.

NUMBER.

1. **Rationals and Irrationals.** The primary objects of study in Arithmetic are the natural numbers or *integers*, 1, 2, 3,... forming an unlimited sequence. Of these any two a and b may be added together, and we find the fundamental law that

$$a+b=b+a.$$

This is known as the commutative law. For more than two numbers, we find that

$$(a+b)+c=a+(b+c).$$

This is known as the associative law. Any two numbers may be multiplied together, and we find that multiplication is subject to the commutative law,

$$ab=ba,$$

to the associative law,

$$(ab)\,c=a\,(bc),$$

and in addition to the distributive law,

$$a\,(b+c)=ab+ac.$$

Defining the operation of subtraction as the inverse of addition, so that c is defined as the result of subtracting b from a if b added to c will give a, we find that the operation of subtraction is possible only if a is greater than b. We are thus led to extend our definition of numbers in such a way as to call that to which b must be added in order to give a, a number, in the case where a is less than b. We are thus led to the conception of the negative

integers, which we can add, multiply, and subtract according to the same laws as the natural numbers.

Defining the operation of division as the inverse of multiplication, so that a divided by b is c, if c multiplied by b will give a, we find that the operation can be performed only when b is a factor of a. We are thus again led to extend our definition of numbers so as to call that which, being multiplied by b, will give a, even when b is not a factor of a, a number. We are thus led to the conception of fractions, which may be operated upon like the positive and negative integers. Every fraction is of the form m/n where m and n are positive or negative integers. This system of numbers suffices for all the ordinary operations of arithmetic, including the solution of equations of the first degree.

Any number may be raised to any power, the process being known as involution. If we define evolution, or the operation of taking a root, as the inverse of involution, so that the bth root of a is c when $c^b = a$, we find that the operation can be performed only when a is one of the series of numbers,

$$c, c^2, c^3, \ldots\ldots$$

If we further extend our definition of number, so that that which raised to the bth power will give a, even in the contrary case, we are led to the conception of *irrational* numbers. *No* irrational number can be expressed as the quotient of two integers, though for any given irrational a we can always find two integers such that their quotient differs from a by an amount that is as small as we please. In symbols, if ϵ is any given positive number as small as we please, we can always find m and n so that

$$\left|\frac{m}{n} - a\right| < \epsilon.$$

By $|a|$ is meant the *absolute*, or arithmetical value of a, irrespective of sign. E.g., the square root of 2 is an irrational, but the rules for the extraction of square roots enable us to find a value that differs from it by as little as we please. The ordinary theory of enumeration shows that we can express any rational number in terms of any integer, b, called the base, as the sum of a *definite* number of terms, each of which is the product of some integer less than b by some power of b with positive or negative exponent, or else as the limit of a sum of such terms, where as the

exponents of the negative powers increase regularly in absolute value, from a certain term on the coefficients are all obtained by the repetition of a certain definite group :

$$a = a_n b^n + a_{n-1} b^{n-1} + \ldots + a_1 b + a_0 + a_{-1} b^{-1} + a_{-2} b^{-2} + \ldots + a_{-m} b^{-m},$$

e.g. $$\frac{16}{5} = 3 \cdot 10^0 + 2 \cdot 10^{-1},$$

$$\frac{61}{6} = 1 \cdot 10^1 + 0 \cdot 10^0 + 1 \cdot 10^{-1} + 6 \cdot 10^{-2} + 6 \cdot 10^{-3} + \ldots\ldots$$

No irrational number can be so expressed, though by taking a sufficient number of terms we may obtain a number differing from the given irrational by as little as we please. The coefficients in this case never repeat indefinitely. Since irrationals can not be expressed by means of a finite number of terms each of which is rational, they are defined by their properties, or as the limit approached by an infinite sequence of rational numbers.

2. Limits. If we have a sequence of rational numbers, $a_1, a_2, a_3, \ldots$, following each other according to a given law, and can find a number A, possessing the property that, corresponding to any arbitrarily given positive number ϵ however small, we can find a number μ such that for all values of n greater than μ,

$$|a_n - A| < \epsilon, \qquad n > \mu,$$

then A is called the *limit* of the sequence.

E.g. the sequence

$$a_1 = 1,\ a_2 = 1 + \frac{1}{2},\ a_3 = 1 + \frac{1}{2} + \frac{1}{4},\ a_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \ldots\ldots$$

has the limit 2, a rational number.

The necessary and sufficient condition for the existence of a limit is that when ϵ is arbitrarily given, we can find a number μ such that for all integral values of n greater than μ, and for any positive integral value of p,

$$|a_n - a_{n+p}| < \epsilon.$$

If the latter condition is fulfilled, even though the sequence has no rational limit, the sequence has a limit, which defines an irrational number.

The sequence

$$a_1 = 1,\ a_2 = 1 + \frac{1}{2},\ a_3 = 1 + \frac{1}{2} + \frac{1}{3},\ a_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4},\ \ldots\ldots$$

does not fulfil the above condition, for

$$|a_n - a_{n+p}| = \frac{1}{n+1} + \frac{1}{n+2} \ldots\ldots + \frac{1}{n+p} > \frac{p}{n+p} = \frac{1}{1+\frac{n}{p}},$$

which cannot be made as small as we please, for all values of p, no matter how great n be taken.

On the other hand, the sequence

$$a_1 = 1,\ a_2 = 1 + \frac{1}{1\,.\,2},\ a_3 = 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3},$$

$$a_4 = 1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1.2.3.4},\ \ldots\ldots$$

does satisfy the condition, for

$$|a_n - a_{n+p}| = \frac{1}{1\,.\,2\,.\,3\ldots\ldots n+1}$$

$$\left\{1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \ldots\ldots \frac{1}{(n+2)\ldots\ldots(n+p)}\right\}$$

$$< \frac{1}{1\,.\,2\,.\,3\ldots\ldots n+1}\left\{1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \ldots\ldots + \frac{1}{(n+2)^{p-1}}\right\}$$

$$= \frac{1}{1\,.\,2\,.\,3\ldots\ldots n+1}\left\{1 - \frac{1}{(n+2)^p}\Big/\left(1 - \frac{1}{n+2}\right)\right\}$$

$$< \frac{(n+2)}{1\,.\,2\,.\,3\ldots\ldots n\,(n+1)^2},$$

which is less than ϵ as soon as n is taken greater than $1/\epsilon$.

This sequence defines the irrational known as e, the natural logarithmic base, which is not the root of any algebraic equation with rational coefficients. The class of irrationals is in fact much larger than the class of algebraic irrationals, which led to their inclusion in the number-system.

3. Complex Numbers. The system composed of the rational and irrational, forming together the real numbers, is still not sufficient for the solution of algebraic equations. For consider the simple equation

$$x^2 + 1 = 0.$$

Since even powers of all real numbers are positive, there is no real number that has a square equal to -1. If we further extend the idea of numbers, so as to call that a number whose square is -1, we have a means of satisfying the equation. If we denote the new number by i, defined by the equation $i^2 = -1$, we may multiply it by any real number, positive or negative, integral, fractional, or irrational, and thus get a class of new numbers, known as pure imaginary numbers. Evidently no imaginary number is equal to a real number, for the quotient of two real numbers is always real.

If we consider the sum of a real and an imaginary number, we arrive at the conception of a complex number (in the narrow sense). Two complex numbers are equal when their real parts are equal and their imaginary parts also. Any equation containing complex numbers is accordingly equivalent to two equations containing only real numbers. In particular the equation

$$a + bi = 0,$$

where a and b are real, is equivalent to the two,

$$a = 0 \text{ and } b = 0.$$

A complex number vanishes only when its real and imaginary parts both vanish.

4. Complex Numbers in the Extended Sense. As we have formed numbers by multiplying the real and imaginary units 1 and i by all real numbers and forming sums therefrom, so we may still further extend the notion of numbers to include sums of terms each formed by multiplying any number n of different units by real numbers. Such numbers are complex numbers in the extended sense, a number involving n units e_s being an n-fold number. The units may have any properties by which we wish to define them. If they are all independent of one another, it is obvious that two complex numbers are equal only when composed of the same number of each unit e_s, so that any equation containing all the units is equivalent to n equations containing only real numbers. In particular, a complex number

$$a = \alpha_1 e_1 + \alpha_2 e_2 \ldots\ldots + \alpha_n e_n,$$

vanishes only when the coefficient α_s of each unit e_s is zero.

Two complex quantities satisfy the associative and commutative laws with respect to addition, and accordingly the sum of

$$a = \alpha_1 e_1 + \alpha_2 e_2 \ldots\ldots + \alpha_n e_n \text{ and } b = \beta_1 e_1 + \beta_2 e_2 \ldots\ldots + \beta_n e_n$$

is defined as

$$a+b=(\alpha_1+\beta_1)\,e_1+(\alpha_2+\beta_2)\,e_2\ \ldots\ldots+(\alpha_n+\beta_n)\,e_n.$$

With respect to multiplication, the units are commutative with respect to real numbers:

$$abe_rce_s=ae_rbce_s=abce_re_s=e_re_sabc, \text{ etc.}$$

The associative and distributive principles also hold, so that

$$(ae_r)\,e_s=a\,(e_re_s) \text{ and either } a\,(e_r+e_s)=ae_r+ae_s,$$

or

$$e_r\,(e_s+e_t)=e_re_s+e_re_t.$$

We may accordingly define the multiplication of any two complex numbers

$$ab=(\alpha_1e_1+\alpha_2e_2\ \ldots\ldots+\alpha_ne_n)\,(\beta_1e_1+\beta_2e_2\ \ldots\ldots+\beta_ne_n)$$
$$=\alpha_1\beta_1e_1^2+\alpha_1\beta_2e_1e_2\ \ldots\ldots+\alpha_1\beta_ne_1e_n+\alpha_2\beta_1e_2e_1+\alpha_2\beta_2e_2^2+\ldots\ldots$$

It will be convenient to consider a system of units of such a nature that instead of the commutative property with respect to multiplication we have $e_re_s=-e_se_r$ where r and s are different, and for any r, $e_r^2=-1$.

If we consider a set of three units, each possessing the above properties, and in addition the property that the product of any two taken in cyclic order is equal to the third, we have the system proposed by Hamilton, and denoted by him by the letters i, j, k. Accordingly by definition

$$ij=-ji=k,\ jk=-kj=i,\ ki=-ik=j.$$

Multiplying each equation by the first unit appearing in it, and observing the associative law, we have

$$iij=i^2j=-iji=-(ij)\,i=-ki=-j,$$
$$jjk=j^2k=-jkj=-(jk)j=-ij=-k,$$
$$kki=k^2i=-kik=-(ki)\,k=-jk=-i,$$

necessitating

$$i^2=j^2=k^2=-1.$$

Even powers of the three units are real, and equal even powers are equal, while odd powers of any unit are equal to real multiples of itself, and equal odd powers of different units are not equal. The product of two threefold complex numbers of this system, a and b,

$$(\alpha i+\beta j+\gamma k)(\alpha' i+\beta' j+\gamma' k)=\alpha\alpha' i^2+\beta\beta' j^2+\gamma\gamma' k^2+\alpha\beta' ij+\beta\alpha' ji$$
$$+\beta\gamma' jk+\gamma\beta' kj+\gamma\alpha' ki+\alpha\gamma' ik=-(\alpha\alpha'+\beta\beta'+\gamma\gamma')$$
$$+(\beta\gamma'-\gamma\beta')\,i+(\gamma\alpha'-\alpha\gamma')\,j+(\alpha\beta'-\beta\alpha')\,k,$$

is accordingly equal to a real number plus a complex number of the same system, and this may be considered as a fourfold complex number compounded of the units $1, i, j, k$. Such a fourfold number was called by Hamilton a *Quaternion*. We shall in this book seldom need the fourfold number, but shall frequently use the threefold one.

5. Geometrical Representation of Numbers. The natural numbers may be represented by an unlimited series of points laid off at equal distances along a straight line. If we take a certain point to represent zero, the positive integers will lie on one side of it and the negative on the other. Points between the integer points will represent fractions and irrationals, and to every real number will correspond a point. For any rational number we may find others lying as near it as we please, and as we have already stated, for any irrational we may find rational numbers lying as near it as we please. It may be shown, however, that between any two rational numbers, however close together, there can always be found an irrational, consequently the rational numbers do not form a continuous series. It may be shown that every point on the line corresponds to either a rational or an irrational number, so that the whole series of real numbers is continuous. Quantities which, like the real numbers, require for their specification but a single given quantity, which may take any of an unlimited series of values, are said to have one degree of freedom. It is also said that there is a single infinity of such quantities.

Complex quantities in the narrow sense, involving two different units, 1 and i, cannot be represented by points on a line. If however we lay off the real numbers on a straight line, we may lay off the pure imaginary numbers on a line at right angles with it through the point representing zero. The point i is to be taken at the same distance from zero on this line that the point 1 is on the other line. The two lines are called respectively the axes of reals and of pure imaginaries, or the axes of X and Y. Any complex number $a=\alpha+\beta i$ may now be represented by a point in the plane whose rectangular x and y coordinates are respectively α

and β. Whatever the values of α and β we may always find a corresponding point, and to every point in the plane there corresponds a single complex number, including the real and pure imaginary numbers as particular cases. As each of the real numbers α and β may independently assume the value of any of the single infinity of real numbers, there is said to be a double infinity of complex numbers, or a complex number has two degrees of freedom. The distance of the representative point from the origin is called the *modulus* of the complex quantity and denoted by $|a| = +\sqrt{\alpha^2+\beta^2}$ since it includes as a particular case the absolute value of a real number. The angle that the radius vector from the origin makes with the X-axis is called the *argument* of the number. This representation of complex numbers in the plane was proposed by Argand and Gauss*.

The threefold complex quantity $a = \alpha i + \beta j + \gamma k$, not being capable of representation in a plane, may in a similar manner be represented in space. If we take three mutually perpendicular axes, points at equal distances from their intersection will represent the three units i, j, k. Multiples of these by real numbers will be represented by points on the axes of X, Y and Z, and any complex number $\alpha i + \beta j + \gamma k$ may be represented by a point having the rectangular x, y and z coordinates α, β, γ. For every complex number we may find a point, and to every point there corresponds a complex number. As each of the real coefficients α, β, γ may independently assume any of a single infinity of values, the complex number has three degrees of freedom, or there is a triple infinity of such complex numbers. The distance of the representative point from the origin was called by Hamilton the *tensor* of the complex number. We may apply the term modulus to the tensor, and use the symbol

$$|a| = +\sqrt{\alpha^2+\beta^2+\gamma^2}.$$

In this book the arrangement of the axes of X, Y, Z will always be such that the motion of a right-handed screw along the axis of X will turn the Y axis toward the Z axis. This will be called right-handed cyclic order, Fig. 1.

* Argand, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, Paris, 1806.

Gauss, "Theoria residuorum biquadraticorum, commentatio secunda." *Werke*, Bd. II., p. 169.

6. Geometric Addition. Instead of the point representing the complex number $\alpha i + \beta j + \gamma k$, we may fix our attention upon

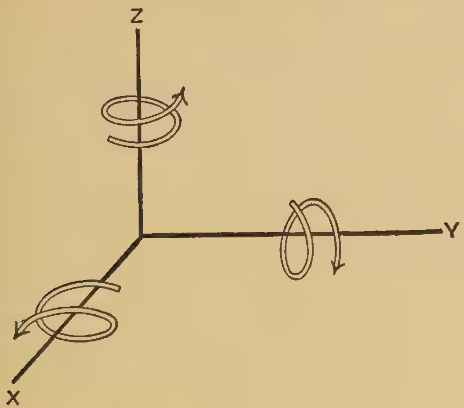

Fig. 1.

the line drawn from the origin to the representative point. This line is a geometrical magnitude, which is completely specified only when its direction as well as its length is given. Such quantities are called *vectors*, the name arising from the significance of the operation of carrying a point from one end of the line to the other. Quantities which do not involve the idea of direction, and are completely specified by a single number, are distinguished by the name *scalars*, the name arising from the possibility of their representation upon a linear scale*. To specify the direction of a vector we must give two angular coordinates, which together with its length make three data. We may otherwise specify the vector symmetrically by giving its projections on three given mutually perpendicular axes. By projection on, resolved part or component along a line, we mean the product of the length of the vector by the cosine of the angle included between the direction of the vector and the positive direction of the line. If the angle is acute, the projection is positive, if obtuse, negative. In particular the projections of the vector on the axes of i, j, k, are the coefficients of i, j, k, in the representation of the vector by the complex number. It follows from the definition of addition of complex numbers that to add two vectors means to find a vector whose components are the sums of the corresponding components of the two given vectors. This vector may be described geometrically as

* If real. Complex scalars may also be used.

the diagonal of the parallelogram formed from the two given vectors as sides, or as given by applying the initial point of the second vector to the terminal point of the first, and constructing a new vector joining the initial point of the first to the terminal point of the second. Either of these geometrical processes shows that the addition of vectors is commutative. The addition of vectors in this manner is known as geometrical addition. When such geometrical addition of vectors is meant, as distinguished from arithmetical addition of their tensors, we shall denote the quantities to be considered as vectors by placing a bar over the quantity otherwise used for the tensor; e.g. the equations

$$\bar{R} = Xi + Yj + Zk,$$

$$R = |\bar{R}| = \sqrt{X^2 + Y^2 + Z^2},$$

are examples of vector and scalar equations respectively.

7. Geometric Multiplication. As we have seen in § 4, by direct multiplication, the product of the two vectors

$$\bar{R}_1 = X_1 i + Y_1 j + Z_1 k \text{ and } \bar{R}_2 = X_2 i + Y_2 j + Z_2 k$$

is $-(X_1X_2 + Y_1Y_2 + Z_1Z_2) + (Y_1Z_2 - Z_1Y_2)\,i$

$$+ (Z_1X_2 - X_1Z_2)\,j + (X_1Y_2 - Y_1X_2)\,k.$$

Of this the scalar part

$$-(X_1X_2 + Y_1Y_2 + Z_1Z_2)$$

has an important geometrical meaning. The direction cosines of the vector R being denoted by $\cos(Rx)$, $\cos(Ry)$, $\cos(Rz)$, we have by the definition of the projections of R,

$$X = R\cos(Rx),\quad Y = R\cos(Ry),\quad Z = R\cos(Rz).$$

Consequently,

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = R_1R_2\,\{\cos(R_1x)\cos(R_2x)$$
$$+ \cos(R_1y)\cos(R_2y) + \cos(R_1z)\cos(R_2z)\}.$$

The factor in the brackets is equal to the cosine of the angle between the directions of R_1 and R_2. Consequently the scalar part of the product of the vectors R_1 and R_2, or the *scalar product* of the two vectors, which will be denoted by the notation $\mathbf{S}\bar{R}_1\bar{R}_2$ is equal to minus the product of the tensor of either multiplied by the projection of the other on its own direction

$$\mathbf{S}\bar{R}_1\bar{R}_2 = -R_1R_2\cos(R_1R_2).$$

In order to avoid the inconvenience of the negative sign in this, Hamilton's notation, we shall call the negative of the scalar product the *geometric product* and denote it by $\widehat{R_1R_2}$, so that

$$(1) \qquad \widehat{R_1R_2} = X_1X_2 + Y_1Y_2 + Z_1Z_2 = R_1R_2 \cos(R_1R_2).$$

If two vectors are perpendicular, their scalar product is zero.

The vector part $\overline{R_3}$ of the product $\overline{R_1}\overline{R_2}$, or the *vector product*, which will be denoted by the symbol $\mathbf{V}R_1R_2$, has the components

$$(2) \quad X_3 = Y_1Z_2 - Z_1Y_2, \quad Y_3 = Z_1X_2 - X_1Z_2, \quad Z_3 = X_1Y_2 - Y_1X_2.$$

It is to be noticed that the suffixes 1, 2, 3, appear in cyclic order, as do the letters in the terms on the left and the first terms on the right. If we multiply these equations by the corresponding components of either R_1 or R_2, we get identically,

$$(3) \qquad \widehat{R_1R_3} = X_1X_3 + Y_1Y_3 + Z_1Z_3 = 0,$$

$$\widehat{R_2R_3} = X_2X_3 + Y_2Y_3 + Z_2Z_3 = 0,$$

showing that the vector product is perpendicular to each of the vectors involved. Squaring and adding the equations (2), we get

$$\begin{aligned} R_3^2 = X_3^2 + Y_3^2 + Z_3^2 = Y_1^2Z_2^2 + Z_1^2Y_2^2 + Z_1^2X_2^2 + X_1^2Z_2^2 \\ + X_1^2Y_2^2 + Y_1^2X_2^2 \\ - 2(Y_1Y_2Z_1Z_2 + Z_1Z_2X_1X_2 + X_1X_2Y_1Y_2) \\ = (X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2) - (X_1X_2 + Y_1Y_2 + Z_1Z_2)^2 \\ = R_1^2R_2^2(1 - \cos^2(R_1R_2)) = R_1^2R_2^2 \sin^2(R_1R_2), \end{aligned}$$

so that

$$(4) \qquad R_3 = |\mathbf{V}\overline{R_1}\overline{R_2}| = R_1R_2 \sin(R_1R_2).$$

The vector product of two vectors is accordingly perpendicular to their plane and its tensor is equal to the product of their tensors and the sine of their included angle, or geometrically, to the area of the parallelogram having them as sides.

The equations (1) and (2) show that

$$\widehat{R_1R_2} = \widehat{R_2R_1}, \quad \mathbf{V}R_1R_2 = -\mathbf{V}R_2R_1.$$

The vector product of two vectors is drawn with respect to their plane in such a manner that the rotation of a right-handed screw advancing in the direction of the vector product would turn the first vector towards the second, as is seen by making the two vectors coincide with two of the unit vectors *i*, *j*, *k*.

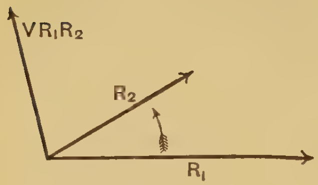

FIG. 2.

If two vectors are parallel, their vector product is zero. The vector and scalar products of two vectors cannot vanish simultaneously unless the tensor of one of the vectors vanishes.

NOTE. Although the above consideration of vectors has been inserted for the sake of logical connection, we shall seldom make use of the conception of a vector as a complex number, and when the term complex number is made use of we shall mean a complex number in the narrow sense. We shall frequently use the terms vector, scalar product and vector product, the latter being defined by the equations (1) and (2) above.

CHAPTER II.

VARIABLES AND FUNCTIONS.

8. **Functions.** A real quantity is said to vary continuously between two values a and b if it assumes successively all real values, rational and irrational, comprised in the interval between and including the values a and b. The notion of continuity was arrived at by considering the motion of a point which at successive instants of time occupies the positions of all the points between those representing a and b, and by the nature of motion cannot omit any intermediate value.

A quantity y is said to be a *function* of a variable x, in an interval from a to b, if for every value that x may take in the interval ab, there is assigned a definite value of y. A function defined in this somewhat restricted manner is called *uniform*, or one-valued. We may extend the definition so that for each value of x, y may have several values, in which case it is said to be a multiform, or many-valued function of x. This definition, due to Dirichlet, is independent of the question whether we can find an analytic expression for the value of y in terms of x or not. For example the analytic expressions

(1) $$a_0 + a_1 x + a_2 x^2 + \ldots\ldots a_n x^n,$$

(2) $$\frac{a_0 + a_1 x + \ldots\ldots + a_n x^n}{b_0 + b_1 x + \ldots\ldots + b_m x^m},$$

(3) $$\sqrt{x - a},$$

(4) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots,$$

(5) $$e^{\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{2!\,x^2} + \frac{1}{3!\,x^3} + \ldots\ldots,$$

(6) $$\sin x = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\ldots,$$

(7) $$\sin\frac{1}{x} = \frac{1}{x} - \frac{1}{2!\,x^2} + \frac{1}{3!\,x^3} - \ldots\ldots,$$

are all functions of x in any interval from a to b, where a and b are *both* positive or negative finite numbers different from zero. Of these the first three are examples of the class called algebraic functions, or such as are defined by an algebraic equation between x and y. An algebraic equation is one in which only a finite number of powers with finite integral exponents and products of such powers of the variables appear. All other functions are called transcendental, and the last four above are examples of such. All the above are uniform functions, except (3), which has two values, one of which is the negative of the other. A function such as (1) is called a polynomial, or a rational entire or integral function. A function such as (2), or the quotient of any two polynomials, is called a rational fractional function. (3) is an example of algebraic irrational functions. (4) and (6), being defined by convergent infinite series of positive powers of x, are called integral transcendental functions, and the quotient of two such is called a fractional transcendental function. The distinction between rational and transcendental functions is similar to that between rational and irrational real numbers, depending on the matter of finiteness or infinity in the method of specification.

A uniform, continuous, integral, rational or transcendental function is called *holomorphic.*

A function taking the value 1 from the value $x=0$, inclusive, to $x=1/2$, exclusive, the value 2 from $x=1/2$, inclusive, to $x=3/4$, exclusive, the value 3 from $x=3/4$, inclusive, to $x=7/8$, exclusive, etc., and a function defined as taking the value 1 for all rational points, and 2 for all irrational points, would be, the first difficult, the second probably impossible to define by analytic expressions. The former, being perfectly defined for every real value of x from zero to 1, excluding the latter, satisfies the definition of a function in that interval, while the latter satisfies it in any interval.

9. **Limit of a Function.** If $y=f(x)$ is a function of the continuous variable x in a certain interval including the value $x=a$, and if there exists a number A having the property that to any

positive number ϵ, however small, there may be found a corresponding number δ such that for *all* values of h of absolute value less than δ,

$$|f(a+h)-A| < \epsilon, \qquad |h| < \delta,$$

then the function is said to approach or converge to the limit A in the neighbourhood of the point $x=a$. This will be denoted by the equation

$$\lim_{x=a} f(x) = A.$$

The necessary and sufficient condition for the existence of a limit at a is that

$$|f(a+h)-f(a+h')| < \epsilon,$$

$$|h| < \delta, \qquad |h'| < \delta,$$

where ϵ, δ have the same significations as before, and h and h' are *any* values whose absolute values are less than δ. If the above condition is satisfied only when h and h' are positive, the function is said to approach the limit on the right of a, if when h and h' are both negative, on the left. A function may approach different limits on the two sides of a point. It is not necessary that a function should be varying always in the same sense in order to approach a limit, e.g. the function

$$y = x \sin x,$$

which alternately increases and decreases, approaches the value zero as a limit in the neighbourhood of $x=0$. The function

$$y = e^{-\frac{1}{x}}$$

approaches the limit zero on the right of $x=0$, but not on the left.

The function

$$y = \sin\frac{1}{x}$$

does not approach any limit whatever in the neighbourhood of $x=0$, for in any interval, however small, from

$$x = \frac{2}{(4n+1)\pi} \text{ to } x = \frac{2}{(4n+3)\pi},$$

where n is any integer, however great, the function takes all values from 1 to -1.

If a function does not approach any limiting value for a certain value of the variable, it must be otherwise defined for such a point; e.g. we may assign to the function defined at all points except $x=0$ by the analytical expression $\sin\frac{1}{x}$ any arbitrary value for the point $x=0$. The function will then be completely defined.

A quantity that approaches the limit zero is called an *infinitesimal.*

If y is a function of x defined in an interval a to b, where b is as large as we please, a number possessing the property that, when M is a given number as large as we please,

$$| f(x)-A | < \epsilon, \qquad x > M,$$

for *all* values of x greater than M, is said to be the limit of y as x increases indefinitely or, briefly, as x approaches infinity. This is denoted as follows:

$$\lim_{x=\infty} f(x) = A;$$

e.g.

$$\lim_{x=\infty} e^{\frac{1}{x}} = 1.$$

If in the above definition, we change M to a negative number whose absolute value is as great as we please, and consider all values of x less than M, we say that A is the limit as x approaches minus infinity.

If in the neighbourhood of a point $x=a$, when M is any number as great as we please, we can find a corresponding number δ such that for *all* values of h, whose absolute value is less than δ,

$$| f(a+h) | > M, \qquad | h | < \delta,$$

then y is said to become infinite for $x=a$. If, as above, we change the definition so that y is less than any negative number, y is said to become negatively infinite, or

$$\lim_{x=a} f(x) = -\infty.$$

The function

$$y = \frac{1}{x}\sin\frac{1}{x}$$

fails to approach any limit, finite or infinite, in the neighbourhood of the point $x=0$, by reason of its continued oscillation between

greater and greater positive and negative values in any interval, however small, including zero.

10. Continuity of Functions. A function is said to be continuous at a point $x=a$, if for any positive ϵ there is a δ such that

$$|f(a+h)-f(a)|<\epsilon, \qquad |h|<\delta$$

for *all* values of h whose absolute value is less than δ.

If the condition holds only for positive values of h, the function is said to be continuous on the right, if for negative, on the left of a. A function may be discontinuous at a point by reason of jumping abruptly from one finite value to another, becoming infinite, or oscillating through a finite or infinite range in an infinitesimal interval. The last function defined in § 8 is nowhere continuous, and the next to the last is discontinuous at the points 1/2, 3/4, 7/8, etc., for the first reason, the function $e^{\frac{1}{x}}$ is discontinuous at $x=0$ for the second reason, and the functions

$$\sin\frac{1}{x}, \quad \frac{1}{x}\sin\frac{1}{x},$$

are discontinuous at the same point for the third reason, $e^{\frac{1}{x}}$ is continuous at the left, discontinuous at the right of the point $x=0$.

A discontinuity arising from a finite jump, or an infinite increase or decrease, is called an ordinary discontinuity, while one arising from an oscillation is called a discontinuity of the second kind, and the value of x at which it occurs is called an essentially singular point for the function.

11. Derivative. In the neighbourhood of any value of the variable x, the difference-quotient

$$\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}$$

is a function of the increment h of the variable. If this quotient approaches a limit as h approaches 0, the value of the limit is called the derivative of the function $f(x)$ at the point x, and is denoted by $f'(x)$ or by

$$\frac{dy}{dx}=\lim_{h=0}\frac{f(x+h)-f(x)}{h}.$$

If a limit exists on one side but not on the other, the function is said to have a derivative on one side. If no limit exists at the point x, the function has no derivative at that point; e.g. the function $\sin\frac{1}{x}$ has no derivative at the point $x=0$, although the derivative at any other point however near is

$$-\frac{1}{x^2}\cos\frac{1}{x},$$

which does not approach a limit as x approaches 0. The last function defined in § 8 has no derivative anywhere. We may also find transcendental functions defined by analytic expressions, which nowhere possess a derivative. The function proposed by Weierstrass,

$$f(x)=\sum_{n=0}^{n=\infty} b^n \cos(a^n \pi x), \text{ where } 0<b<1;\ a \text{ is an odd integer,}$$

may be shown to have nowhere a derivative*.

12. Functions of two or more Variables. If two real variables x and y vary continuously in the respective intervals

$$x_0<x<x_1, \quad y_0<y<y_1,$$

and if to every possible pair of values of x and y is assigned a value of a quantity u, u is said to be a function of x and y. For any particular value of x, u is a function of y, and for any particular value of y, u is a function of x. Suppose that for a certain value y, u considered as a function of x approaches a limit as x approaches a. This limit will in general depend upon the value of y, let us call it

$$\lim_{x=a} u=\Phi(y).$$

It may again approach a limit as y approaches a value b. If we consider the limit approached by u considered as a function of y, we shall have in general a function of x,

$$\lim_{y=b} u=\Psi(x).$$

If x then approaches a, we may have a limit, which is not necessarily the same as before,

$$\lim_{y=b}\left\{\lim_{x=a} u\right\}=\lim_{y=b}\Phi(y)=A, \quad \lim_{x=a}\left\{\lim_{y=b} u\right\}=\lim_{x=a}\Psi(x)=B;$$

* Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 97; Harkness and Morley, *Theory of Functions*, p. 58.

e.g. the function $u = \dfrac{x}{y}$ has

$$\lim_{y=0} \left\{ \lim_{x=0} u \right\} = \lim_{y=0} (0) = 0,$$

since the limit for x does not contain y, while

$$\lim_{x=0} \left\{ \lim_{y=0} u \right\} = \infty.$$

A function of two or more variables is continuous at a point $x=a$, $y=b$, if for any positive value of ϵ, however small, we can find δ_1 and δ_2 so that

$$| f(x+h,\ y+k) - f(x,\ y) | < \epsilon, \qquad | h | < \delta_1, \qquad | k | < \delta_2$$

for *all* values of h and k which satisfy the above inequalities.

A function of two variables is not necessarily continuous if it is a continuous function of either variable; e.g. the function $xy/(x^2+y^2)$ is a continuous function of x for any value of y, even $y=0$, and of y for any value of x, even $x=0$. It is not a continuous function of x and y at $x=0$, $y=0$, since $u=0$ for $x=0$, irrespective of the value of y, and $u=0$ for $y=0$, irrespective of the value of x, but if we select pairs of values of x and y, such that $y=mx$, we have $u = \dfrac{m}{1+m^2}$, which is discontinuous with the value $u=0$ at $x=0$, $y=0$.

Derivative. If u considered as a function of x, for any particular value of y, has a derivative as before defined, this derivative is called the partial derivative of u with respect to x, and is denoted by f_x' or by

$$\frac{\partial f}{\partial x} = \lim_{h=0} \frac{f(x+h,\ y) - f(x,\ y)}{h}.$$

If $\dfrac{\partial f}{\partial x}$ considered as a function of y, say $\phi(y)$, has a derivative, this is called the partial derivative with regard to y of $\dfrac{\partial f}{\partial x}$, and is denoted by

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{k=0} \frac{\phi(y+k) - \phi(y)}{k}$$

$$= \lim_{k=0} \frac{1}{k} \left\{ \lim_{h=0} \frac{f(x+h,\ y+k) - f(x,\ y)}{h} \right\}.$$

It is evident that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y},$$

if f and its first derivatives are continuous functions of x *and* y.

The definitions here given may be extended to functions of any number of variables.

13. Point-Function. If a quantity has for every position of a point in a region of space τ one or more definite values assigned, it is said to be a function of the point, or point-function. This term was introduced by Lamé. If at every point it has a single value, it is a uniform function. Functions of the two or three rectangular coordinates of the point are point-functions. A point-function is continuous at a point A if we can find corresponding to any positive ϵ, however small, a value δ such that when B is *any* point inside a sphere of radius $< \delta$,

$$| f(B) - f(A) | < \epsilon.$$

We may have vector as well as scalar point-functions, the length and direction of the vector being given for every point. A vector point-function is continuous if its components along the coordinate axes are continuous point-functions.

14. Level Surface of Scalar Point-Function. If V is a uniform function of the point M, continuous and without maximum or minimum in a portion of space τ, through any point M in the region τ we may construct a surface having the property that for every point on it V has the same value.

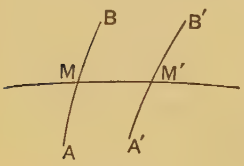

FIG. 3.

For let the value of V at M be c. Then since c is neither a maximum nor minimum, we can find in the neighbourhood of M two points A and B, such that at A, V is less, and at B, greater than c, and that in moving along a line AB through m, V continually increases. If the line AMB is displaced to the position $A'M'B'$, so that

$$| V(A) - V(A') | < c - V(A)$$

and

$$| V(B) - V(B') | < V(B) - c,$$

then $V(A') < c < V(B')$, therefore there is a point M on the line $A'B'$ for which $V = c$.

As AB moves continuously M describes a line, and this line in its motion describes a surface, for every point of which $V = c$. Such a surface is called a *level surface* of the function V. A level surface divides space into two parts, for one of which V is greater, and for the other less, than in the surface.

As examples of point-functions we may take (1) the length of a line drawn from the point M parallel to a given line until it cuts a given plane. Its level surfaces are planes parallel to the given plane. (2) The distance of M from a fixed point O. The level surfaces are spheres with centers at O. (3) The angle that the radius vector OM makes with a fixed line OX. The level surfaces are right circular cones with OX as axis. (4) The dihedral angle made by the plane MOX with a fixed plane through OX. The level surfaces are planes through OX.

15. Coordinates. If a point is restricted to lie on a given surface S, the intersection of that surface with the level surfaces of a function V are the level lines of the function on the surface S; e.g. in examples (3) and (4) above, if S is a sphere with O as center, the level lines are parallels and meridians respectively.

A function $f(V_1, V_2 \dots)$ of several point-functions is itself a point-function. If it is a function of one V only, its level surfaces are the same as those of V, for when V is constant, $f(V)$ is also constant.

Let q_1, q_2, q_3 be three uniform point-functions. Each has a level surface passing through the point M. If these three level surfaces do not coincide or intersect in a common curve, they determine the point M, and we may regard the point-functions q_1, q_2, q_3 as the *coordinates* of the point M. The level surfaces of q_1, q_2, q_3 are the coordinate surfaces, and the intersections of pairs (q_1q_2), (q_2q_3), (q_3q_1), are the coordinate lines. The tangents to the coordinate lines at M are called the coordinate axes at M. If at every point M the coordinate axes are mutually perpendicular, the system is said to be an orthogonal system.

16. Differential Parameter. The consideration of point-functions leads to the introduction of a particular sort of derivative. If V is a uniform point-function, continuous at a point M, and possessing there the value V, and at a point M' the value V', in

virtue of continuity, when the distance MM' is infinitesimal, $V'-V=\Delta V$ is also. The ratio

$$\frac{V'-V}{MM'}=\frac{\Delta V}{\Delta s}$$

is finite, and as $MM'=\Delta s$ approaches 0, the direction of MM' being given, the limit

$$\lim_{\Delta s=0}\frac{\Delta V}{\Delta s}=\frac{\partial V}{\partial s}$$

is defined as the *derivative of V in the direction s*. We may lay off on a line through M in the direction of s a length $MQ=\dfrac{\partial V}{\partial s}$ and as we give s successively all possible directions, we may find the surface that is the locus of Q.

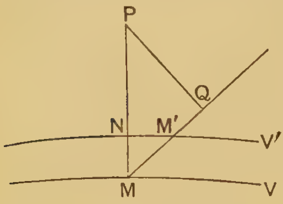

FIG. 4.

Let MN be the direction of the normal to the level surface at M, and let MP represent the derivative in that direction. Let M' and N be the intersections of the same neighbouring level surface, for which $V=V'$, with MQ and MP. Then

$$\frac{\Delta V}{MM'}=\frac{\Delta V}{MN}\,\frac{MN}{MM'}.$$

As MM' approaches zero, we have

$$\lim_{s=0}\frac{\Delta V}{\Delta s}=\frac{\partial V}{\partial s},\quad \lim\frac{MN}{MM'}=\cos PMQ.$$

Hence
$$\frac{\partial V}{\partial s}=\frac{\partial V}{\partial n}\cos PMQ,$$

that is, the derivative in any direction at any point is equal to the projection on that direction of the derivative in the direction of the normal to the level surface at that point. Accordingly all points Q lie on a sphere whose diameter is MP.

The derivative in the direction of the normal to the level surface was called by Lamé* the *first differential parameter* of the function V, and since it has not only magnitude but direction, we shall call it the *vector* differential parameter, or where no ambiguity

* G. Lamé. *Leçons sur les coordonnées curvilignes et leurs diverses applications.* Paris, 1859, p. 6.

will result, simply the parameter, denoted by $\overline{P}$ or $\overline{P}_V$. The above theorem may then be stated by saying that the derivative in any direction is the projection of the vector parameter in that direction. The theorem shows that the parameter gives the direction of the *fastest* increase of the function V.

If V is a function of a point-function q, $V=f(q)$, its level surfaces are those of q, and

$$P=\frac{\partial V}{\partial n}=\frac{dV}{dq}\frac{\partial q}{\partial n}=f'(q)\frac{\partial q}{\partial n},$$

and if

$$\pm\frac{\partial q}{\partial n}=h,\ P=\pm f'(q).h,$$

where the sign $+$ is to be taken if V and q increase in the same, $-$ if in opposite directions.

Suppose now that $V=f(q_1, q_2, q_3 \ldots\ldots)$

$$\frac{\partial V}{\partial s}=\frac{\partial V}{\partial q_1}\frac{\partial q_1}{\partial s}+\frac{\partial V}{\partial q_2}\frac{\partial q_2}{\partial s}+\frac{\partial V}{\partial q_3}\frac{\partial q_3}{\partial s}+\ldots\ldots$$

and if $h_1, h_2, \ldots\ldots$ denote the parameters of $q_1, q_2, \ldots\ldots$ the above theorem gives

$$P\cos(Ps)=\frac{\partial V}{\partial q_1}h_1\cos(h_1 s)+\frac{\partial V}{\partial q_2}h_2\cos(h_2 s)+\ldots\ldots$$

Now $\pm\dfrac{\partial V}{\partial q_i}h_i$ is the parameter of V, considered as a function of q_i, and we may call it the partial parameter P_i, and since P_i and h_i have the same sign if $\dfrac{\partial V}{\partial q_i}>0$, opposite signs if $\dfrac{\partial V}{\partial q_i}<0$, we have in either case

$$\frac{\partial V}{\partial q_i}h_i\cos(h_i s)=P_i\cos(P_i s),$$

and

$$P\cos(Ps)=P_1\cos(P_1 s)+P_2\cos(P_2 s)+\ldots\ldots.$$

This formula holds for any direction s and shows that the parameter P is the geometrical sum, or resultant, of the partial parameters,

$$\overline{P}=\overline{P}_1+\overline{P}_2+\ldots\ldots$$

Hence we have the rule for finding the parameter of any function of several point-functions. If we know the parameters

$h_1, h_2 \ldots\ldots$ of the functions $q_1, q_2 \ldots\ldots$ and the partial derivatives $\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots\ldots$ we lay off the partial parameters

$$P_i = \pm h_i \frac{\partial V}{\partial q_i} \ldots\ldots,$$

in the directions $h_1, h_2 \ldots\ldots$ or their opposites, according as $\frac{\partial V}{\partial q_i} > 0$, or the opposite, and find the resultant of $P_1, P_2, \ldots\ldots$

If the functions $q_1, q_2, \ldots\ldots$ are three in number, and form an *orthogonal* system, the equation

$$\overline{P} = \overline{P}_1 + \overline{P}_2 + \overline{P}_3,$$

gives for the modulus, or numerical value of the parameter

$$P^2 = P_1^2 + P_2^2 + P_3^2.$$

Examples. (1) in § 14. Let the distance of M in the given direction from the plane be u. $\Delta V = \Delta u = \frac{\Delta n}{\cos \alpha}$, where α is the angle between the given direction and that of a perpendicular to the given plane.

$$\frac{\Delta u}{\Delta n} = \frac{1}{\cos \alpha}, \quad P = \frac{1}{\cos \alpha}.$$

If the given direction is perpendicular to the given plane $P = 1$. Accordingly for $q_1 = x$, $q_2 = y$, $q_3 = z$, the rectangular coordinates of a point, we have $P_x = P_y = P_z = 1$, and for any function $f(x, y, z)$

$$P_1 = \pm \frac{\partial f}{\partial x}, \quad P_2 = \pm \frac{\partial f}{\partial y}, \quad P_3 = \pm \frac{\partial f}{\partial z},$$

$$P = \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 \right]^{\frac{1}{2}}.$$

The projections of P on the coordinate axes are the partial parameters

$$P_1 = P \cos (Px) = \frac{\partial f}{\partial x}, \quad P_2 = P \cos (Py) = \frac{\partial f}{\partial y}, \quad P_3 = P \cos (Pz) = \frac{\partial f}{\partial z}.$$

Consequently, if $\cos (sx)$, $\cos (sy)$, $\cos (sz)$ are the direction cosines of a direction s, the derivative in that direction

$$\begin{aligned} \frac{\partial V}{\partial s} &= P_1 \cos (sx) + P_2 \cos (sy) + P_3 \cos (sx) \\ &= \frac{\partial V}{\partial x} \cos (sx) + \frac{\partial V}{\partial y} \cos (sy) + \frac{\partial V}{\partial z} \cos (sz). \end{aligned}$$

P may be written in terms of the unit vectors i, j, k as

$$\overline{P} = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}.$$

The operator $i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}$ which gives the vector differential parameter of a function, was denoted by Hamilton by ∇. (read *Nabla*).

If $f(x, y, z)$ is a homogeneous function of degree n, by Euler's Theorem

$$nf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z},$$

or
$$nf = P\{x\cos(Px) + y\cos(Py) + z\cos(Pz)\}.$$

Now the $\pm$ parenthesis is the distance from the origin of the tangent plane to the level surface at x, y, z. Calling this δ,

$$nf = \pm P\delta, \quad P = \pm\frac{nf}{\delta},$$

or the parameter of a homogeneous function is inversely proportional to the perpendicular from the origin to the tangent plane to the level surface. For example, if $n=1$,

$$V = ax + by + cz,$$

$$P\cos(Px) = a, \quad P\cos(Py) = b, \quad P\cos(Pz) = c, \quad P = \sqrt{a^2+b^2+c^2}.$$

The level surfaces are parallel planes, and the parameter is constant,

$$P = \pm\frac{V}{\delta}, \quad V = \pm\delta\sqrt{a^2+b^2+c^2}.$$

V is proportional to the distance of the level surface from the origin.

If $n=2$,
$$V = \frac{x^2}{a_1} + \frac{y^2}{a_2} + \frac{z^2}{a_3},$$

$$P\cos(Px) = \frac{2x}{a_1}, \quad P\cos(Py) = \frac{2y}{a_2}, \quad P\cos(Pz) = \frac{2z}{a_3},$$

$$P = 2\sqrt{\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2}},$$

$$\delta = \pm\frac{2V}{P} = \pm\frac{V}{\sqrt{\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2}}}.$$

For the surface, $V=1$,

$$\delta = \pm\frac{1}{\sqrt{\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2}}},$$

a familiar result of analytic geometry.

17. Polar Coordinates. If we call the point-functions of Examples 2, 3, and 4, of § 14, r, θ, ϕ, we obtain the system of spherical, or polar coordinates. θ and ϕ may be called the co-latitude and longitude. The level surfaces of r being spheres, the normal coincides with r. Accordingly

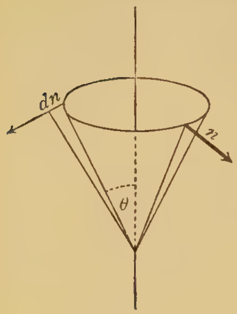

FIG. 5.

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial r} = 1, \quad h_r = 1.$$

The level surface of θ is a circular cone of angular opening θ, (Fig. 5), and

$$dn = r d\theta, \quad \frac{\partial \theta}{\partial n} = \frac{d\theta}{r d\theta} = \frac{1}{r}, \quad h_\theta = \frac{1}{r}.$$

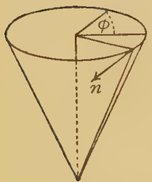

FIG. 6.

The level surfaces of ϕ are meridian planes through the axis of the above cones, (Fig. 6), and

$$dn = r \sin\theta d\phi, \quad \frac{\partial \phi}{\partial n} = \frac{d\phi}{r \sin\theta d\phi} = \frac{1}{r \sin\theta},$$

$$h_\phi = \frac{1}{r \sin\theta}.$$

For any function $f(r, \theta, \phi)$, the partial parameters are

$$P_r = \pm \frac{\partial f}{\partial r} h_r = \pm \frac{\partial f}{\partial r},$$

$$P_\theta = \pm \frac{\partial f}{\partial \theta} h_\theta = \pm \frac{1}{r} \frac{\partial f}{\partial \theta},$$

$$P_\phi = \pm \frac{\partial f}{\partial \phi} h_\phi = \pm \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi}.$$

The total parameter, the resultant of these, is given by

$$P^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2.$$

18. Cylindrical, or Semi-polar Coordinates. If we take the rectangular coordinate z, the perpendicular distance from the Z-axis, ρ, and ω the longitude, or angle made by the plane including the point M and the Z-axis and a fixed plane through that axis, we have the system of semi-polar, cylindrical, or columnar coordinates, for which we have immediately,

$$h_z = 1, \quad h_\rho = 1, \quad h_\omega = \frac{1}{\rho}.$$

The parameter of a function $f(z, \rho, \omega)$ is the resultant of the partial parameters

$$P_z = \pm \frac{\partial f}{\partial z}, \quad P_\rho = \pm \frac{\partial f}{\partial \rho}, \quad P_\omega = \pm \frac{1}{\rho} \frac{\partial f}{\partial \omega},$$

$$P^2 = \left(\frac{\partial f}{\partial z}\right)^2 + \left(\frac{\partial f}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial f}{\partial \omega}\right)^2.$$

19. Ellipsoidal Coordinates. The equation of a central quadric surface referred to its principal axes is

$$\frac{x^2}{a_1} + \frac{y^2}{a_2} + \frac{z^2}{a_3} = 1,$$

where a_1, a_2, a_3 may be positive or negative. If they are *all* negative, the surface is imaginary.

1°. Suppose *one* is negative, say

$$a_3 = -c^2,$$

while
$$a_1 = a^2, \; a_2 = b^2.$$

Let
$$a > b > c.$$

The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. The surface is cut by the XY-plane in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose semi-axes are a and b, and whose foci are at a distance from the center

$$\sqrt{a^2 - b^2} = \sqrt{a_1 - a_2}.$$

The section by the ZX-plane is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

with semi-axes a and c, and foci at distance $\sqrt{a^2 + c^2} = \sqrt{a_1 - a_3}$ on the X-axis.

The section by the YZ-plane is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

with semi-axes b and c and foci at a distance $\sqrt{b^2 + c^2} = \sqrt{a_2 - a_3}$ on the Y-axis.

The surface is an hyperboloid of one sheet.

2°. Let two of the constants a_1, a_2, a_3 be negative, say

$$a_2 = -b^2, \quad a_3 = -c^2.$$

The equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The sections by the coordinate planes and their focal distances are

XY	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$	Hyperbola	$\sqrt{a^2 + b^2}$	$= \sqrt{a_1 - a_2}$,
ZX	$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$	Hyperbola	$\sqrt{a^2 + c^2}$	$= \sqrt{a_1 - a_3}$,
YZ	$\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$	Imaginary Ellipse	$\sqrt{-(b^2 - c^2)}$	$= \sqrt{a_2 - a_3}$.

The surface is an hyperboloid of two sheets.

3°. If a_1, a_2, a_3 are all positive, the sections are ellipses, and the surface is an ellipsoid. In all three cases, the squares of the focal distances of the principal sections are differences of the three constants a_1, a_2, a_3. Accordingly if we add to the three the same number, we get a surface whose principal sections have the same foci as before, or a surface *confocal* with the original. Accordingly

$$(1) \qquad \frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1,$$

represents a quadric confocal with the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

for any real value of ρ.

If $a > b > c$ and $\rho > -c^2$, the surface is an ellipsoid. If $-c^2 > \rho > -b^2$, the surface is an hyperboloid of one sheet, and if $-b^2 > \rho > -a^2$ an hyperboloid of two sheets. If $-a^2 > \rho$, the surface is imaginary.

Suppose we attempt to pass through a given point x, y, z a quadric confocal with the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a > b > c.$$

Its equation is

$$\frac{x^2}{a^2+\rho}+\frac{y^2}{b^2+\rho}+\frac{z^2}{c^2+\rho}=1,$$

where ρ is to be determined. Clearing of fractions, the above is

$$(2)\quad (a^2+\rho)(b^2+\rho)(c^2+\rho)-x^2(b^2+\rho)(c^2+\rho)$$
$$-y^2(c^2+\rho)(a^2+\rho)-z^2(a^2+\rho)(b^2+\rho)=f(\rho)=0,$$

a cubic in ρ. Putting successively ρ equal to ∞, $-c^2$, $-b^2$, $-a^2$, and observing signs of $f(\rho)$,

$$\begin{array}{lll} \rho=\infty, & f(\rho)=+\infty & + \\ \rho=-c^2, & f(\rho)=-z^2(a^2-c^2)(b^2-c^2) & - \\ \rho=-b^2, & f(\rho)=-y^2(c^2-b^2)(a^2-b^2) & + \\ \rho=-a^2, & f(\rho)=-x^2(b^2-a^2)(c^2-a^2) & - \end{array}$$

The changes of sign of $f(\rho)$ show that there are three real roots. Call these λ, μ, ν in order of magnitude. λ lies in the interval $\lambda>-c^2$ necessary in order that the surface may be an ellipsoid, μ in the interval $-c^2>\mu>-b^2$ that it may be an hyperboloid of one sheet, and ν in the interval $-b^2>\nu>-a^2$ that it may be an hyperboloid of two sheets. There pass therefore through every point in space one surface of *each* of the three kinds. If we call

$$(3)\qquad F(\lambda,x,y,z)\equiv\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1,$$

the equation $F=0$ defines λ as a function of x, y, z, and therefore as a point-function. The normal to the surface $\lambda=$ const. has direction cosines proportional to

$$\frac{\partial\lambda}{\partial x},\ \frac{\partial\lambda}{\partial y},\ \frac{\partial\lambda}{\partial z}.$$

Now since identically $F=0$,

$$(4)\qquad \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz+\frac{\partial F}{\partial\lambda}d\lambda=0,$$

and we have

$$\frac{\partial\lambda}{\partial x}=\left(\frac{d\lambda}{dx}\right)_{dy=dz=0}=-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial\lambda}}.$$

Therefore

$$(5)\qquad \frac{\partial\lambda}{\partial x}=\frac{2x}{a^2+\lambda}\Big/\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}$$

$$=-\frac{2x}{(a^2+\lambda)F'(\lambda)},$$

$$\frac{\partial\lambda}{\partial y}=-\frac{2y}{(b^2+\lambda)F'(\lambda)},$$

$$\frac{\partial\lambda}{\partial z}=-\frac{2z}{(c^2+\lambda)F'(\lambda)}.$$

The parameter of the point-function λ is accordingly given by

$$(6)\qquad h_\lambda{}^2=\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(\frac{\delta\lambda}{\partial y}\right)^2+\left(\frac{\partial\lambda}{\partial z}\right)^2$$

$$=\frac{4}{(F'(\lambda))^2}\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\},$$

that is $$=-\frac{4}{F'(\lambda)}.$$

Now the direction cosines of the normal to the surface $\lambda=\text{const.}$ are

$$(7)\qquad \cos(n_\lambda x)=\frac{1}{h_\lambda}\frac{\partial\lambda}{\partial x}$$

$$=\pm\frac{\sqrt{-F'(\lambda)}}{2}\cdot\frac{2x}{(a^2+\lambda)F'\lambda}=\pm\frac{x}{(a^2+\lambda)\sqrt{-F'(\lambda)}},$$

$$\cos(n_\lambda y)=\pm\frac{y}{(b^2+\lambda)\sqrt{-F'(\lambda)}},$$

$$\cos(n_\lambda z)=\pm\frac{z}{(c^2+\lambda)\sqrt{-F'(\lambda)}}.$$

Similarly for the normals to the surface $\mu=\text{const.}$,

$$\cos(n_\mu x)=\pm\frac{x}{(a^2+\mu)\sqrt{-F'(\mu)}},$$

$$\cos(n_\mu y)=\pm\frac{y}{(b^2+\mu)\sqrt{-F'(\lambda)}},$$

$$\cos(n_\mu z)=\pm\frac{z}{(c^2+\mu)\sqrt{-F'(\mu)}}.$$

The angle between the normals to λ and μ is given by

$$(8) \qquad \cos(n_\lambda n_\mu) = \left\{\frac{x^2}{(a^2+\lambda)(a^2+\mu)} + \frac{y^2}{(b^2+\lambda)(b^2+\mu)} + \frac{z^2}{(c^2+\lambda)(c^2+\mu)}\right\} \frac{1}{\sqrt{F'(\lambda)F'(\mu)}}.$$

Now by subtracting from the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

the equation

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1,$$

we get

$$x^2\left\{\frac{1}{a^2+\lambda} - \frac{1}{a^2+\mu}\right\} + y^2\left\{\frac{1}{b^2+\lambda} - \frac{1}{b^2+\mu}\right\} + z^2\left\{\frac{1}{c^2+\lambda} - \frac{1}{c^2+\mu}\right\} = 0,$$

or

$$(9) \qquad (\lambda-\mu)\left\{\frac{x^2}{(a^2+\lambda)(a^2+\mu)} + \frac{y^2}{(b^2+\lambda)(b^2+\mu)} + \frac{z^2}{(c^2+\lambda)(c^2+\mu)}\right\} = 0.$$

Accordingly, unless $\lambda = \mu$, $\cos(n_\lambda n_\mu) = 0$ and the two normals are at right angles. Similarly for the other pairs of surfaces. Accordingly the three surfaces of the confocal system passing through any point cut each other at right angles.

If we give the values of λ, μ, ν, we determine completely the ellipsoid and two hyperboloids, and hence the point of intersection x, y, z (and its seven symmetrical points in the other quadrants). Hence we may take λ, μ, ν for the coordinates of the point, and the family of surfaces forms an orthogonal system. λ, μ, ν are called the ellipsoidal or elliptic coordinates of the point. We shall proceed to find their parameters in a form not containing any coordinates but λ, μ, ν. We must find the rate of change of λ as we go along the normal to the ellipsoid $\lambda = \text{const.}$

Since we have identically

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0,$$

differentiating totally

$$(10)\quad 2\left\{\frac{xdx}{a^2+\lambda}+\frac{ydy}{b^2+\lambda}+\frac{zdz}{c^2+\lambda}\right\} - d\lambda\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}=0.$$

Now if δ_λ is the perpendicular distance of the tangent plane from the origin, we have by the last formula of § 16,

$$\delta_\lambda = 1\Big/\sqrt{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}},$$

so that we may write for the cosines,

$$(11)\quad \cos(n_\lambda x)=\frac{x\delta_\lambda}{a^2+\lambda},\quad \cos(n_\lambda y)=\frac{y\delta_\lambda}{b^2+\lambda},\quad \cos(n_\lambda z)=\frac{z\delta_\lambda}{c^2+\lambda}.$$

Now as we move along the normal, we have

$$dx = dn\cos(n_\lambda x)=\frac{x\delta_\lambda}{a^2+\lambda}dn,$$

$$dy = dn\cos(n_\lambda y)=\frac{y\delta_\lambda}{b^2+\lambda}dn,$$

$$dz = dn\cos(n_\lambda z)=\frac{z\delta_\lambda}{c^2+\lambda}dn.$$

Inserting these values in (10),

$$(12)\quad 2\delta_\lambda\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}dn - \left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}d\lambda=0,$$

so that

$$(13)\quad h_\lambda=\frac{d\lambda}{dn}=2\delta_\lambda=\frac{2}{\sqrt{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}}}.$$

In order to express this result in terms of the elliptic coordinates alone we may express x, y, z, in terms of λ, μ, ν. Observe that the function

$$F(\rho)\equiv\frac{x^2}{\rho+a^2}+\frac{y^2}{\rho+b^2}+\frac{z^2}{\rho+c^2}-1$$

has as roots λ, μ, ν, and being reduced to the common denominator

$$(\rho+a^2)(\rho+b^2)(\rho+c^2)$$

has a numerator of the third degree in ρ. As this vanishes for

$$\rho=\lambda,\ \rho=\mu,\ \rho=\nu$$

it can only be

$$-(\rho-\lambda)(\rho-\mu)(\rho-\nu).$$

Hence we have the identity

$$(14)\quad F(\rho)\equiv\frac{x^2}{\rho+a^2}+\frac{y^2}{\rho+b^2}+\frac{z^2}{\rho+c^2}-1$$
$$\equiv\frac{-(\rho-\lambda)(\rho-\mu)(\rho-\nu)}{(\rho+a^2)(\rho+b^2)(\rho+c^2)}.$$

Multiplying this by $\rho+a^2$ and then putting $\rho=-a^2$ we get

$$(15)\qquad x^2=\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)},$$

and in like manner

$$y^2=\frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2=\frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}.$$

If λ, μ, ν are contained in the intervals above specified, these will all be positive, so that the point will be real.

If we insert these values in δ_λ, we shall have h_λ expressed in terms of λ, μ, ν.

This is more easily accomplished as follows.

Differentiating the above identity (14) according to ρ,

$$(16)\quad -\left\{\frac{x^2}{(a^2+\rho)^2}+\frac{y^2}{(b^2+\rho)^2}+\frac{z^2}{(c^2+\rho)^2}\right\}$$
$$\equiv\frac{-(\rho-\lambda)(\rho-\mu)(\rho-\nu)}{(a^2+\rho)(b^2+\rho)(c^2+\rho)}$$
$$\left\{\frac{1}{\rho-\lambda}-\frac{1}{\rho+a^2}+\frac{1}{\rho-\mu}-\frac{1}{\rho+b^2}+\frac{1}{\rho-\nu}-\frac{1}{\rho+c^2}\right\}.$$

If we put $\rho=\lambda$, all the terms on the right except the first, being multiplied by $\rho-\lambda$, vanish, and we have

$$(17)\quad \frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}=\frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

The expression on the left is $\frac{1}{\delta_\lambda^2}$. Hence

(18) $$h_\lambda = 2\delta_\lambda = 2\sqrt{\frac{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)}}.$$

In a similar manner we find

$$h_\mu = 2\sqrt{\frac{(a^2+\mu)(b^2+\mu)(c^2+\mu)}{(\mu-\nu)(\mu-\lambda)}},$$

$$h_\nu = 2\sqrt{\frac{(a^2+\nu)(b^2+\nu)(c^2+\nu)}{(\nu-\lambda)(\nu-\mu)}},$$

and the parameter of any function $V(\lambda, \mu, \nu)$ is

$$P^2 = \left(\frac{\partial V}{\partial\lambda}\right)^2 h_\lambda^2 + \left(\frac{\partial V}{\partial\mu}\right)^2 h_\mu^2 + \left(\frac{\partial V}{\partial\nu}\right)^2 h_\nu^2.$$

20. Infinitesimal Arc, Area and Volume. If we have any three point-functions q_1, q_2, q_3 forming an orthogonal system of coordinates, since their parameters are

$$h_1 = \frac{\partial q_1}{\partial n_1}, \quad h_2 = \frac{\partial q_2}{\partial n_2}, \quad h_3 = \frac{\partial q_3}{\partial n_3},$$

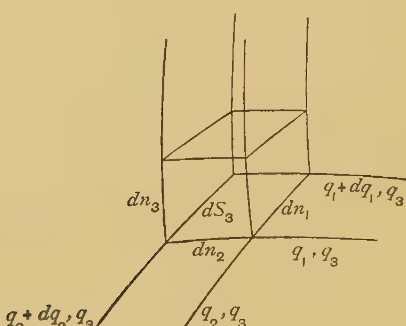

FIG. 7.

the normal distance between two consecutive level surfaces q_1 and $q_1 + dq_1$ is $dn_1 = \frac{dq_1}{h_1}$, consequently if we take six surfaces

$$q_1,\ q_1+dq_1,\ q_2,\ q_2+dq_2,\ q_3,\ q_3+dq_3,$$

the edges of the infinitesimal curvilinear rectangular parallelopiped whose edges are the intersections of the surfaces are

$$\frac{dq_1}{h_1}, \quad \frac{dq_2}{h_2}, \quad \frac{dq_3}{h_3},$$

and since the edges are mutually perpendicular, the diagonal, or

element of arc is

$$ds^2 = \frac{dq_1^2}{h_1^2} + \frac{dq_2^2}{h_2^2} + \frac{dq_3^2}{h_3^2},$$

the elements of area of the surfaces q_1, q_2, q_3 are respectively

$$dS_1 = \frac{dq_2 dq_3}{h_2 h_3}, \quad dS_2 = \frac{dq_3}{h_3}\frac{dq_1}{h_1}, \quad dS_3 = \frac{dq_1}{h_1}\frac{dq_2}{h_2},$$

and the element of volume is

$$d\tau = \frac{dq_1 dq_2 dq_3}{h_1 h_2 h_3}.$$

Examples. Rectangular coordinates x, y, z

$$h_x = h_y = h_z = 1,$$

$$dS_x = dy\,dz, \quad dS_y = dz\,dx, \quad dS_z = dx\,dy, \quad d\tau = dx\,dy\,dz.$$

Polar coordinates r, θ, ϕ,

$$h_r = 1, \quad h_\theta = \frac{1}{r}, \quad h_\phi = \frac{1}{r \sin\theta},$$

$dS_r = r^2 \sin\theta\, d\theta\, d\phi$ element of area of sphere,

$dS_\theta = r \sin\theta\, dr\, d\phi$ element of area of cone,

$dS_\phi = r\, dr\, d\theta$ element of area of plane,

$$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

Cylindrical coordinates, z, ρ, ω,

$$h_z = h_\rho = 1, \quad h_\omega = \frac{1}{\rho},$$

$dS_z = \rho\, d\rho\, d\omega$ element of area of plane,

$dS_\rho = \rho\, d\omega\, dz$ element of area of cylinder,

$dS_\omega = d\rho\, dz$ element of area of meridian plane,

$d\tau = \rho\, d\rho\, d\omega\, dz.$

Elliptic coordinates, λ, μ, ν.

$$dS_\lambda = \frac{d\mu\, d\nu \sqrt{(\mu-\nu)(\mu-\lambda)(\nu-\lambda)(\nu-\mu)}}{4\sqrt{(a^2+\mu)(b^2+\mu)(c^2+\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}} \quad \text{ellipsoid,}$$

$$dS_\mu = \frac{d\nu\, d\lambda \sqrt{(\nu-\lambda)(\nu-\mu)(\lambda-\mu)(\lambda-\nu)}}{4\sqrt{(a^2+\nu)(b^2+\nu)(c^2+\nu)(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \text{hyperboloid,}$$

$$dS_\nu = \frac{d\lambda\, d\mu \sqrt{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)(\mu-\lambda)}}{4\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)}} \quad \text{hyperboloid,}$$

$$d\tau = \frac{d\lambda\, d\mu\, d\nu\, (\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{8\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)(a^2+\mu)(b^2+\mu)(c^2+\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}}.$$

CHAPTER III.

DEFINITE INTEGRALS.

21. Definite Integral of a Function of one Variable. If we consider a continuous function of one real variable, the notion of its definite integral may be illustrated by means of a geometrical representation. If the function $y=f(x)$ be represented as the ordinate of a curve of which x is the abscissa, and if between two points $x=a$, $x=b$, we place any number $n-1$ of points $x_1, x_2, \ldots\ldots x_k, \ldots\ldots x_{n-1}$, and in the intervals between them erect ordinates to the curve at points $\xi_1, \xi_2 \ldots\ldots$ so that

$$a < \xi_1 < x_1,\ x_1 < \xi_2 < x_2 \ldots\ldots x_{k-1} < \xi_k < x_k \ldots\ldots x_{n-1} < \xi_n < b,$$

the sum

$$S = (x_1 - a) f(\xi_1) + (x_2 - x_1) f(\xi_2) \ldots\ldots + (b - x_{n-1}) f(\xi_n),$$

represents the area of the rectangles constructed on the bases

$$\delta_1 = x_1 - a,\ \delta_2 = x_2 - x_1 \ldots\ldots \delta_k = x_k - x_{k-1} \ldots\ldots \delta_n = b - x_{n-1},$$

with the altitudes $f(\xi_k)$. The value of this sum depends on the form of the curve or of the function $f(x)$, on the choice of the points of division, $x_1 \ldots\ldots x_n$, and of the points ξ_k within the intervals. It can be shown, however, that if all the differences δ_k are less than a certain value δ, all the values that S can take are confined between certain limits, and if the number of intervals increases so that δ decreases without limit while $\delta_1 + \delta_2 \ldots\ldots + \delta_n$ remains always equal to $b-a$, that these extreme values of S approach a common limit. This limit will represent the area of the space bounded by the axis of X, the ordinates erected at the points $x=a$ and $x=b$, and the curve representing the function

$$y = f(x).$$

This conception may be extended to any function whether continuous or not, and the limit, *if there be any*, approached by the sum

$$\lim_{n=\infty} S = \lim_{n=\infty} \sum_{k=1}^{k=n} \delta_k f(\xi_k),$$

as the number of intervals is made to increase without limit, is the definition of the definite integral of the function $f(x)$ from a to b. It is denoted by

$$\int_a^b f(x)\,dx,$$

$f(x)$ is called the integrand, a and b the limits, and ab the field of integration. Evidently the letter x in the symbol may be replaced by any other without affecting the integral. If the sum has a limit the function $f(x)$ is said to be *integrable* in the region from a to b.

22. Condition of Integrability. The *oscillation* of a function in a given interval is the difference between the greatest and the least value that it assumes in that interval. It is evident from the definition of continuity that if ϵ is a positive number as small as we please we may always find a number δ such that in *any* interval less than δ and lying in the region ab in which the function is continuous, the oscillation is less than ϵ.

Let $\xi_1 \ldots\ldots \xi_n$ be a system of ordinates for a system of subdivision $x_1 \ldots\ldots x_n$, and let $\xi_1' \ldots\ldots \xi_n'$ be a different set of ordinates contained in the same intervals $\delta_1, \delta_2 \ldots\ldots \delta_n$.

Then

$$\sum_1^n \delta_s f(\xi_s) - \sum_1^n \delta_s f(\xi_s') = \sum_1^n \delta_s (f(\xi_s) - f(\xi_s')).$$

Then we may find δ so that when all δ_s's become less than δ, every

$$|f(\xi_s) - f(\xi_s')| < \epsilon,$$

and consequently

$$\sum_1^n \delta_s (f(\xi_s) - f(\xi_s')) < \sum_1^n \delta_s \epsilon = (b-a)\,\epsilon.$$

As n increases indefinitely, ϵ decreases indefinitely, and

$$\lim_{n=\infty} \sum_1^n \delta_s f(\xi_s) = \lim_{n=\infty} \sum_1^n \delta_s f(\xi_s'),$$

so that the selection of the ordinates in the intervals does not affect the limit, if one exists.

If L_s is the greatest value of $f(x)$ in the interval δ_s, l_s the least, so that the oscillation in that interval is $D_s = L_s - l_s$, the two sums $S_1 = \sum_1^n \delta_s L_s$ and $S_2 = \sum_1^n \delta_s l_s$ must approach limits as n increases indefinitely, for S_1 is always greater than S_2, and as we increase n, S_1 can never increase and S_2 can never decrease. Now the sum $\sum_1^n \delta_s f(\xi_s)$ always lies between S_1 and S_2, therefore if their difference $\sum_1^n \delta_s D_s$ approaches the limit zero the sum $\sum_1^n \delta_s f(\xi_s)$ must approach a limit.

Consider now two different modes of subdivision of the interval ab,

$$x_1,\ x_2 \ldots\ldots x_{n-1} \text{ and } x_1',\ x_2' \ldots\ldots x'_{n'-1},$$

and the corresponding sums

$$S = \sum_1^n \delta_s f_s \text{ and } S' = \sum_1^{n'} \delta_s' f_s'.$$

Let the points $x_1 \ldots\ldots x_{n-1}$ and $x_1' \ldots\ldots x'_{n'-1}$ taken together form the system $r_1 \ldots\ldots r_{p-1}$ and let

$$\rho_1 = r_1 - a$$

$$\rho_2 = r_2 - r_1$$

$$\ldots\ldots\ldots\ldots$$

$$\rho_p = b - r_{p-1}.$$

In the interval x_{s-1}, x_s there may or may not fall an r. In general if $x_{s-1} = r_h$, x_s will be r_{h+t} $(t \geqq 1)$, so that

$$\delta_s = \rho_{h+1} + \rho_{h+2} \ldots\ldots + \rho_{h+t}.$$

Then

$$\delta_s f_s = \rho_{h+1} f''_{h+1} + \rho_{h+2} f''_{h+2} \ldots\ldots + \rho_{h+t} f''_{h+t}$$
$$+ \rho_{h+1}(f_s - f''_{h+1}) + \rho_{h+2}(f_s - f''_{h+2}) \ldots\ldots + \rho_{h+t}(f_s - f''_{h+t}),$$

where the f_k'''s are arbitrarily chosen values of f in the intervals ρ_k, and

$$\sum_1^n \delta_s f_s = \sum_1^p \rho_h f_h'' + P,$$

where

$$P = \sum_{s=1}^{s=n} [\rho_{h+1}(f_s - f''_{h+1}) + \rho_{h+2}(f_s - f''_{h+2}) \ldots\ldots + \rho_{h+t}(f_s - f''_{h+t})].$$

But in this sum P, for every s the greatest possible difference $f_s - f''_{n+k}$ is in absolute value not greater than the oscillation D_s. Therefore

$$|P| \leqq \sum_1^n \delta_s D_s.$$

In like manner

$$\sum_1^{n'} \delta_s' f_s' = \sum_1^p \rho_h f_h'' + P',$$

where

$$|P'| \leqq \sum_1^{n'} \delta_s' D_s'.$$

Then

$$\sum_1^n \delta_s f_s - \sum_1^{n'} \delta_s' f_s' = P - P',$$

and if $\lim\limits_{n=\infty} \sum_1^n \delta_s D_s = 0$ for all systems of division, the limit $\sum_1^n \delta_s f_s$ is the same for all systems of division. It is easy to show that if the condition $\lim\limits_{n=\infty} \sum_1^n \delta_s D_s = 0$ is satisfied for one mode of division, it is satisfied for all. This is then the necessary and sufficient condition that the function $f(x)$ shall be integrable in the interval ab.

23. Properties of Definite Integrals. It results immediately from the definition

$$\int_a^b f(x)\, dx = \lim_{n=\infty} \sum_1^n \delta_s f_s,$$

that if we interchange the limits a, b, since every δ_s changes sign, the sign of the integral is changed. More generally

$$(1) \qquad \int_a^b f(x)\, dx + \int_b^c f(x)\, dx + \int_c^a f(x)\, dx = 0.$$

The arithmetical mean of a number of quantities is defined as their sum divided by their number. If $f(x)$ is finite and integrable in an interval ab, and $\delta_1 \ldots\ldots \delta_n$, $\delta_1' \ldots\ldots \delta_{n'}'$ are two divisions of the interval, from the last equation of § 22,

$$\left|\sum_1^{n'} \delta_s' f_s' - \sum_1^n \delta_s f_s\right| \leqq \sum_1^n \delta_s D_s + \sum_1^{n'} \delta_s' D_s'.$$

Consider n constant and let n' increase without limit.

Then

$$\left|\int_a^b f(x)\, dx - \sum_1^n \delta_s f_s\right| \leqq \sum_1^n \delta_s D_s, \text{ so that } \sum_1^n \delta_s f_s,$$

is an approximate value of the integral whose error is less than

$$\sum_1^n \delta_s D_s.$$

We may put the δ_s's all equal, so that $\delta_s = \dfrac{b-a}{n}$. Then

$$(2) \qquad \frac{1}{b-a}\int_a^b f(x)\,dx = \lim_{n=\infty} \frac{1}{b-a} \sum_1^n \frac{(b-a)}{n} f_s = \lim_{n=\infty} \frac{1}{n} \sum_1^n f_s.$$

That is, the definite integral of a function in a given interval divided by the magnitude of the interval represents the arithmetical mean of all the values of the function taken at equidistant values of the variable throughout the interval, when the number of values taken is increased indefinitely.

From the definition it is evident that if $f(x)$ has the same sign throughout the interval ab, $\int_a^b f(x)\,dx$ has the sign of $(b-a)f(x)$, and if there is in ab a finite interval cd in which $f(x)$ is not zero, then $\int_c^d f(x)\,dx$ is not zero.

In particular, if the function is continuous in a whole interval ab, and the integral between *every* two values of x in the interval is zero, the function must be zero everywhere within the interval. If therefore two continuous functions give in every interval ab the same value of the integral, they must be equal everywhere in the interval.

Suppose that the continuous function $f(x)$ has in the interval ab a greatest value M and a least value m, the integral will have a value lying between $M(b-a)$ and $m(b-a)$ and we may write,

$$\int_a^b f(x)\,dx = A\,(b-a),$$

where

$$M > A > m.$$

Since $f(x)$ is continuous, it will take the value A for at least one value ξ of x between a and b, so that we may write

$$(3) \qquad \int_a^b f(x)\,dx = f(\xi)(b-a), \quad a < \xi < b.$$

The above formula may be generalized. We have always

$$(4) \qquad \left|\int_a^b f(x)\,dx\right| \leqq \int_a^b (\,|f(x)|\,)\,dx.$$

If in the interval ab, $(a<b)$, $f(x)$ and $\phi(x)$ are finite and integrable functions, and $|f(x)|$ lies always between M and m,

$$(5) \qquad \left|\int_a^b f(x)\,\phi(x)\,dx\right| \leqq M\int_a^b (|\,\phi(x)\,|)\,dx.$$

If in the interval $\phi(x)$ always has the same sign, since $M-f(x)$ and $f(x)-m$ are positive,

$$\int_a^b (M-fx)\,\phi(x)\,dx \text{ and } \int_a^b (f(x)-m)\,\phi(x)\,dx,$$

or

$$M\int_a^b \phi(x)\,dx - \int_a^b f(x)\,\phi(x)\,dx$$

and

$$\int_a^b f(x)\,\phi(x)\,dx - m\int_a^b \phi(x)\,dx,$$

have the same sign, and therefore $\int_a^b f(x)\,\phi(x)\,dx$ lies between $M\int_a^b \phi(x)\,dx$ and $m\int_a^b \phi(x)\,dx$ so that $\int_a^b f(x)\,\phi(x)\,dx$ is equal to $\int_a^b \phi(x)\,dx$ multiplied by a factor A lying between M and m.

If $f(x)$ is continuous, there is some point ξ in ab for which $f(\xi)=A$, and accordingly

$$(6) \qquad \int_a^b f(x)\,\phi(x)\,dx = f(\xi)\int_a^b \phi(x)\,dx, \quad a<\xi<b.$$

This important theorem is known as du Bois-Reymond's theorem of the mean.

24. Indefinite Integrals. Let $f(x)$ be integrable between a and b. The integral

$$\int_a^x f(x)\,dx$$

is zero for $x=a$, and for every value of x between a and b it has a definite value. It is therefore a function of its upper limit x. Let us denote it by $F(x)$. If $x+h$ be another value of x in ab,

$$F(x+h) = \int_a^{x+h} f(x)\,dx = \int_a^x f(x)\,dx + \int_x^{x+h} f(x)\,dx,$$

and

$$F(x+h)-F(x) = \int_x^{x+h} f(x)\,dx,$$

$$= hA, \qquad M>A>m.$$

Hence $F(x)$ is a continuous function of x. If α is any number between a and b, $\int_a^x f(x)\,dx$ differs from $\int_\alpha^x f(x)\,dx$ only by a constant, namely the value of the integral $\int_a^\alpha f(x)\,dx = C$. The function $F(x) + C$ is called the indefinite integral of $f(x)$.

Suppose that h approaches zero either positively or negatively, and let $f(x)$ either be continuous at x, or have an ordinary discontinuity, i.e., by making a finite jump.

Then for any positive number ϵ however small we can find a number h_1 of the same sign as h, such that for every x in the interval x, $x + h_1$ (at most excepting x), the value of $f(x)$ for any point differs by less than ϵ from $f(x+0)$ or $f(x-0)$, according as h is positive or negative.

Therefore the value $f(\xi)$ in the expression

$$F(x+h) - F(x) = hf(\xi),$$

differs from $f(x \pm 0)$ by less than ϵ and we have

$$\lim_{h=+0} \frac{F(x+h)-F(x)}{h} = f(x+0) \quad \lim_{h=-0} \frac{F(x+h)-F(x)}{h} = f(x-0).$$

That is, the integral $\int_a^x f(x)\,dx = F(x)$ is not only a finite and continuous function of x in the interval ab, but it has at all points where $f(x)$ is continuous a finite and determined derivative $f(x)$ and where $f(x)$ has an ordinary discontinuity, though not having a determined derivative, $F(x)$ has one on the right and left respectively equal to $f(x+0)$ and $f(x-0)$. If however $f(x)$ has a discontinuity of the second kind, at x, the value of

$$\frac{F(x+h) - F(x)}{h},$$

as h decreases does not approach a limit and $F(x)$ has no derivative at x.

The principle here proved enables us to calculate the definite integral whenever we can find a function $F(x)$ whose derivative is $f(x)$, for then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The definition usually given of the definite integral, as deduced from the indefinite integral by the above formula, is unsatisfactory,

the true nature of the definite integral being that of the limit of a sum.

25. Infinite Integrand or Limit. The definition of the definite integral presupposed that the integrand was finite in the field of integration ab. If there should exist points in the region ab at which $f(x)$ became infinite, the integral would in general have no meaning. In case however there is a single point c for which $f(x)$ becomes infinite, if h_1 and h_2 are positive numbers however small, the integrals $\int_a^{c-h_1} f(x)\,dx$, $\int_{c+h_2}^b f(x)\,dx$ have a definite meaning. If now as h_1 and h_2 approach zero independently of each other the sum

$$\int_a^{c-h_1} f(x)\,dx + \int_{c+h_2}^b f(x)\,dx,$$

approaches a definite finite limit, the value of that limit is what is meant by the definite integral,

$$\int_a^b f(x)\,dx.$$

For example, let

$$f(x) = \frac{1}{(x-c)^k} \qquad k > 0,$$

then for $x = c$, $f(x)$ becomes infinite.

$$\int_a^b \frac{dx}{(x-c)^k} = \lim_{h_1=0} \int_a^{c-h_1} \frac{dx}{(x-c)^k} + \lim_{h_2=0} \int_{c+h_2}^b \frac{dx}{(x-c)^k}$$

$$= \lim \frac{(-h_1)^{1-k} - (a-c)^{1-k} + (b-c)^{1-k} - (h_2)^{1-k}}{1-k}.$$

There is a limit as h_1 and h_2 approach zero only if $1 - k > 0$.

In like manner if the integral $\int_a^x f(x)\,dx$ approaches a finite limit when the limit x increases indefinitely, then this value defines the meaning of the definite integral

$$\int_a^\infty f(x)\,dx.$$

Let, as before,

$$f(x) = \frac{1}{(x-c)^k},$$

$$\int_a^x \frac{dx}{(x-c)^k} = \frac{(x-c)^{1-k} - (a-c)^{1-k}}{1-k}.$$

As x increases indefinitely, this approaches a finite limit only if $k>1$, when

$$\int_a^\infty \frac{dx}{(x-c)^k}=\frac{(a-c)^{1-k}}{k-1}.$$

26. Differentiation of a Definite Integral. Suppose that the integrand is a function of a parameter u as well as of x. Then in the case of a function of x that is capable of representation by a curve, if we change the parameter u we change the curve, and if $f(x, u)$ is a continuous function of u, to an infinitesimal change in u corresponds an infinitesimal change in the curve. The area represented by the definite integral $\int_a^b f(x, u)\,dx$ changes by the area of the narrow strip added to or included between the two curves, and we may find the ratio of this change to the given change in u. We thus get a geometrical notion of the meaning of the derivative of the integral with respect to u. Now by the definition of the derivative

$$\frac{d}{du}\int_a^b f(x, u)\,dx=\lim_{h=0}\frac{\int_a^b f(x, u+h)\,dx-\int_a^b f(x, u)\,dx}{h}$$

$$=\lim_{h=0}\int_a^b\left[\frac{f(x, u+h)-f(x, u)}{h}\right]dx.$$

It now becomes a question whether we may change the order of taking the limits involved in the integration and in making h approach zero. If $f(x, u)$ is a continuous function of x and u we may do this*, and since

$$\lim_{h=0}\frac{f(x, u+h)-f(x, u)}{h}=\frac{\partial f(x, u)}{\partial u},$$

we have

$$\frac{d}{du}\int_a^b f(x, u)\,du=\int_a^b\frac{\partial f(x, u)}{\partial u}\,dx.$$

We have already considered the definite integral as a function of its upper limit, and have found, § 24,

$$\frac{dF(v)}{dv}=\frac{d}{dv}\int_a^v f(x)\,dx=f(v).$$

* So Kronecker, *Theorie der einfachen und der vielfachen Integrale*, p. 26, (the word *gleichmässig* being superfluous, vid. Harkness and Morley, *Theory of Functions*, § 64). For a more careful statement, see Tannéry, *Théorie des Fonctions d'une Variable*, § 166.

In like manner

$$\frac{d}{dw}\int_w^b f(x)\,dx = \frac{d}{dw}\left(-\int_b^w f(x)\,dx\right) = -f(w).$$

If now u, v, w, are all functions of a variable t, we have for the derivative of the definite integral according to t,

$$\frac{dF}{dt} = \frac{d}{dt}\int_w^v f(x, u)\,dx = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} + \frac{\partial F}{\partial w}\frac{dw}{dt}$$

$$= \frac{du}{dt}\int_w^v \frac{\partial f(x, u)}{\partial u}\,dx + f(v, u)\frac{dv}{dt} - f(w, u)\frac{dw}{dt}.$$

27. Double and Multiple Integrals. Suppose we consider a continuous function of two variables, x varying from a to b, and y varying from g to h. We may represent $f(x, y)$ geometrically as the third coordinate z of a surface, erected perpendicular to the plane of xy. If now we subdivide the interval ab by points

$$a < x_1 < x_2 < \ldots\ldots x_s < x_{n-1} < b,$$

and the interval gh by points

$$g < y_1 < y_2 \ldots\ldots < y_r < y_{m-1} < h,$$

and draw through these points lines parallel to the axes of x and y, dividing the plane into rectangles, and at a point in each rectangle erect perpendiculars meeting the surface, the sum

$$\sum_{s=1}^{s=n}\sum_{r=1}^{r=m}(x_s - x_{s-1})(y_r - y_{r-1})f(\xi_s, \eta_r)$$

$$x_{s-1} < \xi_s < x_s$$

$$y_{r-1} < \eta_r < y_r,$$

represents the volume of the rectangular prisms constructed on the rectangles with sides $x_s - x_{s-1}$, $y_r - y_{r-1}$ as bases, and altitudes $f(\xi_s, \eta_r)$.

If as we make the number of points of subdivision increase without limit, the sum approaches a limit, this limit defines the definite integral

$$\int_a^b\int_g^h f(x, y)\,dx\,dy = \lim_{n=\infty}\lim_{m=\infty}\sum_{s=1}^{s=n}\sum_{r=1}^{r=m}(x_s - x_{s-1})(y_r - y_{r-1})f_{s,r}.$$

We shall find by reasoning similar to that used in § 22 that the condition for the existence of a limit is that the sum

$$\sum_{s=1}^{s=n}\sum_{r=1}^{r=m}(x_s - x_{s-1})(y_s - y_{s-1})D_{sr},$$

where D_{sr} is the oscillation in the interval x_{s-1}, x_s, y_{r-1}, y_r, approaches the limit zero.

In forming the double sum we may proceed with the summation according to x first, in which case

$$(y_r - y_{r-1}) \lim_{n=\infty} \sum_1^n (x_s - x_{s-1}) f(\xi_s, \eta_r) = (y_r - y_{r-1}) \int_a^b f(x, \eta_r)\, dx,$$

and the double limit is

$$\int_g^h \left(\int_a^b f(x, y)\, dx \right) dy.$$

Or we may sum first with respect to y, in which case

$$(x_s - x_{s-1}) \lim_{m=\infty} \sum_1^m (y_r - y_{r-1}) f(\xi_s, \eta_r) = (x_s - x_{s-1}) \int_g^h f(\xi_s, y)\, dy,$$

and the double limit is

$$\int_a^b \left(\int_g^h f(x, y)\, dy \right) dx.$$

But we have always

$$\sum_1^m (y_r - y_{r-1}) \left\{ \sum_1^n (x_s - x_{s-1}) f(\xi_s, \eta_r) \right\}$$
$$= \sum_1^n (x_s - x_{s-1}) \left\{ \sum_1^m (y_r - y_{r-1}) f(\xi_s, \eta_r) \right\},$$

however small $x_s - x_{s-1}$ and $y_r - y_{r-1}$. Accordingly,

$$\int_g^h \left\{ \int_a^b f(x, y)\, dx \right\} dy = \int_a^b \left\{ \int_g^h f(x, y)\, dy \right\} dx = \int_a^b \int_g^h f(x, y)\, dx\, dy.$$

(In writing a double or multiple integral we shall write the integral signs with their limits in the same order as the differentials.)

We might now deduce theorems for the double integral similar to those that we have already deduced for the single integral. In particular, the independence of the limit on the mode of subdivision, and the theorem of the mean may be demonstrated, and the extension of the definition made when the integrand or the limits become infinite. The definition of an integral may be extended to triple and multiple integrals in an obvious manner.

28. General Definition of Definite Integral. We have in the preceding definition of a double integral assumed that the limits of integration with respect to x and y were independent.

If instead of a rectangle in the XY-plane we should take any closed curve, given by an equation $\phi(x, y) = 0$, we could in like manner divide its area into rectangles, erecting perpendiculars in each, and define the definite integral as the limit of a similar sum for the new field of integration.

More generally, let $f(M)$ be a function of a point M, moving either in a plane or in space. If we divide any area S in the plane or any volume τ in space up into a number of parts, take the value of $f(M)$ at any point within each of those parts, multiply each value by the area or volume of the part in which it is taken, and add together for all the parts into which the area or volume is divided, the limit approached by this sum as the number of parts increases without limit in such a way that each dimension of every part approaches zero, if such a limit exists, is called the definite integral of $f(M)$ through the region in question. We may write the integrals

$$\iint f(M)\, dS \quad \text{or} \quad \iiint f(M)\, d\tau,$$

respectively. In each case, the field of integration must be expressly specified. It may be easily shown that this definition is equivalent to the preceding.

A particular mode of subdivision is by drawing level lines or surfaces for two or three orthogonal coordinates q_1, q_2, q_3. We have then, (§ 20),

$$dS = \frac{dq_1\, dq_2}{h_1\, h_2} \qquad d\tau = \frac{dq_1\, dq_2\, dq_3}{h_1\, h_2\, h_3}.$$

Suppose that in two different sets of coordinates

q_1, q_2, q_3, and p_1, p_2, p_3 with parameters, h_1, h_2, h_3, and g_1, g_2, g_3,

$$\iiint f(q_1, q_2, q_3)\, dq_1 dq_2 dq_3 = \iiint \phi(p_1, p_2, p_3)\, dp_1 dp_2 dp_3,$$

when taken through *any* equal finite portions of a volume τ.

Then when we consider the meaning of the definite integral and its independence of the manner of subdivision, we see as in (§ 23) that the above integrals, being respectively equal to

$$\iiint h_1 h_2 h_3 f(q_1, q_2, q_3)\, d\tau \text{ and } \iiint g_1 g_2 g_3\, \phi(p_1, p_2, p_3)\, d\tau,$$

can be equal only if the point functions

$$h_1 h_2 h_3 f(q_1, q_2, q_3) = g_1 g_2 g_3 \phi(p_1, p_2, p_3),$$

everywhere in the volume τ for the same point M, denoted by q_1, q_2, q_3 or p_1, p_2, p_3 in the respective coordinates.

29. Calculus of Variations. We shall frequently in what follows have to make use of the calculus of variations, which, since we shall use it always in connection with definite integrals, is introduced here.

In the differential calculus, we have to consider questions of maxima and minima of functions. A function of one variable has a maximum or minimum value at a certain value of the variable if the change in the function is of the same sign for any change in the variable, provided the latter change is small enough. Since if $f(x)$ is continuous at x,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots$$

$$f(x+h) - f(x) = hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots.$$

If h is small enough, the expression on the right will have the sign of the first term, which will change sign with h. Accordingly the condition for a maximum or minimum is

$$f'(x) = 0.$$

Suppose that we change the *form* of the function—such a change may be made to take place gradually. For instance suppose we have a curve given in any way, e.g.

$$x = F_1(t),\ y = F_2(t),\ z = F_3(t),$$

where the F's are any uniform and continuous functions of an independent variable t. If we change the form of the F's we shall change the curve—suppose we change to

$$x = G_1(t),\ y = G_2(t),\ z = G_3(t).$$

To every value of t corresponds one point on each curve, consequently to each point on one curve corresponds a definite point on the other. Such a change from one curve to the other is called a transformation of the curve. The change may be made gradually, e.g.

$$x = F_1(t) + \epsilon(G_1(t) - F_1(t)),$$
$$y = F_2(t) + \epsilon(G_2(t) - F_2(t)),$$
$$z = F_3(t) + \epsilon(G_3(t) - F_3(t)).$$

For every value of ϵ we shall have a particular curve—for $\epsilon=0$ we shall have the original curve, for $\epsilon=1$ the final curve, and for intervening values of ϵ other curves. A small change in ϵ will cause a small change in the curve, and if ϵ is infinitesimal we shall call the transformation an infinitesimal transformation. The changes in the values of x, y, z, or of any functions thereof, for an infinitesimal change ϵ, are called the *variations* of the functions, and are denoted by the sign δ.

Suppose we denote

$$\frac{dx}{dt},\ \frac{d^2x}{dt^2},\ \ldots\ldots \frac{d^kx}{dt^k} \text{ etc.}$$

by the letters

$$x',\ x'',\ x^{(k)},$$

and by ϕ any function

$$\phi\,(t, x, y, z, x', y', z', \ldots\ldots x^{(k)}, y^{(k)}, z^{(k)}, \ldots\ldots x^{(m)}, y^{(m)}, z^{(m)}),$$

and consider the change in ϕ made by an infinitesimal transformation, where we replace x, y, z by

$$x+\epsilon\xi\,(t),$$

$$y+\epsilon\eta\,(t),$$

$$z+\epsilon\zeta\,(t),$$

where ξ, η, ζ are *arbitrary* continuous functions of t.

Then $\frac{dx}{dt}$ or x' is replaced by $\frac{dx}{dt}+\epsilon\frac{d\xi}{dt}$ and $\frac{d^{(k)}x}{dt^k}$ by $\frac{d^kx}{dt^k}+\epsilon\frac{d^k\xi}{dt^k}$,

i.e., by $$x^{(k)}+\epsilon\xi^{(k)}.$$

Hence ϕ becomes

$$\phi\,(t, x+\epsilon\xi, y+\epsilon\eta, z+\epsilon\zeta, x'+\epsilon\xi', y'+\epsilon\eta', \ldots\ldots z^{(m)}+\epsilon\zeta^{(m)}),$$

which developed by Taylor's theorem for any number of variables, gives on collecting terms in equal powers of ϵ

$$\phi\,(t, x, y, z, x'\ldots\ldots)+\epsilon\phi_1+\frac{\epsilon^2}{2!}\phi_2\ldots\ldots+\frac{\epsilon^k}{k!}\phi_k+\ldots\ldots,$$

where

$$\phi_1=\xi\frac{\partial\phi}{\partial x}+\eta\frac{\partial\phi}{\partial y}+\zeta\frac{\partial\phi}{\partial z}+\xi'\frac{\partial\phi}{\partial x'}+\eta'\frac{\partial\phi}{\partial y'}+\zeta'\frac{\partial\phi}{\partial z'}+\ldots\ldots$$

$$\phi_2=\xi^2\frac{\partial^2\phi}{\partial x^2}+\eta^2\frac{\partial^2\phi}{\partial y^2}+\zeta^2\frac{\partial^2\phi}{\partial z^2}+2\xi\eta\frac{\partial^2\phi}{\partial x\partial y}\ldots\ldots+\xi'^2\frac{\partial^2\phi}{\partial x'^2}+\ldots\ldots$$

The terms $\epsilon\phi_1$, $\epsilon^2\phi_2$, $\epsilon^k\phi_k$ are called the first, second, kth variations of ϕ and denoted by

$$\epsilon\phi_1 = \delta\phi,\quad \epsilon^2\phi_2 = \delta^2\phi,\quad \epsilon^k\phi_k = \delta^k\phi.$$

If for ϕ we put successively x, y, z, x', y', z'...... we get

$$\delta x = \epsilon\xi,\qquad \delta y = \epsilon\eta,\qquad \delta z = \epsilon\zeta,\qquad \delta^2 x = \delta^2 y = \delta^2 z = 0,$$
$$\delta x' = \epsilon\xi',\qquad \delta y' = \epsilon\eta',\qquad \delta z' = \epsilon\zeta',\qquad \delta^2 x' = \delta^2 y' = \delta^2 z' = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\qquad\cdots\cdots\cdots\cdots$$
$$\delta x^{(k)} = \epsilon\xi^{(k)},\ \delta y^{(k)} = \epsilon\eta^{(k)},\ \delta z^{(k)} = \epsilon\zeta^{(k)},\ \delta^2 x^{(k)} = \delta^2 y^{(k)} = \delta^2 z^{(k)} = 0.$$

We thus see that the variations of x, y, z are infinitesimal *arbitrary* functions of t, the independent variable, and from the last equation

$$\delta\left(\frac{d^k x}{dt^k}\right) = \epsilon\frac{d^k\xi}{dt^k} = \frac{d^k}{dt^k}(\epsilon\xi) = \frac{d^k}{dt^k}\delta x,$$

that is, the operations of *differentiation* and *variation* are commutative, for the variables x, y, z.

It is evident that ϕ_k is the kth derivative of ϕ with respect to ϵ for the value $\epsilon = 0$.

Since we may always change the order of differentiation, it is evident that the commutative property holds for any function.

Let us now find the variation of the integral

$$I = \int_{t_0}^{t_1} \phi\,(t, x, y, z, x', y', z', \ldots\ldots)\,dt.$$

Changing x to $x + \delta x$, y to $y + \delta y$, x' to $x' + \delta x'$, etc.,

$$I + \delta I + \tfrac{1}{2}\delta^2 I + \ldots\ldots = \int_{t_0}^{t}(\phi + \delta\phi + \tfrac{1}{2}\delta^2\phi + \ldots\ldots)\,dt,$$

and the variations are

$$\delta I = \int_{t_0}^{t_1}\delta\phi\,dt,\quad \delta^k I = \int_{t_0}^{t_1}\delta^k\phi\,dt,$$

that is, the operations of variation and integration are commutative. (The limits have been supposed given, that is unvaried). These two principles of commutativity of δ with d and $\int$ form the basis of the subject.

30. Line and Surface Integrals. If we consider any curve in space joining a point A to a point B, and if on the curve between A and B we place $n-1$ points, $p_1, p_2 \ldots\ldots p_{n-1}$, whose

x-coordinates are $x_1, x_2 \ldots\ldots x_{n-1}$, multiply the length of each chord

$$p_{s-1}p_s, \quad \sqrt{(x_s-x_{s-1})^2+(y_s-y_{s-1})^2+(z_s-z_{s-1})^2},$$

by the value of the point function $f(p)$ at some point π_s in the arc between p_{s-1}, p_s, and take the sum for all the arcs into which the curve has been subdivided, then if this sum approaches a finite limit as the number of subdivisions increases indefinitely, this limit is called the line-integral of the point-function $f(p)$ along the curve AB, and is denoted by

$$\int_A^B f(p)\,ds = \lim_{n=\infty}\sum_1^n f(\pi_s)\sqrt{(x_s-x_{s-1})^2+(y_s-y_{s-1})^2+(z_s-z_{s-1})^2}.$$

If $f(p)=1$, the integral represents the length of the curve AB

$$\int_A^B ds = s_{AB}.$$

If in forming the line-integral we had multiplied the values of $f(\pi_s)$ by the x-projection of the chord, instead of by the chord itself, we should have arrived at the integral already defined,

$$\int f(p)\,dx = \lim_{n=\infty}\sum_1^n f(\pi_s)(x_s-x_{s-1}),$$

except that $f(p)$ is a function not of x alone, but of the point on the given curve corresponding to x. It will in general happen that as we go continuously along the curve from A, x will not increase continuously but will increase to a certain value C, and then decrease. As x decreases, however, reassuming previous values, we are still continuing along the curve and reaching new points and corresponding values of $f(p)$ which are to be used in the integral. The function f, which would otherwise not be uniform in x, becomes uniform when defined in this manner, so that if we interpret in the ordinary manner the integral $\int f(p)\,dx$, we must separate it into several integrals, in each of which x varies in one direction throughout, taking in each the values of f belonging to the corresponding part of the curve. If however we write

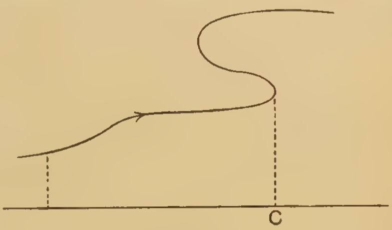

FIG. 8.

$$(x_s-x_{s-1}) = \frac{x_s-x_{s-1}}{l_s}\,l_s,$$

where $$l_s = \sqrt{(x_s - x_{s-1})^2 + (y_s - y_{s-1})^2 + (z_s - z_{s-1})^2},$$

$$\lim_{l_s=0} \frac{x_s - x_{s-1}}{l_s} = \frac{dx}{ds} = \cos(ds, x),$$

and take s for the independent variable, the above integral becomes

$$\int f\,dx = \int \left(f\frac{dx}{ds}\right) ds = \int_A^B f \cos(ds, x) \,.\, ds,$$

in which there can be no ambiguity.

In like manner if we divide the area of any surface S into parts, multiply the area of each by the value of a point-function $f(p)$ at some point on that part, and sum for all the parts, the

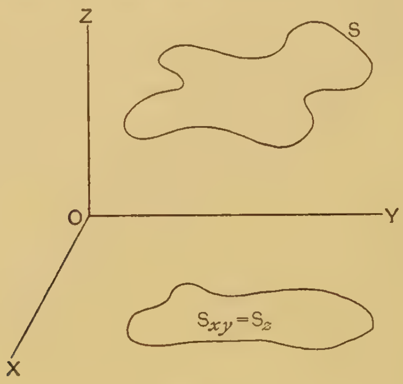

FIG. 9.

limit approached by the sum, if any, as both the dimensions of the parts approach zero, is called the surface-integral of $f(p)$ over the given portion of surface and denoted by

$$\iint f(p)\,dS.$$

Here if we multiply $f(p)$ not by the area of the part of the surface S, but by its projection on the XY-plane, we reduce the surface-integral to the double integral $\iint f(p)\,dS_{xy}$ already treated, with the exception that the point-function depends not only upon x and y but upon the surface S. If, as is generally the case, several regions of the surface project upon the same part of the XY-plane, the integral must be interpreted in an analogous manner to that used in the case of the line-integral. If n is the

normal to the surface S drawn always toward the same side of the surface, it is easily seen that the area of the projection of the element dS on the XY-plane is dS_{xy} or $dS_z = dS \cos(nz)$. We may accordingly write the surface integral,

$$\iint f(p)\cos(nz)\,dS = \iint f(p)\,dS_z = \iint f(p)\,dx\,dy,$$

with the understanding that in the last form the integral is to be taken over the projection of the surface S on the XY-plane, in such a manner that the projection is to be divided into regions for each of which the normal to S in the corresponding portions of S points either always towards the XY-plane or always away from it, and that those parts of the integral for which the normal points in opposite directions are given opposite signs. It will be seen that this corresponds exactly to the interpretation of the line-integral in terms of x, when x changes its direction of variation. The first form of the integral above, with S as the variable of integration, is preferable, its meaning being unambiguous.

31. Dependence of Line Integral on Path. Stokes's Theorem. Curl. The line integrals with which we shall have most to do are integrals of a vector point-function. If R is a vector function of the point, whose projections are X, Y, Z, functions of x, y, z, the component of R along the tangent to the curve AB at any point is, since the direction cosines of the tangent are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$,

$$R\cos(R, ds) = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}.$$

The line integral of this resolved component

$$I = \int_A^B R\cos(R, ds)\,.\,ds = \int_A^B \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds$$

may be written

$$\int_A^B (Xdx + Ydy + Zdz),$$

with the understanding of the previous section.

The functions X, Y, Z, being given for every point x, y, z, the integral I will in general depend on the form of the curve AB. If we make an infinitesimal transformation of the curve, the

integral will change, and we shall now seek an expression for the variation. We have

$$\delta I = \int\left(\delta X \cdot \frac{dx}{ds} + X\delta\frac{dx}{ds} + \delta Y \cdot \frac{dy}{ds} + Y\delta\frac{dy}{ds} + \delta Z\frac{dz}{ds} + Z\delta\frac{dz}{ds}\right) ds.$$

Now $$\delta X = \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y + \frac{\partial X}{\partial z}\delta z,$$

and $$\delta\frac{dx}{ds} = \frac{d(\delta x)}{ds}.$$

We may perform upon the term

$$\int_A^B X\frac{d(\delta x)}{ds}\,ds$$

an integration by parts

$$\int_A^B X\frac{d(\delta x)}{ds}\,ds = X\delta x\Big|_A^B - \int_A^B \delta x\frac{dX}{ds}\,ds,$$

where $X\delta x\Big|_A^B$ signifies that from the value of the function $X\delta x$ at the point B we subtract its value at A. Now

$$\frac{dX}{ds} = \frac{\partial X}{\partial x}\frac{dx}{ds} + \frac{\partial X}{\partial y}\frac{dy}{ds} + \frac{\partial X}{\partial z}\frac{dz}{ds}.$$

Performing similar operations on the other terms we have

$$\begin{aligned}\delta I = (X\delta x + Y\delta y + Z\delta z)\Big|_A^B &+ \int\Big[\left(\frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y + \frac{\partial X}{\partial z}\delta z\right)\frac{dx}{ds}\\
&+ \left(\frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y + \frac{\partial Y}{\partial z}\delta z\right)\frac{dy}{ds} + \left(\frac{\partial Z}{\partial x}\delta x + \frac{\partial Z}{\partial y}\delta y + \frac{\partial Z}{\partial z}\delta z\right)\frac{dz}{ds}\\
&- \delta x\left(\frac{\partial X}{\partial x}\frac{dx}{ds} + \frac{\partial X}{\partial y}\frac{dy}{ds} + \frac{\partial X}{\partial z}\frac{dz}{ds}\right)\\
&- \delta y\left(\frac{\partial Y}{\partial x}\frac{dx}{ds} + \frac{\partial Y}{\partial y}\frac{dy}{ds} + \frac{\partial Y}{\partial z}\frac{dz}{ds}\right)\\
&- \delta z\left(\frac{\partial Z}{\partial x}\frac{dx}{ds} + \frac{\partial Z}{\partial y}\frac{dy}{ds} + \frac{\partial Z}{\partial z}\frac{dz}{ds}\right)\Big]\,ds.\end{aligned}$$

Now if in the variation the ends of the curve A and B are fixed, δx, δy, δz vanish for A and B, and the integrated part

$X\delta x + Y\delta y + Z\delta z \Big|_A^B$ vanishes. Collecting those terms under the sign of integration that do not cancel, we have,

$$\delta I = \int\Big[(\delta y dz - \delta z dy)\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + (\delta z dx - \delta x dz)\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + (\delta x dy - \delta y dx)\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\Big].$$

Now the determinant $\delta y dz - \delta z dy$ is the area of a parallelogram in the YZ-plane the projections of whose sides on the Y- and Z-axes are dy, dz, δy, δz. That is, if we consider the infinitesimal parallelogram whose vertices are the points s, $s+ds$ and their transformed positions, the above determinant is the area of its projection on the YZ-plane. If the area of the parallelogram is dS and n is the direction of its normal, we have as in § 30

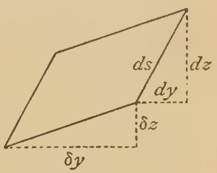

FIG. 10.

$$\delta y dz - \delta z\, dy = dS \cos(nx),$$
$$\delta z\, dx - \delta x dz = dS \cos(ny),$$
$$\delta x dy - \delta y dx = dS \cos(nz),$$

and $$\delta I = \int\left\{\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(nx) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\cos(ny) + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\cos(nz)\right\} dS,$$

which is in the form of a surface integral over the strip of infinitesimal width.

If we again make an infinitesimal transformation, and so continue until the path has swept out any finite portion of a

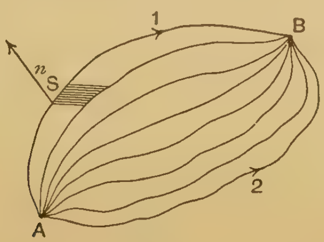

FIG. 11.

surface S, and sum all the variations of I, we get for the final result that the difference in I for the two extreme paths 1 and 2 is the surface integral

$$\lim \Sigma\delta I = I_2 - I_1 = \iint\left\{\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(nx) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\cos(ny) + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\cos(nz)\right\} dS$$

taken over the portion of the surface bounded by the paths 1 and 2 from A to B. Now $-I_1$ may be considered the integral from B to A along the path 1, so that $I_2 - I_1$ is the integral around the closed path which forms the contour of the portion of surface S. We accordingly get the following, known as

STOKES'S THEOREM*. The line integral, around any closed contour, of the tangential component of a vector R, whose components are X, Y, Z, is equal to the surface integral over any portion of surface bounded by the contour, of the normal component of a vector ω, whose components ξ, η, ζ are related to X, Y, Z by the relations

$$\xi = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z},$$

$$\eta = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x},$$

$$\zeta = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}.$$

The normal must be drawn toward that side of the surface that shall make the rotation of a right-handed screw advancing along the normal agree with the direction of traversing the closed contour of integration.

$$\int \overline{R}\overline{ds} = \int X dx + Y dy + Z dz = \iint \omega \cos(\omega n)\, dS$$

$$= \iint (\xi \cos(nx) + \eta \cos(ny) + \zeta \cos(nz))\, dS.$$

The vector ω related to the vector point-function R by the differential equations above is called the *rotation*, *spin* (Clifford), or *curl* (Maxwell and Heaviside) of R. Such vectors are of frequent

* The proof here given is from the author's notes on the lectures of Professor von Helmholtz. A similar treatment is given by Picard, *Traité d'Analyse*, Tom. I, p. 73.

occurrence in mathematical physics. The curl may be derived from the primitive vector by the application of Hamilton's vector differential operator ∇ (§ 16) to a vector point-function $\bar{R}$,

$$\nabla R = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(iX + jY + kZ)$$

$$= -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + i\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)$$

$$+ j\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + k\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right),$$

$$\text{curl } R = \mathbf{V}\nabla\bar{R} = i\xi + j\eta + k\zeta = \omega.$$

So that the vector part resulting from the application of the operation ∇ to a *vector* point-function gives its curl. The scalar part

$$-\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$

has an important interpretation to be given shortly.

[The significance of the geometrical term *curl* can be seen from the physical example in which the vector R represents the velocity of a point instantaneously occupying the position x, y, z in a rigid body turning about the Z-axis with an angular velocity ω. Then the vector $R = \omega\rho$ is perpendicular to the radius ρ and its components are

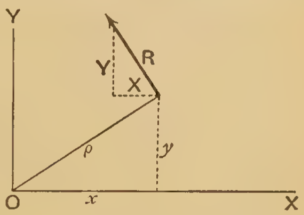

FIG. 12.

$$X = R\cos(Rx) = -R\sin(\rho x) = -R\frac{y}{\rho} = -y\omega,$$

$$Y = R\cos(Ry) = \quad R\cos(\rho x) = \quad R\frac{x}{\rho} = \quad x\omega,$$

where ω is constant, and

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 2\omega.$$

So that the z-component of the curl of the linear velocity is twice the angular velocity about the Z-axis.]

32. Lamellar Vectors. In finding the variation of the integral I in the previous section, since the variations δx, δy, δz are perfectly arbitrary functions of s, if the integral is to be inde-

pendent of the path, δI must vanish, which can only happen for all possible choices of δx, δy, δz if

$$\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}=0,$$

that is, if the curl of R vanishes everywhere. In case this condition is satisfied, I depends only on the positions of the limiting points A and B, and not on the path of integration. If A is given, I is a point-function of its upper limit B, let us say ϕ. If B is displaced a distance s in a given direction to B', the change in the function ϕ is

$$\phi_{B'}-\phi_B=\int_B^{B'}(X dx+Y dy+Z dz),$$

and the limit of the ratio of the change to the distance

$$\lim_{s=0}\frac{\phi_{B'}-\phi_B}{s}=\frac{\partial\phi}{\partial s}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}$$

is the derivative of ϕ in the direction s.

If we take s successively in the directions of the axes of coordinates,

$$\frac{\partial\phi}{\partial x}=X,\quad\frac{\partial\phi}{\partial y}=Y,\quad\frac{\partial\phi}{\partial z}=Z,$$

that is, R is the vector differential parameter of the scalar function ϕ.

Accordingly the three equations of condition equivalent to curl $R=0$ are simply the conditions that X, Y, Z may be represented as the derivatives of a point-function. In this case the expression

$$X dx+Y dy+Z dz=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi$$

is called a perfect differential.

From the definition of the parameter of a scalar point-function, we see that the magnitude of the parameter is inversely proportional to the normal distance between two infinitely near level surfaces of the function. Such a pair of surfaces will be called a thin *level sheet* or lamina. For this reason a vector point-function that may be represented everywhere in a certain region as the

vector parameter of a scalar point-function will be called a *laminar*, or *lamellar* vector (Maxwell).

The scalar function ϕ (or its negative) will sometimes be termed the *potential* of the vector R.

33. Connectivity of Space. Green's Theorem. We have supposed in § 31 that it was possible to change the path 1 from A to B into the path 2 by continuous deformation, without passing out of the space considered. A portion of space in which any path between two points may be thus changed into any other between the same two points is said to be singly-connected. For instance, in the case of a two-dimensional space, any area bounded by a single closed contour will have this property. If, however, we consider an area bounded externally by a closed contour C, and internally by one or more closed contours I, Fig. 13, such as the surface of a lake containing islands, it will be possible to go from any point A to any other point B by two routes which cannot be continuously changed into each other without passing out of the space considered, that is traversing the shaded part.

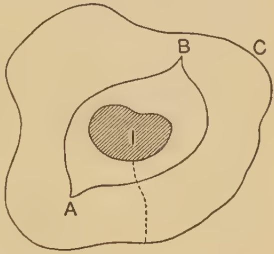

FIG. 13.

The space in Fig. 13 between the contour C and the island I is said to be doubly-connected. We may make it singly-connected by drawing a barrier connecting the island with the contour C, represented by the dotted line. If no path is allowed which crosses the barrier the space is singly-connected.

A three-dimensional space bounded externally by a single closed surface is not made doubly-connected by containing an inner closed boundary. For instance, the space lying between two concentric spheres allows all paths between two given points to be deformed into each other, avoiding the inner sphere. But the space bounded by an endless tubular surface, Fig. 14, is doubly-connected, because we may go from A to B in either direction of the tube, and the two paths cannot be deformed into each other. We may make the space singly-connected by the insertion of

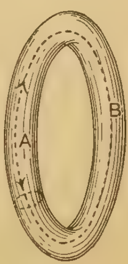

FIG. 14.

a barrier in the shape of a diaphragm, closing the tube so that one of the paths is inadmissible.

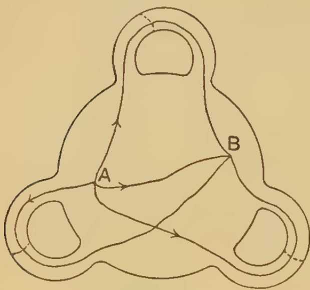

Fig. 15.

The connectivity of a portion of space is defined as one more than the least number of barriers or diaphragms necessary to make it singly-connected. Thus the space in a closed vase with three hollow handles, Fig. 15, is quadruply-connected. *We shall always suppose the spaces with which we deal in this book to be singly-connected, or to be made so by the insertion of diaphragms, unless the contrary is expressly stated.*

Suppose that W is a point-function which, together with its derivative in any direction, is uniform and continuous in a certain portion of space τ bounded by a closed surface S. Then its derivative $\partial W/\partial x$ is finite in the whole region, and if we multiply it by the element of volume $d\tau$ and integrate throughout the volume τ, the integral is finite, being less than the maximum value attained by $\partial W/\partial x$ in the space τ multiplied by the volume τ. We have at once

$$J = \iiint \frac{\partial W}{\partial x}\, dx\, dy\, dz = \iint dy\, dz \left[\int \frac{\partial W}{\partial x}\, dx \right].$$

If, keeping y and z constant, we perform the integration with respect to x, the volume is divided into elementary prisms whose

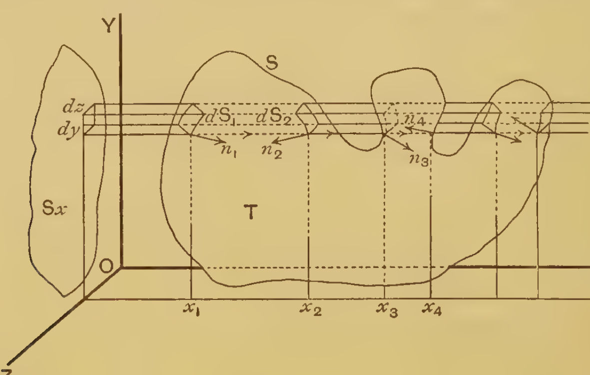

Fig. 16.

sides are parallel to the X-axis, and whose bases are rectangles with sides dy, dz.

The portion of the integral due to one such prism is

$$dydz\int\frac{\partial W}{\partial x}\,dx.$$

Now the integral is to be taken between the values of x where the edge of the elementary prism cuts into the surface S and where it cuts out from the surface. If it cuts in more than once, it will, since the surface is closed, cut out the same number of times. Let the values of x, at the successive points of cutting, be

$$x_1,\ x_2\ \ldots\ldots\ x_{2n},$$

then $$\int\frac{\partial W}{\partial x}\,dx = W_2 - W_1 + W_4 - W_3 \ldots + W_{2n} - W_{2n-1},$$

W_k being the value of W for x_k, and

$$\iiint\frac{\partial W}{\partial x}\,dx\,dy\,dz$$
$$= \iint[W_2 - W_1 + W_4 - W_3 \ldots + W_{2n} - W_{2n-1}]\,dy\,dz.$$

Now let dS_1, $dS_2 \ldots dS_{2n}$ denote the areas of the elements of the surface S cut out by the prism in question at x_1, x_2, ... x_{2n}—these all have the same projection on the YZ-plane, namely $dydz$. Now if all these elements are considered positive, and if n be the normal always drawn *inward* from the surface S toward the space τ, at each point of cutting into the surface S, n makes an acute angle with the positive direction of the axis of X, and the projection of dS is

$$dydz = dS\cos(nx),$$

but where the edge cuts out n makes an obtuse angle, with negative cosine, and therefore

$$dydz = -\,dS\cos(nx).$$

We may accordingly write

$$\begin{aligned}
dydz\,W_1 &= W_1\ \cos(n_1x)\,dS_1,\\
-\,dydz\,W_2 &= W_2\ \cos(n_2x)\,dS_2,\\
dydz\,W_3 &= W_3\ \cos(n_3x)\,dS_3,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
-\,dydz\,W_{2n} &= W_{2n}\cos(n_{2n}x)\,dS_{2n},
\end{aligned}$$

and in integrating with respect to y and z we cover the whole of the projection of the surface S on the YZ-plane. On the other hand we cover the whole of the surface S, so that the volume integral is transformed into a surface integral,

$$J = \iiint \frac{\partial W}{\partial x}\, d\tau = \iint dy\,dz\,[W_2 - W_1 + \ldots - W_{2n-1}]$$

$$= -\iint W \cos(nx)\, dS$$

taken all over the surface S.

In like manner we may transform the two similar integrals

$$\iiint \frac{\partial W}{\partial y}\, d\tau = -\iint W \cos(ny)\, dS,$$

$$\iiint \frac{\partial W}{\partial z}\, d\tau = -\iint W \cos(nz)\, dS.$$

Applying this lemma to the function

$$W = U\frac{\partial V}{\partial x},$$

where both U, V and their derivatives in any direction are uniform and continuous point-functions in the space τ, we have

$$\iiint \frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial x}\right) d\tau = -\iint U\frac{\partial V}{\partial x}\cos(nx)\, dS.$$

Similarly for
$$W = U\frac{\partial V}{\partial y},$$

$$\iiint \frac{\partial}{\partial y}\left(U\frac{\partial V}{\partial y}\right) d\tau = -\iint U\frac{\partial V}{\partial y}\cos(ny)\, dS;$$

and for
$$W = U\frac{\partial V}{\partial z},$$

$$\iiint \frac{\partial}{\partial z}\left(U\frac{\partial V}{\partial z}\right) d\tau = -\iint U\frac{\partial V}{\partial z}\cos(nz)\, dS.$$

Adding these three, and performing the differentiations,

$$\iiint \left[U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) + \frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right] d\tau$$

$$= -\iint U\left(\frac{\partial V}{\partial x}\cos(nx) + \frac{\partial V}{\partial y}\cos(ny) + \frac{\partial V}{\partial z}\cos(nz)\right) dS,$$

or, transposing, and denoting the symmetrical integral by J,

$$(1)\quad J = \iiint \left[\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right] d\tau$$

$$= -\iint U \left\{ \frac{\partial V}{\partial x}\cos(nx) + \frac{\partial V}{\partial y}\cos(ny) + \frac{\partial V}{\partial z}\cos(nz) \right\} dS$$

$$- \iiint U \left\{ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right\} d\tau.$$

This result is known as GREEN'S THEOREM*.

By the definition of differentiation in any direction the parenthesis in the surface integral on the right is

$$\frac{\partial V}{\partial n} = P_V \cos(P_V n),$$

if P_V is the parameter of V.

Since the integral on the left is symmetrical in U and V, we may interchange them on the right, so that

$$J = -\iint V \frac{\partial U}{\partial n} dS - \iiint V \left\{ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right\} d\tau.$$

Writing this equal to the former value, and transposing, we obtain

$$(2)\quad \iint \left[U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right] dS$$

$$= \iiint \left\{ V\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) - U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) \right\} d\tau,$$

which will be referred to as Green's theorem in its second form.

It will be noticed that the integrand on the left in the first form is the geometric product of the parameters of the functions U and V,

$$\widehat{P_U P_V}.$$

We shall, unless the contrary is stated, always mean by n the *internal* normal to a closed surface, but if necessary shall distinguish the normals drawn internally and externally as n_i and n_e. If we do not care to distinguish the inside from the outside we shall denote the normals toward opposite sides by n_1 and n_2.

* *An Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism.* Nottingham, 1828. Geo. Green, Reprint of papers, p. 25.

34. Second Differential Parameter.

If for the function U we take a constant, say 1,

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial y}=\frac{\partial U}{\partial z}=0, \quad P_U=0, \quad \widehat{P_U P_V}=0,$$

and we have simply

$$-\iint P_V \cos(P_V n)\, dS=-\iint \frac{\partial V}{\partial n} dS=\iiint\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right) d\tau.$$

The function

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2},$$

which, following the usage of the majority of writers, we shall denote by ΔV, was termed by Lamé* the second differential parameter of V. As it is a scalar quantity it will be sufficiently distinguished from the first parameter if we call it the scalar parameter. We have accordingly the theorem giving the relation between the two:—

The volume integral of the scalar differential parameter of a uniform continuous point-function throughout any volume is equal to the surface integral of the vector parameter resolved along the *outward* normal to the surface S bounding the volume.

We may obtain a geometrical notion of the significance of ΔV in a number of ways. Applying the above theorem to the volume enclosed by a small sphere of radius R, we have, since n is in the direction of the radius, but drawn *inwards*,

$$-\frac{\partial V}{\partial n}=\lim_{R=0}\frac{V_S-V_0}{R},$$

$$-\iint\frac{\partial V}{\partial n}dS=\lim_{R=0}\iint\frac{V_S-V_0}{R}dS$$

$$=\lim_{R=0}\frac{1}{R}\left\{\iint V_S dS-V_0 S\right\},$$

where V_0 is the value of V at the centre of the sphere, V_S on the surface. Now remembering the significance of a definite integral as a mean, we have

$$\lim_{R=0}\frac{1}{R}\{\text{Mean of } V \text{ on surface} - V \text{ at center}\}\times\text{Area of surface}$$

$$=\iiint\Delta V d\tau=(\text{Mean of } \Delta V \text{ in sphere})\times\text{Volume of sphere}.$$

* G. Lamé. *Leçons sur les Coordonnées curvilignes et leurs diverses Applications.* Paris, 1859, p. 6.

Now since the volume of a sphere is $\frac{1}{3}$ the product of the surface by the radius, we have, on making R approach zero,

$$\Delta V = 3 \operatorname*{Lim}_{R=0} \frac{\{\text{Mean } V \text{ on surface} - V \text{ at center}\}}{R^2}.$$

The negative scalar parameter $-\Delta V$ was accordingly called by Maxwell the *concentration* of V, being proportional to the excess of the value of V at any point over the mean of the surrounding values. It is evident from this interpretation of ΔV that if the concentration of a function vanishes throughout a certain region, then about any point in the region the values at neighbouring points are partly greater and partly less than at the point itself, so that the function cannot have at any point in the region either a maximum or minimum with respect to surrounding points. A function that in a certain region is uniform, continuous, and has no concentration is said to be *harmonic* in that region. The study of such functions constitutes one of the most important parts, not only of the theory of functions, but also of mathematical physics.

By means of the same theorem we may obtain another representation of ΔV. Let us apply the theorem to the space included between two small concentric spheres of radii R_1 and $R_2 = R_1 + h$. Then at the outer sphere

$$\left(\frac{\partial V}{\partial r}\right)_{R_2} = \left(\frac{\partial V}{\partial r}\right)_{R_1} + \left(\frac{\partial^2 V}{\partial r^2}\right)_{R_1} h,$$

and the surface integral being taken over the surface of both spheres, with the normal pointing in each case into the space between them,

$$-\iint \frac{\partial V}{\partial n} dS = -\iint_{R_1} \frac{\partial V}{\partial r} dS + \iint_{R_2} \left\{\left(\frac{\partial V}{\partial r}\right)_{R_1} + \left(\frac{\partial^2 V}{\partial r^2}\right)_{R_1} h\right\} dS.$$

As we make h approach zero, the first term of the second integral destroys the first, and

$$-\iint \frac{\partial V}{\partial n} dS = \operatorname*{Lim}_{h=0} \iint_{R_1} \frac{\partial^2 V}{\partial r^2} h\,dS,$$

so that

$$\iiint \Delta V d\tau = \operatorname*{Lim}_{h=0} \iint_{R_1} \frac{\partial^2 V}{\partial r^2} h\,dS.$$

Now hdS is the element of volume $d\tau$, so that ΔV may be defined as the mean value of the second derivative $\frac{\partial^2 V}{\partial r^2}$ for all

directions as we leave the point. This interpretation is due to Boussinesq*.

We may derive the parameter ΔV by applying Hamilton's operator ∇ twice to V,

$$\nabla^2 V = \nabla(\nabla V) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right) = -\Delta V.$$

35. Divergence. Solenoidal Vectors. If the components of the vector parameter are

$$P\cos(Px) = X = \frac{\partial V}{\partial x},$$

$$P\cos(Py) = Y = \frac{\partial V}{\partial y},$$

$$P\cos(Pz) = Z = \frac{\partial V}{\partial z},$$

we have

$$\Delta V = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z},$$

and the above theorem becomes

$$-\iint P\cos(Pn)\,dS = -\iint (X\cos(nx) + Y\cos(ny) + Z\cos(nz))\,dS$$

$$= \iiint\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)d\tau.$$

If P is everywhere directed outward from the surface S, the integral is positive, and

$$\text{mean}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) > 0.$$

Accordingly $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ is called the *divergence* of the vector point-function whose components are X, Y, Z, and will be denoted by div. R. Comparing with § 31 we find that the divergence of a vector is minus the *scalar* part of the ∇ of the vector,

$$\text{div.}\,R = -\mathbf{S}\nabla R.$$

The theorem as just given may be stated as follows, and will be referred to as the DIVERGENCE THEOREM: *The mean value of the normal component of any vector point-function* outward *from*

* Boussinesq, *Application des Potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*, p. 45.

any closed surface S *within which the function is uniform and continuous, multiplied by the area of the surface, is equal to the mean value of the divergence of the vector in the space within* S *multiplied by its volume.* The theorem was proved for a vector which was the parameter of a scalar point-function V, but it is evident that it may be proved directly by partial integration whether this is the case or not.

Let us consider the geometrical nature of a vector point-function R whose divergence vanishes in a certain region. In the neighbourhood of any point, the vector will at some points be directed toward the point and at others away. We may then draw curves of such a nature that at every point of any curve the tangent is in the direction of the vector point-function R at that point. Such curves will be called *lines* of the vector function. Suppose that such lines be drawn through all points of a closed curve, they will generate a tubular surface, which will be called a *tube* of the vector function. Let us now construct any two surfaces S_1 and S_2 cutting across the vector tube and apply the divergence theorem to the portion of space inclosed by the tube and the two surfaces or caps S_1 and S_2. Since at every point on the surface of the tube, R is tangent to the tube, the normal component vanishes. The only parts contributing anything to the surface integral are accordingly the caps, and since the divergence everywhere vanishes in τ, we have

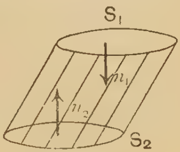

FIG. 17.

$$\iint_{S_1} R\cos(Rn_1)\,dS_1 + \iint_{S_2} R\cos(Rn_2)\,dS_2 = 0.$$

If we draw the normal to S_2 in the other direction, so that as we move the cap along the tube the direction of the normal is continuous, the above formula becomes

$$\iint_{S_1} R\cos(Rn_1)\,dS_1 - \iint_{S_2} R\cos(Rn_2)\,dS_2 = 0,$$

or the surface integral of the normal component of R over *any* cap cutting the same vector tube is constant.

Such a vector will be termed *solenoidal*, or tubular, and the condition $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0$ will be termed the solenoidal condition (Maxwell). We may abbreviate it, div. $R=0$. If a vector point-

function R is lamellar as well as solenoidal, the scalar function V of which it is the vector parameter is harmonic, for

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=\text{div. } R=\Delta V=0.$$

A solenoidal vector may be represented by its tubes, its direction being given by the tangent to an infinitesimal tube, and its magnitude being inversely proportional to its cross-section. As an example of a solenoidal vector we may take the velocity of particles of a moving fluid. If the velocity is R, with components X, Y, Z, the amount of liquid flowing through an element of surface dS in unit time is that contained in a prism of slant height R, and base dS, whose volume is

$$R\cos(Rn)\,dS.$$

The total flux, or quantity flowing in unit time through a surface S, is the surface integral

$$\iint R\cos(Rn)\,dS=\iint(X\cos(nx)+Y\cos(ny)+Z\cos(nz))\,dS.$$

Such a surface integral may accordingly be called the *flux* of the vector R through S.

A tube of vector R is a tube such that no fluid flows across its sides, such as a material tube through which liquid flows, and the divergence theorem shows that as much liquid flows in through one cross-section as out through another, if the solenoidal condition holds. If the liquid is incompressible, this must of course be true. As a second example of solenoidal vectors we have any vector which is the curl of another vector, for

$$\frac{\partial}{\partial x}\left\{\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right\}=0$$

identically.

The equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\Delta V=0$$

is called Laplace's equation, and the operator $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ Laplace's operator.

36. Representation of Solenoidal Vector. Multiplier. We have obtained in § 32 a means of representing a lamellar vector-function by means of the level surfaces of its potential function. By means of Jacobi's multiplier we may find a somewhat similar representation for a solenoidal vector. If we suppose the curves drawn whose tangent at every point has the direction of the vector function R whose components are X, Y, Z, since the direction cosines of the tangent are

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds},$$

the curve is defined by the differential equations

$$dx : dy : dz = X : Y : Z. \tag{1}$$

The integrals of these equations will each contain an arbitrary constant. Let us suppose that an integral is of the form

$$\lambda\,(x, y, z) = \text{const.}$$

Then we must have

$$\frac{\partial\lambda}{\partial x}dx + \frac{\partial\lambda}{\partial y}dy + \frac{\partial\lambda}{\partial z}dz = 0,$$

and since dx, dy, dz are proportional to X, Y, Z,

$$X\frac{\partial\lambda}{\partial x} + Y\frac{\partial\lambda}{\partial y} + Z\frac{\partial\lambda}{\partial z} = 0. \tag{2}$$

This partial differential equation may serve as a definition of an integral of the system of differential equations (1). Geometrically it shows that the vector R is perpendicular to the normal to the surface $\lambda = \text{const.}$, that is, is tangent to the surface. If $\mu = \text{const.}$ is a second integral, then

$$X\frac{\partial\mu}{\partial x} + Y\frac{\partial\mu}{\partial y} + Z\frac{\partial\mu}{\partial z} = 0, \tag{3}$$

and since R is tangent to a surface of each family $\lambda = \text{const.}$, $\mu = \text{const.}$, the lines of the vector R are the intersections of the surfaces λ with the surfaces μ. From (2) and (3), linear equations in X, Y, Z, we may determine their ratios. We obtain

$$X : Y : Z = \begin{vmatrix} \frac{\partial\lambda}{\partial y}, & \frac{\partial\lambda}{\partial z} \\ \frac{\partial\mu}{\partial y}, & \frac{\partial\mu}{\partial z} \end{vmatrix} : \begin{vmatrix} \frac{\partial\lambda}{\partial z}, & \frac{\partial\lambda}{\partial x} \\ \frac{\partial\mu}{\partial z}, & \frac{\partial\mu}{\partial x} \end{vmatrix} : \begin{vmatrix} \frac{\partial\lambda}{\partial x}, & \frac{\partial\lambda}{\partial y} \\ \frac{\partial\mu}{\partial x}, & \frac{\partial\mu}{\partial y} \end{vmatrix}. \tag{4}$$

If M be a factor to be determined, we may put

$$MX = A,\ MY = B,\ MZ = C, \tag{5}$$

where A, B, C are the above determinants.

But the determinants A, B, C, if differentiated by x, y, z, respectively and added, are found to satisfy identically the solenoidal condition

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0, \tag{6}$$

so that we have the equation for M,

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MY)}{\partial y} + \frac{\partial (MZ)}{\partial z} = 0. \tag{7}$$

Consequently for any continuous vector function R it is possible to find a scalar multiplier M that shall make the vector whose components are MX, MY, MZ, solenoidal. If the vector R is itself solenoidal, the equation for M is satisfied by any constant, say 1, so that in this case we have

$$\begin{aligned} X &= \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial y}, \\ Y &= \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial x} - \frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial z}, \\ Z &= \frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial y} - \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial x}. \end{aligned} \tag{8}$$

But if P_λ, P_μ denote the vector parameters of the functions λ, μ we see by the definition of the vector product,

$$\overline{R} = \mathbf{V}\overline{P}_\lambda \overline{P}_\mu,$$

$$R = P_\lambda P_\mu \sin (P_\lambda P_\mu).$$

If we consider two infinitely near surfaces of the first family for which λ has the values λ and $\lambda + d\lambda$ respectively, the normal distance between which is dn_λ, we have by §§ 16 and 20

$$dn_\lambda = \frac{d\lambda}{P_\lambda}.$$

Considering two infinitely near surfaces of the other family μ and $\mu + d\mu$, we have in like manner for their normal distance

$$dn_\mu = \frac{d\mu}{P_\mu}.$$

The area of a right section of the four-sided tube thus formed (Fig. 18) is

$$dS = \frac{dn_\lambda \, dn_\mu}{\sin (n_\lambda \, n_\mu)} = \frac{dn_\lambda \, dn_\mu}{\sin (P_\lambda \, P_\mu)},$$

and multiplying this by the value of R,

$$RdS = P_\lambda \, P_\mu \, dn_\lambda \, dn_\mu = d\lambda \, d\mu,$$

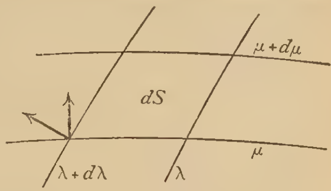

FIG. 18.

which is constant for the whole tube. Consequently we obtain a new proof of the fundamental property of a solenoidal vector, for any tube may be divided up into infinitesimal tubes defined by surfaces of the two families.

37. Principle of the Last Multiplier. If we have two functions M and N, *each* of which is a multiplier for the equations (1), they must each satisfy the partial differential equation (7) so that

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + Z\frac{\partial M}{\partial z} + M\left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} = 0,$$

$$X\frac{\partial N}{\partial x} + Y\frac{\partial N}{\partial y} + Z\frac{\partial N}{\partial z} + N\left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} = 0.$$

Multiplying the second of these by M, the first by N, and subtracting,

$$X\left\{M\frac{\partial N}{\partial x} - N\frac{\partial M}{\partial x}\right\} + Y\left\{M\frac{\partial N}{\partial y} - N\frac{\partial M}{\partial y}\right\} + Z\left\{M\frac{\partial N}{\partial z} - N\frac{\partial M}{\partial z}\right\} = 0,$$

and dividing by M^2,

$$X\frac{\partial\left(\frac{N}{M}\right)}{\partial x} + Y\frac{\partial\left(\frac{N}{M}\right)}{\partial y} + Z\frac{\partial\left(\frac{N}{M}\right)}{dz} = 0.$$

That is, the quotient of the two multipliers is an integral of the differential equations (1). This result is of particular importance when we have found one integral $\lambda = \text{const.}$ and any multiplier, for we may then find a last multiplier, which shall give us at once the remaining integral. By means of the integral equation $\lambda(x, y, z) = \text{const.}$ let us, by solving for one of the variables, say z, express z as a function of x, y, λ,

$$z = z(x, y, \lambda).$$

If $\mu = \text{const.}$ is a new integral, let us by introducing the value of z just found, express μ in terms of x, y, λ,

$$\mu = \mu(x, y, \lambda).$$

We shall distinguish the partial derivatives of μ thus expressed from its partial derivatives when expressed in x, y, z, by brackets, so that we have

$$\frac{\partial\mu}{\partial x} = \left[\frac{\partial\mu}{\partial x}\right] + \left[\frac{\partial\mu}{\partial\lambda}\right]\frac{\partial\lambda}{\partial x}, \quad \frac{\partial\mu}{\partial y} = \left[\frac{\partial\mu}{\partial y}\right] + \left[\frac{\partial\mu}{\partial\lambda}\right]\frac{\partial\lambda}{\partial y}, \quad \frac{\partial\mu}{\partial z} = \left[\frac{\partial\mu}{\partial\lambda}\right]\frac{\partial\lambda}{\partial z}.$$

Accordingly we obtain for the values of A, B, C

$$A = -\frac{\partial\lambda}{\partial z}\left[\frac{\partial\mu}{\partial y}\right], \quad B = \frac{\partial\lambda}{\partial z}\left[\frac{\partial\mu}{\partial x}\right].$$

Now μ being expressed in terms of x, y, λ, we have

$$d\mu = \left[\frac{\partial\mu}{\partial x}\right]dx + \left[\frac{\partial\mu}{\partial y}\right]dy + \left[\frac{\partial\mu}{\partial\lambda}\right]d\lambda,$$

and since $\lambda = \text{const.}$ is an integral, $d\lambda = 0$,

$$d\mu = \left[\frac{\partial\mu}{\partial x}\right]dx + \left[\frac{\partial\mu}{\partial y}\right]dy.$$

But from the values of A and B

$$\left[\frac{\partial\mu}{\partial x}\right] = \frac{B}{\frac{\partial\lambda}{\partial z}}, \quad \left[\frac{\partial\mu}{\partial y}\right] = -\frac{A}{\frac{\partial\lambda}{\partial z}},$$

so that
$$d\mu = \frac{Bdx - Ady}{\frac{\partial\lambda}{\partial z}}.$$

But since
$$A = MX, \quad B = MY,$$

this becomes
$$d\mu = \frac{M}{\frac{\partial\lambda}{\partial z}}(Ydx - Xdy).$$

Accordingly although the expression

$$Ydx - Xdy$$

is not a perfect differential, the factor

$$\frac{M}{\frac{\partial\lambda}{\partial z}}$$

makes it the differential of a function μ, and

$$\mu = \int \frac{M}{\dfrac{\partial \lambda}{\partial z}} (Ydx - Xdy) = \text{const.}$$

is a second integral of the equations (1). X, Y and $\dfrac{\partial \lambda}{\partial z}$ must of course be expressed in terms of x, y, λ.

Consequently if we have the system of differential equations

$$dx : dy : dz = X : Y : Z,$$

and we have found one integral $\lambda = \text{const.}$ together with a multiplier satisfying the partial differential equation

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MY)}{\partial y} + \frac{\partial (MZ)}{\partial z} = 0,$$

then the expression

$$\frac{M}{\dfrac{\partial \lambda}{\partial z}}$$

is an integrating factor, or *last multiplier** for the equation

$$Ydx - Xdy = 0.$$

When X, Y, Z satisfy the solenoidal condition, the last multiplier is

$$\frac{1}{\dfrac{\partial \lambda}{\partial z}}.$$

This result will be used in § 103.

38. Variation of a Multiple Integral. In illustrating the method of the Calculus of Variations we have found the variation of a single integral, and in the example taken the functions varied were the coordinates x, y, z, of points of a curve, the variable of integration being t. We may in a similar manner vary a surface or volume integral, by causing the functions entering into the integrand to change their forms by an infinitesimal transformation, while the variables of integration are unchanged. For instance let

$$I = \iiint_\tau V dx dy dz$$

* Jacobi, *Vorlesungen über Dynamik*, p. 78.

be a volume integral, we may define its variation by the equation

$$I + \delta I = \iiint_\tau (V + \delta V)\, dx dy dz,$$

where δV is any arbitrary function of x, y, z multiplied by an infinitesimal constant ϵ. We may also vary an integral in another manner. Suppose we consider the volume in question to be occupied by material substance, and that to each material point belongs a value of the function V. Now let every material point be displaced in any manner by an infinitesimal amount defined by the projections δx, δy, δz. The material point which arrives at x, y, z brings with it a different value of V, and the value of the integral through the same portion of space, since the latter is filled with different material points, is different. It is to be noticed that this is the exact converse of the process exemplified in §§ 29, 31 for there the functions X, Y, Z were associated with fixed points in space, while the integral was over a field which was varied, whereas here the function V goes with the varied point, while the field of integration is fixed. As an example, let us consider the integral

$$m = \iiint_\tau \rho\, dx dy dz$$

representing the mass of a body τ whose density at any point is ρ, the density being defined as the limit of the ratio of the mass of a portion of the body to its volume, both being decreased indefinitely. Let us consider the mass in an infinitesimal rectangular parallelopiped, whose sides are dx, dy, dz, and whose mass is $dm = \rho\, dx dy dz$. When all points are displaced by the amounts δx, δy, δz, particles in the face normal to the X-axis and nearest the origin move to the right a distance δx, and the volume of new matter that enters the parallelopiped through that face is $dy dz \delta x$, whose mass is $\rho\, dy dz \delta x$, ρ and δx having the values belonging to the face in question. At the opposite parallel face, farthest from the origin, $\rho \delta x$ has the value

$$\rho \delta x + \frac{\partial (\rho \delta x)}{\partial x} dx,$$

and the amount of matter that moves *out* of the parallelopiped to the right is

$$dy dz \left\{ \rho \delta x + \frac{\partial (\rho \delta x)}{\partial x} dx \right\}.$$

The total *gain* through these two sides is, accordingly, the difference

$$-\frac{\partial(\rho\,\delta x)}{\partial x}\,dx\,dy\,dz.$$

Similarly through the sides normal to the Y-axis the gain is

$$-\frac{\partial(\rho\,\delta y)}{\partial y}\,dx\,dy\,dz,$$

and through the sides normal to the Z-axis

$$-\frac{\partial(\rho\,\delta z)}{\partial z}\,dx\,dy\,dz.$$

The total increase of the mass in the parallelopiped is therefore

$$\delta\,dm = -\left\{\frac{\partial(\rho\,\delta x)}{\partial x} + \frac{\partial(\rho\,\delta y)}{\partial y} + \frac{\partial(\rho\,\delta z)}{\partial z}\right\} dx\,dy\,dz,$$

and this being taken for an element of our integral, the total increase of mass, or variation of the integral, is

$$\delta m = -\iiint_\tau \left\{\frac{\partial(\rho\,\delta x)}{\partial x} + \frac{\partial(\rho\,\delta y)}{\partial y} + \frac{\partial(\rho\,\delta z)}{\partial z}\right\} dx\,dy\,dz.$$

We may obtain this result in a more rigid manner by the use of Green's Theorem. Through each element of surface dS of the boundary of the space in question there moves inwards an infinitesimal prism of matter whose volume is

$$dS\,\{\delta x \cos(nx) + \delta y \cos(ny) + \delta z \cos(nz)\}.$$

The mass of this is

$$\{\rho\,\delta x \cos(nx) + \rho\,\delta y \cos(ny) + \rho\,\delta z \cos(nz)\}\,dS,$$

so that the total *gain* of mass in the space τ is

$$\delta m = \iint_S \{\rho\,\delta x \cos(nx) + \rho\,\delta y \cos(ny) + \rho\,\delta z \cos(nz)\}\,dS.$$

But by Green's Theorem this is equal to

$$\delta m = -\iiint_\tau \left\{\frac{\partial(\rho\,\delta x)}{\partial x} + \frac{\partial(\rho\,\delta y)}{\partial y} + \frac{\partial(\rho\,\delta z)}{\partial z}\right\} dx\,dy\,dz.$$

This result will be of frequent use.

39. Reciprocal Distance. Gauss's Theorem. Consider the scalar point-function, $V = \frac{1}{r}$, where r is the distance from a

fixed point or *pole* O. Then the level surfaces are spheres, and the parameter is

$$R = \pm \frac{d}{dr}\left(\frac{1}{r}\right) h_r,$$

and since $h_r = 1$, $$R = \frac{1}{r^2},$$

drawn toward O. (§ 16.)

Consider the surface integral of the normal component of R directed into the volume bounded by a closed surface S not containing O, or as we shall call it, the flux of R into S,

(1) $$\iint R \cos(Rn)\, dS = -\iint \frac{1}{r^2} \cos(rn)\, dS.$$

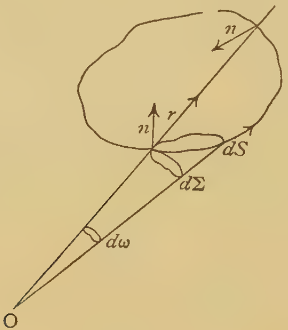

Fig. 19.

The latter geometrical integral was reduced by Gauss. If to each point in the boundary of an element dS we draw a radius and thus get an infinitesimal cone with vertex O, and call the part of the surface of a sphere of radius r cut by this cone $d\Sigma$, $d\Sigma$ is the projection of dS on the sphere, and as the normal to the sphere is in the direction of r, we have

$$d\Sigma = \pm\, dS \cos(rn),$$

the upper sign, for r cutting in, the lower for r cutting out. If now we draw about O a sphere of radius 1, whose area is 4π, and call the portion of its area cut by the above-mentioned cone $d\omega$, we have from the similarity of the right sections of the cone

$$\frac{d\Sigma}{d\omega} = \frac{r^2}{1},$$

$$d\Sigma = r^2 d\omega.$$

The ratio $d\omega$ is called the *solid angle* subtended by the infinitesimal cone.

Accordingly

(2)
$$\frac{dS\cos(rn)}{r^2} = \pm\frac{d\Sigma}{r^2} = \pm\, d\omega,$$

and

(3)
$$\iint \frac{dS\cos(rn)}{r^2} = \iint \pm\, d\omega.$$

Now for every element $d\omega$, where r cuts into S, there is another equal one, $-d\omega$, where r cuts out, and the two annul each other. Hence for O outside S,

(4)
$$\iint \frac{\cos(rn)}{r^2}\,dS = 0.$$

If on the contrary, O lies inside S, the integral is to be taken over the whole of the unit sphere with the same sign, and consequently gives the area 4π. Hence for O within S,

(5)
$$\iint \frac{\cos(rn)}{r^2}\,dS = -\,4\pi.$$

These two results are known as Gauss's theorem, and the integral (3) will be called Gauss's integral*.

These results could have been obtained as direct results of the divergence theorem. For the tubes of the vector function R are cones with vertex O. If O is outside S, R is continuous in every point within S, and since the area of any two spheres cut out by a cone are proportional to the squares of the radii of the spheres, we have the normal flux of

$$R = \frac{1}{r^2}$$

equal for all spherical caps. Consequently R is solenoidal, and the flux through *any* closed surface is zero. If O is within S, R is solenoidal in the space between S and any sphere with center O lying entirely within S, and the flux through S is the same as the flux through the sphere, which is evidently $-\,4\pi$.

The fact that R is solenoidal and V harmonic may be directly shown by differentiation. If the coordinates of O are a, b, c,

(6)
$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2,$$

$$\frac{\partial(r^2)}{\partial x} = 2r\frac{\partial r}{\partial x} = 2(x-a);$$

* Gauss, *Theoria Attractionis Corporum Sphaeroidicorum Ellipticorum homogeneorum Methodo nova tractata.* Werke, Bd. v., p. 9.

(7) $$\frac{\partial r}{\partial x} = \frac{x-a}{r}, \quad \frac{\partial r}{\partial y} = \frac{y-b}{r}, \quad \frac{\partial r}{\partial z} = \frac{z-c}{r};$$

(8) $$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{x-a}{r^3};$$

(9) $$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-a)}{r^4}\frac{\partial r}{\partial x} = \frac{3(x-a)^2 - r^2}{r^5},$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{3(y-b)^2 - r^2}{r^5}, \quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{3(z-c)^2 - r^2}{r^5};$$

(10) $$\Delta\left(\frac{1}{r}\right) = \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2}$$

$$= \frac{3\{(x-a)^2 + (y-b)^2 + (z-c)^2\} - 3r^2}{r^5} = 0,$$

and $\frac{1}{r}$ is harmonic, except where $r = 0$.

CHAPTER IV.

FUNCTIONS OF A COMPLEX VARIABLE.

40. Multiplication of Complex Numbers. We have seen in (5) how the two-dimensional complex number $a+ib$ may be represented in the plane by Argand's diagram. From the definition of addition of complex numbers it follows that two complex numbers are added by the parallelogram construction, that is the sum of the two complex numbers $p=a_1+ib_1$ and $q=a_2+ib_2$ is represented by the diagonal of the parallelogram constructed on lines whose lengths are equal to the moduli of p and q,

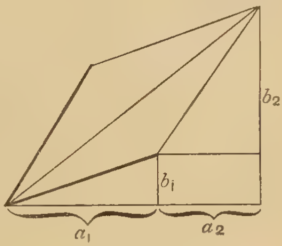

FIG. 20.

$$|p|=\sqrt{a_1^2+b_1^2}, \quad |q|=\sqrt{a_2^2+b_2^2},$$

and which make angles with the X-axis equal to the arguments of p and q.

Hence $$|p \pm q| \leqq |p| + |q|.$$

If we introduce the polar coordinates

$$r=|p|, \ \phi=\tan^{-1}\frac{b}{a},$$

we have

$$a=r\cos\phi,$$

$$b=r\sin\phi,$$

$$p=a+ib=r(\cos\phi+i\sin\phi).$$

Now since

$$e^{\phi} = 1 + \frac{\phi}{1!} + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \dots\dots$$

$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \dots\dots$$

$$\sin \phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots\dots,$$

it follows that

$$\cos \phi + i \sin \phi = e^{i\phi},$$

$$p = |p| e^{i\phi}.$$

It is easy to show that the modulus of the product of two complex numbers is equal to the product of their moduli, and that the argument is equal to the sum of their arguments. For if

$$p = a_1 + ib_1 = r_1 (\cos \phi_1 + i \sin \phi_1) = r_1 e^{i\phi_1},$$

$$q = a_2 + ib_2 = r_2 (\cos \phi_2 + i \sin \phi_2) = r_2 e^{i\phi_2},$$

then $\quad pq = r_1 r_2 e^{i(\phi_1 + \phi_2)} = r_1 r_2 [\cos (\phi_1 + \phi_2) + i \sin (\phi_1 + \phi_2)].$

In like manner for the quotient, substituting the words quotient for product, and difference for sum. A complex number vanishes only when its modulus vanishes, and is considered infinite when its modulus is infinite, whatever its argument.

41. Function of Complex Variable. A function of the complex variable $z = x + iy$, if given as an analytic expression containing z, will be a certain function of the two real variables x and y and will contain a real part, which we shall denote by $u(x, y)$, and an imaginary part, which we shall denote by $iv(x, y)$. Hence the study of functions of a complex variable may be made to depend on the study of functions of two real variables. Let

$$w = f(z) = u + iv.$$

The representation of variable and function by means of abscissa and ordinate of a curve is not here applicable, for both variable and function have two degrees of freedom. The function may be otherwise represented by means of another plane in which we mark off lengths u and v as the rectangular coordinates of another point representing w on another Argand's diagram. To every point x, y in the first plane will then correspond a point u, v in the second plane. As the point x, y moves, so will the point u, v. As the point x, y representing the variable z, describes any curve, u, v,

representing $w=f(z)$ describes another curve, if $f(z)$ is continuous, otherwise the point u, v may jump from one point to another. The definition of continuity is that two points on the function curve may be made to approach each other as nearly as we please by taking the corresponding points on the curve of the variable sufficiently near. Or, a function is continuous in a region of the z-plane continuing z_0 if to every real positive quantity ϵ as small as we please, we can find a corresponding quantity δ such that

$$|f(z)-f(z_0)| < \epsilon \text{ if } |z-z_0| < \delta.$$

In considering the representation by means of curves, it is of importance to inquire whether, if the curve of z starting from an arbitrary point z_0, returns to it after describing a closed curve, the curve representing $w=f(z)$ also returns to its point of departure. If this is the case, the function $f(z)$ within the region in which this property holds, is said to be uniform, or single-valued, for to every value of z corresponds one value of w.

42. Derivative. Analytic Function. Let us examine the relation between an infinitesimal change in z and the corresponding change in $f(z)$. The change $dz=dx+idy$ has the modulus $|dz|=\sqrt{dx^2+dy^2}$, and the argument $w=\tan^{-1}\dfrac{dy}{dx}$. The change $dw=du+idv$ has the modulus $|dw|=\sqrt{du^2+dv^2}$ and the argument $\theta=\tan^{-1}\dfrac{dv}{du}$.

Also

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy,$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy,$$

$$dw=du+idv=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+i\left\{\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right\}.$$

The ratio

$$\lim_{|dz|=0}\frac{dw}{dz}=\frac{du+idv}{dx+idy}=\frac{\dfrac{\partial u}{\partial x}dx+\dfrac{\partial u}{\partial y}dy+i\left\{\dfrac{\partial v}{\partial x}dx+\dfrac{\partial v}{\partial y}dy\right\}}{dx+idy}$$

$$=\frac{\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}+\left(\dfrac{\partial u}{\partial y}+i\dfrac{\partial v}{\partial y}\right)\dfrac{dy}{dx}}{1+i\dfrac{dy}{dx}},$$

is in general dependent on $\frac{dy}{dx}$, that is on the direction in which we leave the point z. The value of the derivative will not then be determined for the point z irrespective of the direction of leaving it unless the numerator is a multiple of the denominator and the expression containing $\frac{dy}{dz}$ divides out.

In order that this may be true we must have

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) : 1 = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) : i,$$

that is
$$i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Putting real and imaginary parts on both sides equal,

$$(A) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

and
$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

$$\left|\frac{dw}{dz}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2.$$

In this case the function w has a definite derivative, and it is only when the functions u and v satisfy these conditions that $u+iv$ is said to be an *analytic function* of z. This is Riemann's definition of a function of a complex variable*. (Cauchy says monogenic instead of analytic.) The real functions u and v are said to be *conjugate* functions of the real variables x, y.

It is obvious that if w is given as an analytic expression involving z, $w = f(z)$, then w always satisfies this condition. For

$$\frac{\partial w}{\partial x} = \frac{df(z)}{dz}\frac{\partial z}{\partial x} = f'(z), \quad \frac{\partial w}{\partial y} = \frac{df(z)}{dz}\frac{\partial z}{\partial y} = if'(z).$$

Accordingly
$$i\frac{\partial w}{\partial x} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

* Riemann, *Mathematische Werke*, p. 5.

43. Orthogonal Coordinates. Conformal Representation. We may apply the considerations of §§ 15 and 20 to the case of orthogonal coordinates in a plane. If a set of point-functions are independent of one rectangular coordinate, the geometry of all planes perpendicular to the axis of that coordinate is the same, and we have the uniplanar, or two-dimensional case involving only two variables which we will take as x, y. If we take u and v as any two point-functions, whose parameters are h_u, h_v

$$h_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2, \quad h_v^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2,$$

their level lines $u = \text{constant}$ and $v = \text{constant}$ may be taken for coordinate lines.

Their normals have the direction cosines

$$\cos(n_u x) = \frac{1}{h_u}\frac{\partial u}{\partial x}, \qquad \cos(n_u y) = \frac{1}{h_u}\frac{\partial u}{\partial y},$$

$$\cos(n_v x) = \frac{1}{h_v}\frac{\partial v}{\partial x}, \qquad \cos(n_v y) = \frac{1}{h_v}\frac{\partial v}{\partial y},$$

and the condition that u and v shall form an orthogonal system is

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.$$

The lengths of infinitesimal arcs of curves, forming the sides of a rectangle whose opposite vertices have coordinates $u, v, u + du, v + dv$, are as in § 20

$$\frac{du}{h_u}, \qquad \frac{dv}{h_v},$$

and the length of the diagonal ds, or element of length of a curve whose ends have the above coordinates, is given by

$$ds^2 = \frac{du^2}{h_u^2} + \frac{dv^2}{h_v^2}.$$

If now we take for curvilinear coordinates in the x, y plane two functions u and v such that $u + iv$ is an analytic function of $x + iy$, in virtue of the equations (A) of § 42 we have

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0,$$

and u and v form an orthogonal system. Now in any orthogonal system if we construct a set of level curves for equal small increments of u and v, they will divide the plane up into small

curvilinear rectangles the ratios of whose sides at any point are given by the ratio of the parameters h_u and h_v. But from the equations (A), we have

$$h_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = h_v^2,$$

$$h_u^2 = h_v^2 = \left|\frac{dw}{dz}\right|^2,$$

so that in this case the plane is divided into small squares. Let us now construct in the second plane, in which u and v are

FIG. 21.

rectangular coordinates, the curves corresponding to $u = \text{constant}$ and $v = \text{constant}$. These are of course straight lines dividing their plane into small squares. Moreover the length of any arc $d\sigma$ of a curve in their plane, is given by

$$d\sigma^2 = du^2 + dv^2.$$

But in virtue of the above relations, this gives

$$d\sigma^2 = h^2 ds^2,$$

$h = \left|\frac{dw}{dz}\right|$ is accordingly the ratio of *magnification* at the point in question, and varies for different points of the plane.

Let us now construct, (Fig. 21,) at a point in the x, y plane an infinitesimal triangle made by the intersection of any three curves, and let the lengths of its sides be ds_1, ds_2, ds_3. Construct the corresponding curves in the u, v plane, intersecting to form an infinitesimal triangle with sides

$$d\sigma_1,\ d\sigma_2,\ d\sigma_3.$$

Now we have

$$d\sigma_1 = h\, ds_1,\ d\sigma_2 = h\, ds_2,\ d\sigma_3 = h\, ds_3,$$

and therefore

$$d\sigma_1 : d\sigma_2 : d\sigma_3 = ds_1 : ds_2 : ds_3,$$

and the infinitesimal triangles are similar. Consequently *corresponding curves intersect each other in the same angle* in both corresponding planes. Such a relation as this is called a *Conformal Relation**, and it is of fundamental importance in the theory of functions and in mathematical physics. The two planes are said to be conformal representations of each other. The relation is sometimes specified by saying that the conformal representations are similar in their infinitely small parts.

It is easy to show that if the functions u and v give a conformal representation of the plane, they must satisfy the equations (A).

44. Laplace's Equation. Conjugate Functions. If we differentiate the equations (A), the first by x and the second by y and add, since $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that the function u satisfies Laplace's equation in two variables, or is harmonic.

Differentiating the other way and adding we show that v also satisfies the same equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus every conformal development or every analytic function of a complex variable gives us two harmonic functions. The question arises whether the converse is true. It obviously will not do to take *any* two harmonic functions for u and v, for they must be related so as to satisfy the equations (A). But if one function is given, we may find the conjugate, for we must have

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy,$$

which by the first equation (A) is

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy.$$

* Professor Cayley has called it *orthomorphosis*.

Now if we call this $Xdx + Ydy$ it satisfies the condition for a perfect differential

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \quad \text{i.e.} \; -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}.$$

Consequently the line integral $\int -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ from a given point x_0, y_0 to a variable point x, y is a function only of its upper limit, and represents v. Similarly if v is given,

$$u = \int \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \int \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy.$$

Furthermore the first of the equations (A) is the condition that $vdx + udy$ is a perfect differential, and the second that $udx - vdy$ is such.

Accordingly the line integrals

$$\phi = \int vdx + udy,$$

$$\psi = \int udx - vdy,$$

give two new point-functions ϕ, ψ which in virtue of the equations

$$v = \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \quad u = \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x},$$

are conjugate to each other, and give a new analytic function of z, $\phi + i\psi$, or $\psi + i\phi$. From these by new integrations we may obtain any number.

Examples. The function

$$z^2 = (x + iy)^2 = x^2 + 2ixy - y^2,$$

gives
$$u = x^2 - y^2, \; v = 2xy,$$

both harmonic functions.

The curves $u = x^2 - y^2 = \text{const.}$ and $v = 2xy = \text{const.}$ give two sets of equilateral hyperbolas, which intersect everywhere at right angles, Fig. 22.

The function
$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2},$$

gives
$$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}.$$

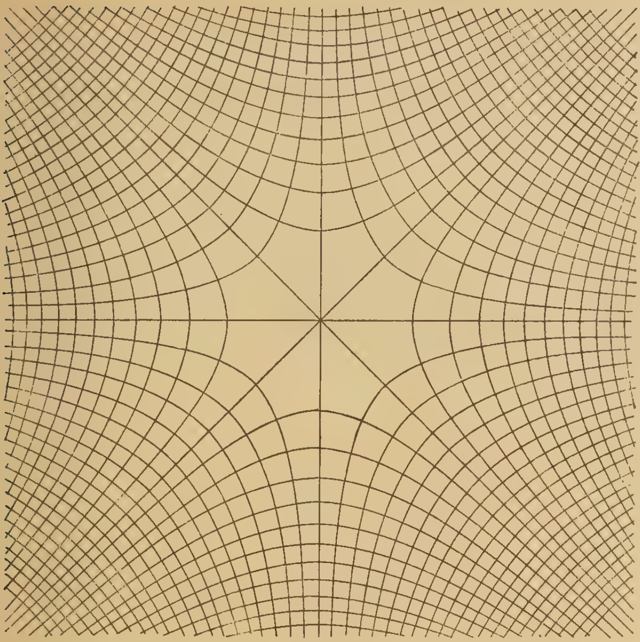

Fig. 22.

The curves

$$u = \frac{x}{x^2+y^2} = \text{const. and } v = -\frac{y}{x^2+y^2} = \text{const.},$$

$$x^2 + y^2 - C_1x = 0, \qquad x^2 + y^2 + C_2y = 0,$$

give two sets of circles, the first all tangent to the Y-axis at the origin, the second all tangent to the X-axis, Fig. 23.

The power

$$z^n = (x+iy)^n = r^n \{\cos n\phi + i \sin n\phi\},$$

gives the two functions

$$u = r^n \cos n\phi, \quad v = r^n \sin n\phi,$$

and a sum of any number of such terms each multiplied by a constant

$$\Sigma r^n \{A_n \cos n\phi + B_n \sin n\phi\},$$

therefore gives a harmonic function. If a function can be developed in such a trigonometric series it accordingly is harmonic. Terms such as these may be called circular harmonic functions.

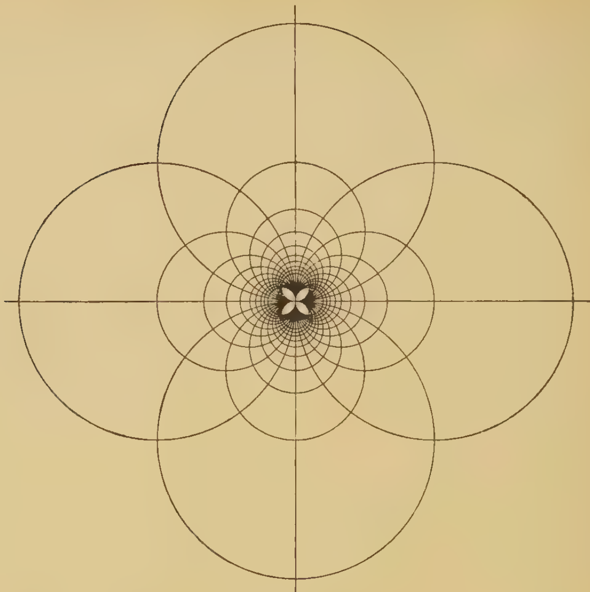

FIG. 23.

45. Integral of a Function of a Complex Variable. Since the complex variable has two degrees of freedom, its integral is not of so simple a nature as that of a single real variable. Suppose the variable z moves from a point A to a point B along any continuous path. The definite integral of $f(z)=u+iv$ along this path will be defined as the line integral

$$F=\int_A^B f(z)\,dz=\int_A^B (u+iv)\,(dx+idy)$$

$$=\int_A^B udx-vdy+i\int_A^B vdx+udy.$$

Now in virtue of the equations (A) both the integrals above are independent of the path, so that F is a function of z. It is evidently $\psi+i\phi$ of the last section. This is on the supposition that the functions u, v are uniform and continuous in the whole region considered. If this is the case the function $w=u+iv$ is called *holomorphic.*

If w becomes discontinuous in the region considered it ceases to be true that the integral is the same over two paths AB between which lies a point of discontinuity of the function w.

For example the function $w = \frac{1}{z}$ is discontinuous at the point $z = 0$. Accordingly the integral $\int \frac{dz}{z}$ around a closed contour containing the origin within it is not zero, for it may be taken as the difference between the integrals between two points AB on the contour along two paths between which lies the point of discontinuity of the function w. The integral around any closed contour embracing the origin is however the same as around a circle of radius R with center at the origin, for between the two curves there is no point of discontinuity of the function. Now since $z = x + iy = re^{i\phi}$, if r is constant $= R$,

$$dz = iRe^{i\phi}\, d\phi,$$

and the integral from $z = 1$

$$\int \frac{dz}{z} = i \int \frac{Re^{i\phi}\, d\phi}{Re^{i\phi}} = i \int d\phi = i\phi,$$

which taken around the circle is $2\pi i$.

The integral $\int_1^z \frac{dz}{z}$ is defined as the logarithm of z, and it possesses the property that as z describes any closed path enclosing the origin, the function instead of returning to its original value increases by a constant $2\pi i$. The function is then not uniform, but has at any point an unlimited number of values, depending upon the path by which we arrive at the point. These values all differ by integral multiples of the constant $2\pi i$.

We see that this accords with the ordinary definition of the logarithm,

$$\log z = \log(x + iy) = \log(re^{i(\phi + 2n\pi)}) = \log r + i\phi + 2n\pi i,$$

for if we increase the argument ϕ of a complex number z by any multiple of 2π, the number is unchanged. A point such that a function $f(z)$ assumes a new value when the variable traverses a closed circuit about the point is called a *critical*, or *branch point*. In this case the conformal representation given by the function $f(z)$ is multiple in character, for in the UV-plane we are to take a point for each of the values of the function $f(z)$. Each of these representative points gives a conformal representation of the whole of the XY-plane on a part of the UV-plane.

For instance, in the case of the logarithm

$$\log z = \log r + i\phi + 2n\pi i,$$
$$u = \log r, \quad v = \phi + 2n\pi,$$

as z takes all possible values in the XY-plane, $u = \log r$ varies from $-\infty$ to $+\infty$ but v varies only in limits differing by 2π, so that the whole XY-plane is entirely represented on a strip of the UV-plane infinite in one direction but of the finite width 2π in the other. This strip is repeated an infinite number of times each giving the same conformal representation of the whole XY-plane. For instance the radii $\phi = \text{const.}$ and the circles $r = \text{const.}$ cutting them orthogonally in the XY-plane correspond to the lines $v = \text{const.}$ $u = \text{const.}$ in the UV-plane.

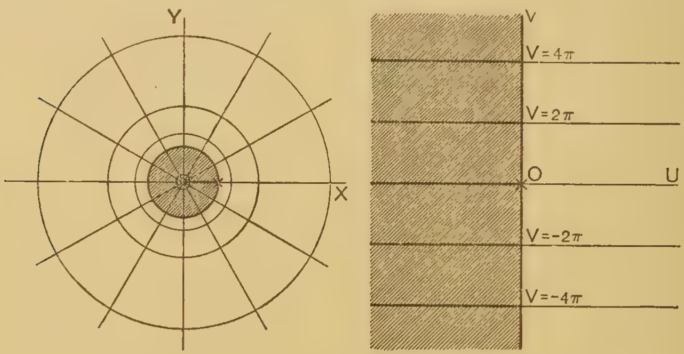

FIG. 24.

Corresponding regions of the figures are similarly shaded.

PART I.

THEORY OF NEWTONIAN FORCES.

CHAPTER I.

PRINCIPLES OF MECHANICS. UNITS AND DIMENSIONS.

46. Matter and Energy. Dynamics. Physics is the science of Matter and of Energy. Its laws are found to be invariable and capable of exact statement, that is of presentation in the language of Mathematics. The application of mathematical analysis to the treatment of physical phenomena, enabling us to deduce general laws from the results of experiment, and to infer the consequences of general laws, forms the subject of Mathematical or Theoretical Physics.

Matter has the essential property of occupying space. It has in addition universally only the property of Inertia, to be defined below. In order to define Energy, we must consider the motion of matter in space. That portion of mathematical physics which treats of the motion of matter is called Mechanics, or Dynamics. It is the object of physicists to reduce the explanation of all physical phenomena to descriptions of motion of matter, and accordingly the study of the principles of Dynamics is indispensable to the study of any branch of theoretical physics. Before considering the nature of electrical and magnetic phenomena we shall therefore devote a few chapters to Dynamics.

47. Scalar and Vector Quantities. Physical quantities are of two kinds. Quantities whose complete specification involves

no idea of direction are called scalar quantities, for they may be conceived as arranged on a scale according to their magnitude. Such are time, temperature, size, density.

Quantities whose specification involves the idea of direction as well as of magnitude are called vector quantities. They may be represented by geometrical directed lines, and all that has been said of vector quantities and their addition, etc. applies to them.

48. Degrees of Freedom. A set of magnitudes or parameters which completely specify a quantity are called its coordinates. The number of coordinates required is called the number of degrees of freedom of the quantity. For instance, a point in a plane may be defined by two rectangular, or two polar coordinates, and has two degrees of freedom. We may also say that there is a double infinity or ∞^2 of points in a plane. A point in space requires three coordinates of any sort, and has three degrees of freedom. Every independent relation that the coordinates of a quantity are made to satisfy diminishes the number of its degrees of freedom by one. For instance, a relation between the rectangular coordinates of a point restricts it to lie on a certain surface,—it then has two degrees of freedom instead of three, and requires but two coordinates to specify it. For example, a point satisfies the condition $x^2+y^2+z^2=a^2$. It lies on the sphere of radius a, and may be fully specified by giving its latitude and longitude.

For the coordinates of a vector R we may take its projections on the three coordinate axes. If we choose its length, or modulus, and its three direction cosines,

$$\alpha=\cos(Rx),\quad \beta=\cos(Ry),\quad \gamma=\cos(Rz),$$

one of the four coordinates R, α, β, γ is redundant, for the latter three satisfy the identical relation

$$\alpha^2+\beta^2+\gamma^2=1.$$

This furnishes us an example of the general case where we give n coordinates of a quantity satisfying k independent identical relations, or equations of condition. The quantity then has only $n-k$ degrees of freedom, and we may find $n-k$ independent coordinates which completely specify it.

49. Velocities. If a point change its position in space, its motion may be described by giving the values of its coordinates for every instant of time, by means of equations such as

$$x = f_1(t), \; y = f_2(t), \; z = f_3(t).$$

The functions f must be continuous, since the point cannot jump from one position to another.

We may describe the motion otherwise by giving two equations $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$, which denote the curve of intersection of two surfaces along which the point moves. This curve is called the path of the point. We must further give the distance s measured along the curve, which the point has traversed, counting from a fixed point on the curve. We must know s at all times t, which is expressed by giving s as a continuous function of t, $s = \phi(t)$. This, with the two equations of the path, gives as before three equations to completely define the motion.

The velocity of the point is defined as the limit of the ratio of the distance Δs traversed in an interval of time Δt to the time Δt when both decrease without limit,

$$v = \operatorname*{Lim}_{\Delta t = 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

A point travelling with a given numerical velocity may however be moving in any of an indefinite number of directions, accordingly a velocity is completely specified only when we give its direction and magnitude, or velocities are vector quantities. The direction of the velocity is that of the tangent to its path. Its direction cosines are accordingly

$$\alpha = \frac{dx}{ds}, \quad \beta = \frac{dy}{ds}, \quad \gamma = \frac{dz}{ds}.$$

Velocities are resolved and compounded like vectors in general,—in particular the projections of $\bar{v}$ on the coordinate axes are

$$v_x = v\alpha = v\frac{dx}{ds} = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

$$v_y = v\beta = v\frac{dy}{ds} = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt},$$

$$v_z = v\gamma = v\frac{dz}{ds} = \frac{ds}{dt}\frac{dz}{ds} = \frac{dz}{dt},$$

and $$v^2 = v_x^2 + v_y^2 + v_z^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

We might have defined the vector velocity as the resultant of the three vectors

$$\bar{v}_x = \frac{\overline{dx}}{dt}, \quad \bar{v}_y = \frac{\overline{dy}}{dt}, \quad \bar{v}_z = \frac{\overline{dz}}{dt}.$$

50. **Acceleratıons.** If the velocity of a point is variable with the time we define the acceleration of the point as the limit of the ratio of the increment of velocity Δv to the increment of time Δt, as both approach zero. We may consider either the numerical change

$$\operatorname*{Lim}_{\Delta t=0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 s}{dt^2},$$

or the geometrical change. If we draw a vector $\overline{AB}$ to represent the velocity at the time t and the vector $\overline{AC}$ to represent the velocity at the time $t + \Delta t$, and draw the arc of a circle BD, DC will represent the numerical change of velocity, Δv, not considering its direction, while $\overline{BC}$ represents its geometrical, or vector change, $\Delta \bar{v}$, for

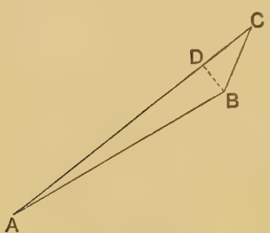

FIG. 25.

$$\overline{AB} + \overline{BC} = \overline{AC},$$

$$\overline{BC} = \overline{AC} - \overline{AB} = \Delta \bar{v}.$$

Accordingly $$\operatorname*{Lim}_{\Delta t=0} \frac{\Delta \bar{v}}{\Delta t} = \operatorname*{Lim}_{\Delta t=0} \frac{\overline{BC}}{\Delta t}$$

is the vector acceleration $\bar{a}$.

Since the projections of the geometrical difference of two vectors are the differences of the projections, the components of $\bar{a}$ in any direction will be proportional to the changes of the corresponding components of the velocities, that is

$$a_x = \frac{dv_x}{dt} = \frac{d^2 x}{dt^2},$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2 y}{dt^2},$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2 z}{dt^2}.$$

The vector acceleration, $\bar{a}$, may be defined as the resultant of the components a_x, a_y, a_z, and accordingly its modulus is

$$a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}.$$

This is *not* in general equal to $\frac{d^2s}{dt^2}$ which is the acceleration of the scalar velocity. The direction of $\bar{a}$ is given by its direction cosines

$$\alpha = \frac{\frac{d^2x}{dt^2}}{a}, \quad \beta = \frac{\frac{d^2x}{dt^2}}{a}, \quad \gamma = \frac{\frac{d^2y}{dt^2}}{a}.$$

51. Physical Axioms. The results of universal experience with regard to motion are summed up by Newton in his three Laws of Motion or Axioms of Physics. An axiom is defined by Thomson and Tait* as a proposition, the truth of which must be admitted as soon as the terms in which it is expressed are clearly understood. These physical axioms rest, not on intuitive perception, but on convictions drawn from observation and experiment.

LEX I. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum suum mutare.*

Every body persists in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

The property of persistence thus defined is called *Inertia.*

This gives a criterion for finding whether a force is acting on a body or not, or in other words a negative definition of force. Force is acting on a body when its motion is not uniform. By uniform we mean such motion that the vector velocity is constant. If the body be a material point, that is a body so small that the distances apart of its different parts may be neglected, the motion is uniform if

(1) $$\frac{dx}{dt} = c_1, \quad \frac{dy}{dt} = c_2, \quad \frac{dz}{dt} = c_3,$$

that is

(2) $$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0.$$

* Thomson and Tait, *Natural Philosophy*, § 243.

Accordingly we see that the force and acceleration vanish together. Integrating the equations (1),

$$x = c_1 t + d_1, \quad y = c_2 t + d_2, \quad z = c_3 t + d_3,$$

$$\frac{x - d_1}{c_1} = \frac{y - d_2}{c_2} = \frac{z - d_3}{c_3},$$

the path is a straight line, and since

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{c_1^2 + c_2^2 + c_3^2},$$

it is traversed with constant velocity. We may on the other hand interpret the statement as giving us a means of measuring time. Intervals of time are proportional to the corresponding distances traversed by a point not acted on by forces.

The second law gives the measure of a force.

Lex II. *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*

Change of motion is proportional to force applied, and takes place in the direction of the straight line in which the force acts.

By change of motion is meant acceleration. If we have to do with different bodies, however, the factor of proportionality will be different for each.

Lex III. *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.*

To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed.

If we have an action between two bodies 1 and 2, if the forces were proportional only to the accelerations, we should have

$$\frac{d^2x_1}{dt^2} = -\frac{d^2x_2}{dt^2}, \quad \frac{d^2y_1}{dt^2} = -\frac{d^2y_2}{dt^2}, \quad \frac{d^2z_1}{dt^2} = -\frac{d^2z_2}{dt^2}.$$

This is not the case, but we must introduce factors of proportionality, so that

$$m_1 \frac{d^2x_1}{dt^2} = - m_2 \frac{d^2x_2}{dt^2},$$

$$m_1 \frac{d^2y_1}{dt^2} = - m_2 \frac{d^2y_2}{dt^2},$$

$$m_1 \frac{d^2z_1}{dt^2} = - m_2 \frac{d^2z_2}{dt^2}.$$

The factors m_1, m_2, are called the *masses* of the bodies 1 and 2. This gives us a means of comparing masses. If we make two bodies act upon each other in any manner, their masses are inversely proportional to the accelerations they have at the same instant. The vector whose components are

$$X = m \frac{d^2x}{dt^2}, \quad Y = m \frac{d^2y}{dt^2}, \quad Z = m \frac{d^2z}{dt^2}$$

is called the *impressed force* acting on the mass m. If the quantities X, Y, Z are given functions, the above are called the *differential equations of motion* of the material point m.

52. Units. The specification of any quantity, scalar or vector, involves two factors, first a numerical quantity (integer, fraction or irrational) or *numeric*, and secondly a concrete quantity in terms of which all quantities of that kind are numerically expressed, called a unit. The simplest unit is that of the geometrical quantity, length. We shall adopt as the unit of length the *centimeter*, defined as the one-hundredth part of the distance at temperature zero degrees Centigrade, and pressure 760 millimeters of mercury, between two parallel lines engraved on a certain bar of platinum-iridium alloy, deposited in a vault in the laboratory of the "*Comité International des Poids et Mesures*," at Sèvres, near Paris. This bar is known as the "*Mètre Prototype*," and serves as the basis of length measurements for the civilized world (except the British Empire and Russia*).

It was proposed by Maxwell to use a natural unit of length, namely the length of a wave of light corresponding to some well-defined line in the spectrum of some element, at a definite temperature and pressure, as it is extremely probable that such a wave-length is extremely constant. Measurements were carried out at Sèvres by Michelson, with this end in view, which established the ratio between the above meter and the wave-length in air of a red cadmium ray as 1,553,164†.

* The United States yard is defined as 3600/3937 metres.

† Michelson, *Journal de Physique*, Jan. 1894.

The unit of mass will be assumed to be the *gram*, defined as the one-thousandth part of a piece of platinum-iridium, deposited at the place above mentioned and known as the "*Kilogramme Prototype.*"

As the unit of time we shall take the mean solar second, obtained from astronomical observations on the rotation of the earth. The unit of time cannot be preserved and compared as in the case of the units of length and mass, but is fortunately preserved for us by nature, in the nearly constant rotation of the earth. As the earth is gradually rotating more slowly, however, this unit is not absolutely constant, and it has been proposed to take for the unit of time the period of vibration of a molecule of the substance giving off light of the standard wave-length. To obtain such a unit would involve a measurement of the velocity of light, which cannot at present be made with sufficient accuracy to warrant the change.

53. Derived Units. Dimensions. It can be shown that the measurements of all physical quantities with which we are acquainted may be made in terms of three independent units. These are known as fundamental units, and are most conveniently taken as those of length, mass, and time. Other units, which depend on these, are known as derived units. If the same quantity is expressed in terms of two different units of the same kind, the numerics are inversely proportional to the size of the units. Thus six feet is otherwise expressed as two yards, the numerics 6 and 2 being in the ratio 3, that of a yard to a foot. If we change the magnitude of one of the fundamental units in any ratio r, the numeric of a quantity expressed in derived units will vary proportionately to a certain power of r, r^{-n}; the derived unit is then said to be of *dimensions** n in the fundamental unit in question. For instance, if we change the fundamental unit of length from the foot to the yard, $r = 3$, an area of 27 sq. ft. becomes expressed as 3 sq. yds., the numeric has changed in the ratio $3 : 27 = 1 : 3^2 = r^{-2}$, and the unit of area is of dimensions 2 in the unit of length. We may express this by writing

$$[\text{Area}] = [L^2].$$

* The idea of dimensions of units originated with Fourier: vid. *Théorie analytique de la Chaleur*, Section IX.

The derived unit increases in the same ratio that the numeric of the quantity decreases. In our system the unit of area is the square centimeter, written $1\,cm^2$. In like manner the unit of volume is of the dimensions $[L^3]$ and the unit is $1\,cm^3$. The dimensions of velocity are $\left[\frac{L}{T}\right]$, or as we write for convenience,

$$\text{velocity} = \text{length}/\text{time}.$$

Two quantities of different sorts do not have a ratio in the ordinary arithmetical sense, but such equations as the above are of great use in physics, and give rise to an extended meaning of the terms ratio and product.

The above equation is to be interpreted as follows. If any velocity be specified in terms of units of length and time the numerical factor is greater in proportion directly as the unit of length is smaller, and as the unit of time is greater. For instance we may write the equation expressing the fact that a velocity of 30 feet per second is the same as a velocity of 10 yards per second or 1800 feet per minute

$$30\,\frac{\text{ft.}}{\text{sec.}} = 10\,\frac{\text{yd.}}{\text{sec.}} = 1800\,\frac{\text{ft.}}{\text{min.}}.$$

We may operate on such equations precisely as if the units were ordinary arithmetical quantities, for the ratio of two quantities of the same kind is always a number. For instance

$$\frac{30}{10} = \frac{\text{yd.}}{\text{ft.}}\,\frac{\text{sec.}}{\text{sec.}}.$$

The ratio $\frac{\text{yd.}}{\text{ft.}}$ is the number 3, while $\frac{\text{sec.}}{\text{sec.}} = 1$. Also

$$\frac{1800}{10} = \frac{\text{yd.}}{\text{ft.}}\,\frac{\text{min.}}{\text{sec.}} = 3 \times 60.$$

Such an expression as $\frac{\text{ft.}}{\text{sec.}}$ is read *feet per second.*

The unit of velocity is one centimeter-per-second,

$$\frac{\text{cm.}}{\text{sec.}} = \text{cm. sec}^{-1}.$$

Since acceleration is defined as a ratio of increment of velocity to increment of time, we have

$$[\text{Acceleration}] = \frac{[\text{Velocity}]}{[\text{Time}]} = \frac{[\text{Length}]}{[\text{Time}^2]} = \left[\frac{L}{T^2}\right];$$

or the numeric of a certain acceleration varies inversely as the magnitude of the unit of length, and directly as the square of the unit of time. For instance, an acceleration in which a velocity of 10 feet per second is gained in 2 seconds is equal to one in which a velocity of 9000 feet per minute is gained in a minute,

$$\frac{10 \text{ ft.}}{(2 \text{sec.})^2} = \frac{10}{4} \frac{\text{ft.}}{\text{sec.}^2} = 9000 \frac{\text{ft.}}{\text{min.}^2}.$$

The unit of acceleration is one centimeter-per-second per second.

Since force = mass × acceleration,

$$[\text{Force}] = \frac{[\text{Mass}].[\text{Length}]}{[(\text{Time})^2]} = \left[\frac{ML}{T^2}\right].$$

The unit of force is one gram-centimeter-per-second-per-second. It is called a *dyne*.

All physical equations must be homogeneous in the various units, that is, the dimensions of every term must be the same. This gives us a valuable check on the correctness of our equations.

54. Absolute Systems. The above system of units, which has for its fundamental units the centimeter, gram, and second, is called the C.G.S. system, and was recommended by a committee of the British Association for the Advancement of Science in 1861. It is sometimes incorrectly spoken of as *the absolute system* of units. An absolute system is any system, irrespective of the magnitudes of the units, by which physical quantities can be specified in terms of the least number of fundamental units, which shall be independent of time or place, and reproducible by copying from standards. A system based on the foot, pound, and minute is just as much an absolute system as the C.G.S. system. The idea of an absolute system is due to Gauss*.

The ordinary method of measuring force, used by non-scientific persons and (or including) engineers, does not belong to the absolute system of measurements. The unit of force is taken as the weight of, or downward force exerted by the earth upon, the mass of a standard piece of metal, such as the standard pound or kilogram. To measure the force in absolute units, we must know

* Gauss. *Intensitas vis magneticae terrestris ad mensuram absolutam revocata.* Göttingen, 1832. Ges. Werke, v. p. 80.

what acceleration the earth's pull would cause this mass to receive, if allowed to fall. Experiment shows that in a given locality on the earth's surface all bodies fall in vacuo with the same acceleration. The value of this acceleration is denoted by g, and its value at the sea-level in latitude 45° is

$$g = 980{\cdot}606 \frac{\text{cm.}}{\text{sec.}^2}.$$

Accordingly the force exerted by the earth on a mass of m grams is mg dynes, or the

weight of a kilogram in latitude 45° = 980,606 *dynes.*

Now the value of the acceleration g is not constant, but varies as we go from place to place on the earth's surface, ascend mountains or descend into mines. Accordingly, the weight of a kilogram is not an invariable, or *absolute* standard of force. At the center of the earth, a kilogram would weigh nothing. Its mass is, however, invariable. The value of g at points on the earth in latitude λ and h centimeters above the sea-level, is given by the formula, originally given by Clairaut*,

$$g = 980{\cdot}6056 - 2{\cdot}508 \cos 2\lambda - {\cdot}000003h.$$

For further information with regard to units, the reader may consult Everett's *Units and Physical Constants.*

* Everett, *Units and Physical Constants*, Chap. III.

CHAPTER II.

WORK AND ENERGY.

55. Work. If a point be displaced in a straight line, under the action of a force which is constant in magnitude and direction, the product of the length of the displacement and the resolved part of the force in the direction of the displacement, that is, the geometrical product of the force and the displacement (§ 7), is called the *work* done by the force in producing the displacement. If the components of the force F are X, Y, Z, and those of the displacement s are s_x, s_y, s_z, the work W is

$$W = sF\cos(Fs) = \widehat{Fs} = Xs_x + Ys_y + Zs_z. \tag{1}$$

Since work is defined as force × distance, we have for its dimensions,

$$[\text{Work}] = [L]\left[\frac{ML}{T^2}\right] = [ML^2T^{-2}].$$

The C.G.S. unit of work is the work done when a force of one dyne produces a displacement of one centimeter in its own direction. This unit is called the *erg* $=$ gm . cm^2 . sec^{-2}.

If the displacement be not in a straight line, and the force be not constant, the work done in an infinitesimal displacement ds is

$$dW = \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds, \tag{2}$$

and the work done in a displacement along any path AB is the line integral

$$W_{AB} = \int_A^B \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)ds. \tag{3}$$

The components of the force are supposed to be given as functions of s and the derivatives $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are known as functions of s from the equations of the path.

Understanding this, we may write

$$W_{AB} = \int_A^B X dx + Y dy + Z dz. \tag{4}$$

56. Virtual Work. Suppose that we have a system of n material points. If they are entirely free to move, they require $3n$ coordinates for their specification. They may be mechanically constrained, however, in such a manner that there must be certain relations satisfied by their coordinates. Let these *equations of condition* or *constraint* be

$$(5)\quad \begin{aligned} &\phi_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots x_n, y_n, z_n) = 0, \\ &\phi_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots x_n, y_n, z_n) = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\phi_k(x_1, y_1, z_1, x_2, y_2, z_2, \ldots\ldots x_n, y_n, z_n) = 0. \end{aligned}$$

Such constraints may be imposed by causing the particles to lie on certain surfaces. For instance, if two particles 1 and 2 are connected by a rigid rod of length l, either particle must move on a sphere of radius l of which the other is the center, and we have the equation of condition

$$^*\phi \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - l^2 = 0.$$

(We might have constraints defined by inequalities, e.g., if a particle were obliged to stay on or within a spherical surface of radius l the constraint would be only from without, and we should have

$$(x-a)^2 + (y-b)^2 + (z-c)^2 - l^2 \leqq 0.$$

We shall assume that the constraint is toward both sides, and is defined by an equation.)

If any particle at x_r, y_r, z_r is displaced by a small amount so that it has the coordinates

$$x_r + \delta x_r,\ y_r + \delta y_r,\ z_r + \delta z_r,$$

* The sign $\equiv$ is to be read—*is identically*—i.e., is for all possible values of the variables, or *is defined as*.

in order that the constraint may hold we must have for each ϕ

$$\text{(6)} \qquad \begin{aligned} &\phi(x_r, y_r, z_r \ldots\ldots) = 0, \\ &\phi(x_r + \delta x_r,\ y_r + \delta y_r,\ z_r + \delta z_r \ldots\ldots) = 0, \end{aligned}$$

and if ϕ be a continuous function, developing by Taylor's Theorem, $\phi(x_r + \delta x_r,\ y_r + \delta y_r,\ z_r + \delta z_r \ldots\ldots) =$

$$\phi(x_r, y_r, z_r) + \delta x_r \frac{\partial \phi}{\partial x_r} + \delta y_r \frac{\partial \phi}{\partial y_r} + \delta z_r \frac{\partial \phi}{\partial z_r} + \tfrac{1}{2}\delta x_r^2 \frac{\partial^2 \phi}{\partial x_r^2} + \ldots\ldots,$$

and accordingly, taking account only of the terms of the first order in the small quantities δx_r, δy_r, δz_r, and using equations (6), we have

$$\text{(7)} \qquad \frac{\partial \phi}{\partial x_r}\delta x_r + \frac{\partial \phi}{\partial y_r}\delta y_r + \frac{\partial \phi}{\partial z_r}\delta z_r = 0.$$

If a number of particles are displaced, we must take the sum of expressions like the above for all the particles, or

$$\text{(8)} \qquad \sum_{r=1}^{r=n} \left\{ \frac{\partial \phi}{\partial x_r}\delta x_r + \frac{\partial \phi}{\partial y_r}\delta y_r + \frac{\partial \phi}{\partial z_r}\delta z_r \right\} = 0,$$

as the conditions which must be satisfied by all the displacements δx_r, δy_r, δz_r. There must be one such equation for each function ϕ. Such displacements, which are purely arbitrary, except that they satisfy the equations of condition, are called *virtual*, being possible, as opposed to the displacements that *actually take place* in a motion of the system.

The Principle of Virtual Work is an analytical statement of the conditions for equilibrium of a system. A system is in equilibrium when the forces acting on its various particles, together with the constraints, balance each other in such a way that there is no tendency toward motion of any part of the system. If the system consists of a single free point, in order for it to be in equilibrium, the resultant of all the forces applied to it, whose components are X, Y, Z, must vanish,

$$\text{(9)} \qquad X = Y = Z = 0.$$

If we multiply these equations respectively by the arbitrary small quantities δx, δy, δz and add, we get

$$\text{(10)} \qquad X\delta x + Y\delta y + Z\delta z = 0,$$

which expresses that the work done in an infinitesimal displacement of a point from its position of equilibrium vanishes. The

equation (10) is equivalent to the equation (9), for since the quantities δx, δy, δz, are arbitrary, if X, Y, Z, are different from zero, we may take δx, δy, δz respectively of the same sign as X, Y, Z,—each product will then be positive, and the sum will not vanish. If the sum is to vanish for all possible choices of δx, δy, δz, X, Y, Z must vanish.

If the particle is not free, but constrained to lie on a surface $\phi = 0$, δx, δy, δz are not entirely arbitrary, but must satisfy

$$(7) \qquad \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = 0.$$

Let us multiply this by a quantity λ and add it to (10), obtaining

$$(11) \qquad \left(X + \lambda\frac{\partial \phi}{\partial x}\right)\delta x + \left(Y + \lambda\frac{\partial \phi}{\partial y}\right)\delta y + \left(Z + \lambda\frac{\partial \phi}{\partial z}\right)\delta z = 0.$$

We may no longer conclude that the coefficients of δx, δy, δz must vanish, for δx, δy, δz are not arbitrary, being connected by the equation (7). Two of them are however arbitrary, say δy and δz, λ has not yet been fixed—suppose it determined so that

$$X + \lambda\frac{\partial \phi}{\partial x} = 0.$$

Then we have $\left(Y + \lambda\frac{\partial \phi}{\partial y}\right)\delta y + \left(Z + \lambda\frac{\partial \phi}{\partial z}\right)\delta z = 0,$

in which δy and δz are perfectly arbitrary, it therefore follows of necessity that the coefficients vanish.

$$Y + \lambda\frac{\partial \phi}{\partial y} = 0, \qquad Z + \lambda\frac{\partial \phi}{\partial z} = 0.$$

By the introduction of the multiplier λ we are accordingly enabled to draw the same conclusion as if δx, δy, δz were arbitrary. Eliminating λ from the above equations we get

$$\frac{X}{\frac{\partial \phi}{\partial x}} = \frac{Y}{\frac{\partial \phi}{\partial y}} = \frac{Z}{\frac{\partial \phi}{\partial z}}.$$

Now the direction cosines of the normal to the surface $\phi = 0$ are proportional to $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial \phi}{\partial z}$, consequently, the components X, Y, Z being proportional to these direction cosines, the resultant

is in the direction of the normal to the surface. But under these conditions the particle is in equilibrium.

In like manner we may show that if the forces X_1, Y_1, Z_1, act upon the particle 1, X_2, Y_2, Z_2, upon the particle 2, etc., the condition of equilibrium is

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 + X_2\delta x_2 + Y_2\delta y_2 + Z_2\delta z_2 \ldots\ldots + Z_n\delta z_n = 0, \tag{12}$$

where the displacements satisfy

$$\begin{aligned}
&\frac{\partial\phi_1}{\partial x_1}\delta x_1 + \frac{\partial\phi_1}{\partial y_1}\delta y_1 + \frac{\partial\phi_1}{\partial z_1}\delta z_1 + \frac{\partial\phi_1}{\partial x_2}\delta x_2 + \frac{\partial\phi_1}{\partial y_2}\delta y_2 \ldots + \frac{\partial\phi_1}{\partial z_n}\delta z_n = 0,\\
&\frac{\partial\phi_2}{\partial x_1}\delta x_1 + \frac{\partial\phi_2}{\partial y_1}\delta y_1 + \frac{\partial\phi_2}{\partial z_1}\delta z_1 + \frac{\partial\phi_2}{\partial x_2}\delta x_2 + \frac{\partial\phi_2}{\partial y_2}\delta y_2 \ldots + \frac{\partial\phi_2}{\partial z_n}\delta z_n = 0,\\
(13)\quad &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{\partial\phi_k}{\partial x_1}\delta x_1 + \frac{\partial\phi_k}{\partial y_1}\delta y_1 + \frac{\partial\phi_k}{\partial z_1}\delta z_1 + \frac{\partial\phi_k}{\partial x_2}\delta x_2 + \frac{\partial\phi_k}{\partial y_2}\delta y_2 \ldots + \frac{\partial\phi_k}{\partial z_n}\delta z_n = 0.
\end{aligned}$$

Multiplying the equations (13) respectively by λ_1, λ_2, ... λ_k, and adding to (12) we have

$$\begin{aligned}
&\left(X_1 + \lambda_1\frac{\partial\phi_1}{\partial x_1} + \lambda_2\frac{\partial\phi_2}{\partial x_1}\ldots\ldots + \lambda_k\frac{\partial\phi_k}{\partial x_1}\right)\delta x_1\\
&+\left(X_2 + \lambda_1\frac{\partial\phi_1}{\partial x_2} + \lambda_2\frac{\partial\phi_2}{\partial x_2}\ldots\ldots + \lambda_k\frac{\partial\phi_k}{\partial x_2}\right)\delta x_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(14)\quad &+\left(Y_1 + \lambda_1\frac{\partial\phi_1}{\partial y_1} + \lambda_2\frac{\partial\phi_2}{\partial y_1}\ldots\ldots + \lambda_k\frac{\partial\phi_k}{\partial y_1}\right)\delta y_1\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\left(Z_n + \lambda_1\frac{\partial\phi_1}{\partial z_n} + \lambda_2\frac{\partial\phi_2}{\partial z_n}\ldots\ldots + \lambda_k\frac{\partial\phi_k}{\partial z_n}\right)\delta z_n = 0.
\end{aligned}$$

Of the $3n$ quantities δx_1, ... δz_n, only $3n-k$ are arbitrary, we may however determine the k multipliers λ so that the coefficients of the k other δ's vanish, then the coefficients of the $3n-k$ *arbitrary* δ's *must* vanish, so that we get the $3n$ equations

$$
(15) \quad \begin{aligned}
&X_1 + \lambda_1 \frac{\partial \phi_1}{\partial x_1} + \lambda_2 \frac{\partial \phi_2}{\partial x_1} \cdots\cdots + \lambda_k \frac{\partial \phi_k}{\partial x_1} = 0, \\
&X_2 + \lambda_1 \frac{\partial \phi_1}{\partial x_2} + \lambda_2 \frac{\partial \phi_2}{\partial x_2} \cdots\cdots + \lambda_k \frac{\partial \phi_k}{\partial x_2} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&Z_n + \lambda_1 \frac{\partial \phi_1}{\partial z_n} + \lambda_2 \frac{\partial \phi_2}{\partial z_n} \cdots\cdots + \lambda_k \frac{\partial \phi_k}{\partial z_n} = 0.
\end{aligned}
$$

Eliminating from these the k quantities λ, we have $3n-k$ equations expressing the conditions of equilibrium, being as many as the system has degrees of freedom. The equation (12), or as we may write it

$$(16) \qquad \Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0,$$

expresses the fact that the work in a virtual displacement vanishes, and is the condition for equilibrium. This is the Principle of Virtual Work.

57. D'Alembert's Principle. The equations of motion of a point are (§ 51)

$$
(17) \qquad \begin{aligned}
m_r \frac{d^2 x_r}{dt^2} &= X_r, \\
m_r \frac{d^2 y_r}{dt^2} &= Y_r, \\
m_r \frac{d^2 z_r}{dt^2} &= Z_r.
\end{aligned}
$$

or

$$
(18) \qquad \begin{aligned}
m_r \frac{d^2 x_r}{dt^2} - X_r &= 0, \\
m_r \frac{d^2 y_r}{dt^2} - Y_r &= 0, \\
m_r \frac{d^2 z_r}{dt^2} - Z_r &= 0.
\end{aligned}
$$

Multiplying the equations (18) respectively by the arbitrary quantities δx_r, δy_r, δz_r adding, and taking the sum for all values of the suffix r,

$$
(19) \quad \Sigma_r \left\{ \left(m_r \frac{d^2 x_r}{dt^2} - X_r \right) \delta x_r + \left(m_r \frac{d^2 y_r}{dt^2} - Y_r \right) \delta y_r + \left(m_r \frac{d^2 z_r}{dt^2} - Z_r \right) \delta z_r \right\} = 0.
$$

This equation may be called the fundamental equation of dynamics, and is the analytical statement of what is known as *d'Alembert's Principle.* Lagrange made it the basis of the entire subject of dynamics*. Interpreted by means of the principle of virtual work, equation (19) states:—

If, the motion of a system of particles being given, we find the acceleration of every particle, and apply to each particle a force whose components are

$$X_r' = -m_r \frac{d^2x_r}{dt^2}, \quad Y_r' = -m_r \frac{d^2y_r}{dt^2}, \quad Z_r' = -m_r \frac{d^2z_r}{dt^2},$$

then the system of forces X', Y', Z', together with the impressed forces X, Y, Z, will form a system in equilibrium.

The forces X', Y', Z' are called the *forces of inertia*, or the reversed effective forces. D'Alembert's principle is thus only another form of stating Newton's third law of motion.

We have now a measure of the inertia of a body, namely the force of inertia above defined†. We may now define matter as whatever can exert forces of inertia.

58. **Energy. Conservative Systems.** If in the equation of d'Alembert's principle, (19), we put for δx, δy, δz the displacements which take place in the actual motion of the system in the time dt,

$$\delta x_r = \frac{dx_r}{dt} dt, \quad \delta y_r = \frac{dy_r}{dt} dt, \quad \delta z_r = \frac{dz_r}{dt} dt,$$

we obtain

$$(20) \quad \Sigma_r \left\{ m_r \left(\frac{d^2x_r}{dt^2} \frac{dx_r}{dt} + \frac{d^2y_r}{dt^2} \frac{dy_r}{dt} + \frac{d^2z_r}{dt^2} \frac{dz_r}{dt} \right) \right.$$
$$\left. - X_r \frac{dx_r}{dt} - Y_r \frac{dy_r}{dt} - Z_r \frac{dz_r}{dt} \right\} dt = 0.$$

Since
$$m_r \frac{d^2x_r}{dt^2} \frac{dx_r}{dt} = \tfrac{1}{2} \frac{d}{dt} \left(m_r \left(\frac{dx_r}{dt} \right)^2 \right),$$

the sum of the first three terms is the derivative of the sum

$$\tfrac{1}{2} \Sigma_r m_r \left\{ \left(\frac{dx_r}{dt} \right)^2 + \left(\frac{dy_r}{dt} \right)^2 + \left(\frac{dz_r}{dt} \right)^2 \right\},$$

* Lagrange, *Mécanique Analytique. Œuvres*, t. 11, p. 267.

† The *inertia* of a body is sometimes considered as the factor of the negative acceleration in the expression for the force of inertia, thus making inertia synonymous with mass.

and the equation may be written

$$(21)\qquad \frac{d}{dt}\left[\tfrac{1}{2}\Sigma_r m_r\left\{\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right\}\right]dt$$

$$=\Sigma_r\left\{X_r\frac{dx_r}{dt}+Y_r\frac{dy_r}{dt}+Z_r\frac{dz_r}{dt}\right\}dt.$$

Integrating with respect to t between the limits t_0 and t_1,

$$(22)\qquad \left[\tfrac{1}{2}\Sigma m_r\left\{\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right\}\right]_{t_0}^{t_1}$$

$$=\Sigma_r\int_{t_0}^{t_1}\left(X_r\frac{dx_r}{dt}+Y_r\frac{dy_r}{dt}+Z_r\frac{dz_r}{dt}\right)dt.$$

The square brackets with the affixes t_0, t_1 denote that the value of the expression in brackets for $t=t_0$ is to be subtracted from the value for $t=t_1$.

The integral on the right of (22), which may be written

$$\int X_r dx_r + Y_r dy_r + Z_r dz_r,$$

denotes the work done by the forces of the system on the particle m_r during the motion from t_0 to t_1, and the sum of such integrals denotes the total work done by the forces acting on the system during the motion.

The expression

$$\tfrac{1}{2}\Sigma m_r\left\{\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right\}=\tfrac{1}{2}\Sigma_r m_r v_r^2,$$

the half-sum of the products of the mass of each particle by the square of its velocity, is called the *Kinetic Energy* of the system. If we denote it by T, the equation (22) becomes

$$(23)\qquad T_{t_1}-T_{t_0}=\Sigma_r\int_{t_0}^{t_1}(X_r dx_r+Y_r dy_r+Z_r dz_r).$$

This is called the *equation of energy*, and states that the gain of kinetic energy is equal to the work done by the forces during the motion.

The equation of energy assumes an important form in the particular case that the forces acting on the particles depend only on the positions of the particles, and that the components may be

represented by the partial derivatives of a single function of the coordinates

$$U(x_1, y_1, z_1, x_2, y_2, z_2, \dots z_n),$$

(24)
$$X_r = \frac{\partial U}{\partial x_r}, \quad Y_r = \frac{\partial U}{\partial y_r}, \quad Z_r = \frac{\partial U}{\partial z_r}.$$

In this case the expression

$$\Sigma_r \{X_r dx_r + Y_r dy_r + Z_r dz_r\}$$
$$= \Sigma_r \left\{\frac{\partial U}{\partial x_r} dx_r + \frac{\partial U}{\partial y_r} dy_r + \frac{\partial U}{\partial z_r} dz_r\right\},$$

is the exact differential of the function U, and the integral

$$\int_{t_0}^{t_1} \Sigma (X_r dx_r + Y_r dy_r + Z_r dz_r) = U_{t_1} - U_{t_0}.$$

The equation of energy then is

(25)
$$T_{t_1} - T_{t_0} = U_{t_1} - U_{t_0}.$$

The function U is called the force-function, and its negative $W = -U$ is called the *Potential Energy* of the system. Inserting W in (25) we have

(26)
$$T_{t_1} + W_{t_1} = T_{t_0} + W_{t_0}.$$

The sum of the kinetic and potential energies of a system possessing a force-function is the same at all instants of time. This is the principle of *Conservation of Energy.*

Systems for which the conditions (24) are satisfied are accordingly called *conservative systems.*

The potential energy, being defined by its derivatives, contains an arbitrary constant. Conservative systems possess the property, since W depends only on the coordinates, and $T + W$ is constant, that T, the kinetic energy, depends only on the coordinates, or if in the course of the motion all the points of the system pass simultaneously through positions that they have before occupied, the kinetic energy will be the same as at the previous instant, irrespective of the directions in which the points may be moving. For instance, a particle thrown vertically upwards, or a pendulum swinging, have the same velocity when passing a given point whether rising or falling.

The principle of virtual work, § 56, may evidently be expressed by saying that for equilibrium the potential energy of the system

is a maximum or minimum, and a little consideration shows that for *stable* equilibrium it is a *minimum.*

Examples of non-conservative systems are found whenever the forces depend upon the velocities as well as upon the coordinates; for example, bodies moving through the air or other resisting medium or bodies whose motion is opposed by friction of any sort, form non-conservative systems. Even if the friction be constant in magnitude, its direction will depend on the direction of the velocities, being in such a direction as always to oppose the motion, and to diminish the total energy of the system. The dynamical theory of heat accounts for the energy that apparently disappears in non-conservative systems.

Kinetic energy being defined as $\Sigma\frac{1}{2}mv^2$ is of the dimensions $\left[\frac{ML^2}{T^2}\right]$, the same as those of work. Potential energy is defined as work. The unit of energy is, therefore, the *erg.*

59. Particular case of Force-function. Newtonian Forces. In the particular case in which the only forces acting on the system are attractions or repulsions by the several particles directed along the lines joining them and depending only on their mutual distances, a force-function always exists.

For let the force between two particles m_r and m_s at a distance apart r_{rs} be

$$F = \phi\,(r_{rs}).$$

It will be convenient to consider F positive if the force is a *repulsion.*

Consider now the force $F_s^{(r)}$ acting on m_s and acting in the direction from m_r to m_s. Its direction cosines are those of the vector r_{rs},

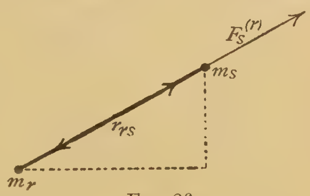

Fig. 26.

$$\frac{X_s^{(r)}}{F_s^{(r)}} = \frac{x_s - x_r}{r_{rs}},$$

$$\frac{Y_s^{(r)}}{F_s^{(r)}} = \frac{y_s - y_r}{r_{rs}},$$

$$\frac{Z_s^{(r)}}{F_s^{(r)}} = \frac{z_s - z_r}{r_{rs}}.$$

Now since

$$r_{rs}^2 = (x_s - x_r)^2 + (y_s - y_r)^2 + (z_s - z_r)^2,$$

differentiating partially by x_s,

$$2r_{rs}\frac{\partial r_{rs}}{\partial x_s} = 2(x_s - x_r),$$

(28) $$\frac{\partial r_{rs}}{\partial x_s} = \frac{x_s - x_r}{r_{rs}}, \quad \frac{\partial r_{rs}}{\partial y_s} = \frac{y_s - y_r}{r_{rs}}, \quad \frac{\partial r_{rs}}{\partial z_s} = \frac{z_s - z_r}{r_{rs}},$$

and accordingly

$$\frac{X_s^{(r)}}{F_s^{(r)}} = \frac{\partial r_{rs}}{\partial x_s}, \quad \frac{Y_s^{(r)}}{F_s^{(r)}} = \frac{\partial r_{rs}}{\partial y_s}, \quad \frac{Z_s^{(r)}}{F_s^{(r)}} = \frac{\partial r_{rs}}{\partial z_s};$$

$$X_s^{(r)} = F_s^{(r)}\frac{\partial r_{rs}}{\partial x_s} = \phi(r_{rs})\frac{\partial r_{rs}}{\partial x_s},$$

$$Y_s^{(r)} = F_s^{(r)}\frac{\partial r_{rs}}{\partial y_s} = \phi(r_{rs})\frac{\partial r_{rs}}{\partial y_s},$$

$$Z_s^{(r)} = F_s^{(r)}\frac{\partial r_{rs}}{\partial z_s} = \phi(r_{rs})\frac{\partial r_{rs}}{\partial z_s}.$$

If we put U_{rs} such a function of r_{rs} that

$$\frac{dU_{rs}}{dr_{rs}} = \phi(r_{rs}),$$

$$X_s^{(r)} = \frac{dU_{rs}}{dr_{rs}}\frac{\partial r_{rs}}{\partial x_s} = \frac{\partial U_{rs}}{\partial x_s},$$

$$Y_s^{(r)} = \frac{dU_{rs}}{dr_{rs}}\frac{\partial r_{rs}}{\partial y_s} = \frac{\partial U_{rs}}{\partial y_s},$$

$$Z_s^{(r)} = \frac{dU_{rs}}{dr_{rs}}\frac{\partial r_{rs}}{\partial z_s} = \frac{\partial U_{rs}}{\partial z_s}.$$

If now we find the resultant F_s of all the forces acting on m_s due to the repulsions by all the particles m_r, we shall have

$$X_s = \frac{\partial U_{1s}}{\partial x_s} + \frac{\partial U_{2s}}{\partial x_s} + \dots + \frac{\partial U_{ns}}{\partial x_s} = \frac{\partial U_s}{\partial x_s}$$

(29) $$Y_s = \frac{\partial U_{1s}}{\partial y_s} + \frac{\partial U_{2s}}{\partial y_s} + \dots + \frac{\partial U_{ns}}{\partial y_s} = \frac{\partial U_s}{\partial y_s},$$

$$Z_s = \frac{\partial U_{1s}}{\partial z_s} + \frac{\partial U_{2s}}{\partial z_s} + \dots + \frac{\partial U_{ns}}{\partial z_s} = \frac{\partial U_s}{\partial z_s},$$

if we write $U_s = U_{1s} + U_{2s} \dots + U_{ns}$. Thus U_s satisfies the conditions for a force-function as far as concerns the point m_s. In the summation s does not occur as the first index.

It is evident that the function U_{rs} serves the same purpose for m_r as for m_s. For the force F exerted on m_r by m_s is equal and opposite to that exerted on m_s by m_r. But r_{rs} is the same function of $(-x_r)$ that it is of x_s, therefore

$$\frac{\partial r_{rs}}{\partial x_r} = -\frac{\partial r_{rs}}{\partial x_s},$$

and $$X_r^{(s)} = \phi(r_{rs})\frac{\partial r_{rs}}{\partial x_r} = \frac{\partial U_{rs}}{\partial x_r} = -\frac{\partial U_{rs}}{\partial x_s}.$$

We may add to U_s terms independent of x_s, y_s, z_s, without affecting the values of X_s, Y_s, Z_s. If we make U a symmetrical function of all the coordinates, containing x_s, y_s, z_s as U_s does, then U will serve as the force function for all the coordinates.

In particular, let the force of repulsion vary as the product of the masses of the particles divided by the square of their distance apart $\phi(r_{rs}) = \dfrac{m_r m_s}{r_{rs}^2}$. Such forces are called *Newtonian forces*, the most familiar examples of which are the mutual attractions of the sun and the planets. Then

(30) $$\phi(r_{rs}) = \frac{m_r m_s}{r_{rs}^2}, \qquad U_{rs} = -\frac{m_r m_s}{r_{rs}},$$

(31) $$U_s = -\left\{\frac{m_1 m_s}{r_{1s}} + \frac{m_2 m_s}{r_{2s}} + \dots\dots + \frac{m_n m_s}{r_{ns}}\right\},$$

and the symmetrical function U will be

(32) $$\begin{aligned} U = -\tfrac{1}{2}\Bigg\{ & \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \dots\dots + \frac{m_1 m_n}{r_{1n}} \\ & + \frac{m_2 m_1}{r_{21}} + \frac{m_2 m_3}{r_{23}} + \dots\dots + \frac{m_2 m_n}{r_{2n}} \\ & + \frac{m_3 m_1}{r_{31}} + \frac{m_3 m_2}{r_{32}} + \dots\dots + \frac{m_3 m_n}{r_{3n}} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \frac{m_n m_1}{r_{n1}} + \frac{m_n m_2}{r_{n2}} + \dots\dots + \frac{m_n m_{n-1}}{r_{n,\,n-1}} \Bigg\}, \end{aligned}$$

or more briefly

$$U = -\tfrac{1}{2}\sum_{r=1}^{r=n}\sum_{s=1}^{s=n}\frac{m_r m_s}{r_{rs}},$$

understanding that terms in which $r = s$ are to be omitted.

The factor $\frac{1}{2}$ is introduced because in the above summation every term appears twice. But in U each pair of particles is to appear only once.

If no constant be added to U as defined above, both it and the potential energy

$$W = \tfrac{1}{2}\,\Sigma_r \Sigma_s \frac{m_r m_s}{r_{rs}}, \tag{33}$$

will vanish when every r_{rs} is infinite, that is when no two particles are within a finite distance of each other. This furnishes a convenient zero configuration for the potential energy, and is the one generally adopted. We may accordingly define the potential energy of the system in any given configuration as the work that must be done against the mutual repulsions of the particles in order to bring them from a state of infinite dispersion to the given configuration.

CHAPTER III.

HAMILTON'S PRINCIPLE.

GENERALIZED EQUATIONS OF MOTION. CYCLIC SYSTEMS.

60. Hamilton's Principle. If in d'Alembert's equation

$$\Sigma\left\{\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right\}=0,$$

we consider δx, δy, δz variations consistent with the equations of condition, we have

$$\frac{d^2x}{dt^2}\delta x=\frac{d}{dt}\left(\frac{dx}{dt}\delta x\right)-\frac{dx}{dt}\frac{d\delta x}{dt}$$

$$=\frac{d}{dt}\left(\frac{dx}{dt}\delta x\right)-\frac{dx}{dt}\frac{\delta dx}{dt}$$

$$=\frac{d}{dt}\left(\frac{dx}{dt}\delta x\right)-\delta\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2.$$

Treating each term in this manner,

$$(1)\quad \frac{d}{dt}\Sigma\left\{m\left(\frac{dx}{dt}\delta x+\frac{dy}{dt}\delta y+\frac{dz}{dt}\delta z\right)\right\}$$

$$=\delta\tfrac{1}{2}\Sigma\left\{m\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right)\right\}$$

$$+\Sigma(X\delta x+Y\delta y+Z\delta z).$$

If there is a force function U we have

$$\Sigma(X\delta x+Y\delta y+Z\delta z)=\delta U,$$

hence the right-hand member of (1) is

$$\delta T+\delta U.$$

The left-hand member being an exact derivative we may integrate with respect to t,

$$(2)\qquad \left[\Sigma\left\{m\left(\frac{dx}{dt}\,\delta x+\frac{dy}{dt}\,\delta y+\frac{dz}{dt}\,\delta z\right)\right\}\right]_{t_0}^{t_1}$$
$$=\int_{t_0}^{t_1}\delta(T+U)\,dt=\delta\int_{t_0}^{t_1}(T+U)\,dt.$$

If the positions are given for t_0 and t_1, that is if the variations δx, δy, δz vanish for t_0 and t_1, then the integrated parts vanish, and

$$\delta\int_{t_0}^{t_1}(T+U)\,dt=0,$$

or

$$(3)\qquad \delta\int_{t_0}^{t_1}(T-W)\,dt=0.$$

This is known as *Hamilton's Principle**. It may be stated by saying that if the configuration of the system is given at two instants t_0 and t_1, then the value of the time-integral of $T+U$ is less (or greater) for the paths actually described in the natural motion than in any other infinitely near motion.

Hamilton's principle is broader than the principle of energy, inasmuch as U may contain the time as well as the coordinates. It is true even for non-conservative systems (where a force-function U does not exist), if we write instead of δU

$$X\delta x+Y\delta y+Z\delta z.$$

61. Lagrange's Generalized Equations. By means of Hamilton's Principle we may deduce the generalized equations of motion.

Suppose that by means of the equations of condition, if there are any, we express all the coordinates as functions of $m=3n-k$ parameters q_1, q_2, ... q_m, which are known as the *generalized coordinates* of the system,

$$x_1=x_1(q_1,\,q_2,\dots q_m)$$
$$y_1=y_1(q_1,\,q_2,\dots q_m)$$
$$\dots\dots\dots\dots\dots\dots\dots$$

Then W, if the system is conservative, becomes a function of the parameters q.

* Hamilton. *On a General Method in Dynamics.* *Phil. Trans.* 1834.

Differentiating the above by t,

$$\text{(1)}\qquad \frac{dx_1}{dt}=\frac{\partial x_1}{\partial q_1}\frac{dq_1}{dt}+\frac{\partial x_1}{\partial q_2}\frac{dq_2}{dt}+\ldots+\frac{\partial x_1}{\partial q_m}\frac{dq_m}{dt}$$

$$=\frac{\partial x_1}{\partial q_1}q_1'+\frac{\partial x_1}{\partial q_2}q_2'+\ldots+\frac{\partial x_1}{\partial q_m}q_m', \text{ if } \frac{dq_r}{dt}=q_r'$$

$$\frac{dy_1}{dt}=\frac{\partial y_1}{\partial q_1}q_1'+\frac{\partial y_1}{\partial q_2}q_2'+\ldots+\frac{\partial y_1}{\partial q_m}q_m',$$

$$\frac{dz_1}{dt}=\frac{\partial z_1}{\partial q_1}q_1'+\frac{\partial z_1}{\partial q_2}q_2'+\ldots+\frac{\partial z_1}{\partial q_m}q_m'.$$

Since the x, y, z's are given as functions of $q_1, q_2, \ldots q_m$ alone, every $\frac{\partial x_r}{\partial q_s}$ is given also as a function of the q's. Hence every velocity-component is a *linear function of the* q''s, whose coefficients are certain functions of the q's. The q''s are called the *velocities* corresponding to the *coordinates* q.

Squaring, adding, and summing, we get

$$\text{(2)}\qquad T=\tfrac{1}{2}\Sigma_r m_r\left\{\left(\frac{dx_r}{dt}\right)^2+\left(\frac{dy_r}{dt}\right)^2+\left(\frac{dz_r}{dt}\right)^2\right\}$$

$$=\tfrac{1}{2}\Sigma_r m_r\left\{\left(\frac{\partial x_r}{\partial q_1}\right)^2 q_1'^2+\ldots\right\},$$

a homogeneous quadratic function of the q''s, whose coefficients are certain functions of the q's, so that we may write

$$\text{(3)}\qquad T=\tfrac{1}{2}Q_{11}q_1'^2+\tfrac{1}{2}Q_{22}q_2'^2+\ldots\ldots+Q_{12}q_1'q_2'+\ldots\ldots,$$

where

$$Q_{rs}=\sum_{p=1}^{p=n} m_p\left\{\frac{\partial x_p}{\partial q_r}\frac{\partial x_p}{\partial q_s}+\frac{\partial y_p}{\partial q_r}\frac{\partial y_p}{\partial q_s}+\frac{\partial z_p}{\partial q_r}\frac{\partial z_p}{\partial q_s}\right\}.$$

Performing the operation of variation upon the integral occurring in Hamilton's Principle, we obtain

$$\text{(4)}\qquad \int_{t_0}^{t_1}\left[\sum_{r=1}^{r=m}\left\{\frac{\partial(T-W)}{\partial q_r}\delta q_r+\frac{\partial(T-W)}{\partial q_r'}\delta q_r'\right\}\right]dt=0,$$

and since

$$\delta q_r'=\delta\frac{dq_r}{dt}=\frac{d}{dt}\delta q_r,$$

we may integrate the second term by parts. Since the initial and final configuration of the system is supposed given, the δq's

vanish at $t=t_0$ and $t=t_1$, so that the integrated part vanishes, and

$$(5)\qquad \int_{t_0}^{t_1}\left[\sum_{r=1}^{r=m}\left\{\frac{\partial(T-W)}{\partial q_r}-\frac{d}{dt}\left(\frac{\partial(T-W)}{\partial q_r'}\right)\right\}\delta q_r\right]dt=0.$$

Now if all the δq's are arbitrary, the integral vanishes only if the coefficient of every δq_r is equal to zero.

$$(6)\qquad \frac{\partial(T-W)}{\partial q_r}-\frac{d}{dt}\left(\frac{\partial(T-W)}{\partial q_r'}\right)=0,$$

or if we write L for the Lagrangian function $T-W$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_r'}\right)=\frac{\partial L}{\partial q_r}.$$

Since the potential energy depends only on the coordinates, $\frac{\partial W}{\partial q_r'}=0$, and we may write the equation (6)

$$(7)\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_r'}\right)-\frac{\partial T}{\partial q_r}=-\frac{\partial W}{\partial q_r}=P_r.$$

P_r is the generalized component of impressed force tending to increase the coordinate q_r.

If the system is not conservative, we must write, instead of $-\delta W$

$$\Sigma_r\{X_r\,\delta x_r+Y_r\delta y_r+Z_r\delta z_r\},$$

and the integral is

$$(8)\qquad \delta\int T dt+\int\Sigma\{X_r\delta x_r+Y_r\delta y_r+Z_r\delta z_r\}\,dt=0$$
$$=\delta\int(T+\Sigma X_r x_r+Y_r y_r+Z_r z_r)\,dt.$$

Now $\qquad \delta x_r=\frac{\partial x_r}{\partial q_1}\delta q_1+\frac{\partial x_r}{\partial q_2}\delta q_2+\ldots\ldots+\frac{\partial x_r}{\partial q_m}\delta q_m,$

so that if we write

$$(9)\qquad P_r=\Sigma_s\left\{X_s\frac{\partial x_s}{\partial q_r}+Y_s\frac{\partial y_s}{\partial q_r}+Z_s\frac{\partial z_s}{\partial q_r}\right\},$$

we get the same equations as before. If there is a force-function

$$X_s=-\frac{\partial W}{\partial x_s}$$

and

$$P_r=-\Sigma_s\left\{\frac{\partial W}{\partial x_s}\frac{\partial x_s}{\partial q_r}+\frac{\partial W}{\partial y_s}\frac{\partial y_s}{\partial q_r}+\frac{\partial W}{\partial z_s}\frac{\partial z_s}{\partial q_r}\right\}$$
$$=-\frac{\partial W}{\partial q_r}.$$

There are m of the equations (7), one for each coordinate q.

These are Lagrange's equations of motion in generalized coordinates.

62. Proof independent of Hamilton's Principle.

We will verify these equations by direct transformation of the equations in rectangular coordinates

$$(10)\qquad \begin{aligned} m_r\frac{d^2x_r}{dt^2} &= X_r + \lambda_1\frac{\partial\phi_1}{\partial x_r} + \lambda_2\frac{\partial\phi_2}{\partial x_r} + \ldots\ldots, \\ m_r\frac{d^2y_r}{dt^2} &= Y_r + \lambda_1\frac{\partial\phi_1}{\partial y_r} + \lambda_2\frac{\partial\phi_2}{\partial y_r} + \ldots\ldots, \\ m_r\frac{d^2z_r}{dt^2} &= Z_r + \lambda_1\frac{\partial\phi_1}{\partial z_r} + \lambda_2\frac{\partial\phi_2}{\partial z_r} + \ldots\ldots, \end{aligned}$$

which are obtained from equation (15) of Chapter II. by means of d'Alembert's principle.

Multiplying these respectively by

$$\frac{\partial x_r}{\partial q_s},\ \frac{\partial y_r}{\partial q_s},\ \frac{\partial z_r}{\partial q_s},$$

adding and summing for all values of r, the coefficient of λ_1 becomes

$$\Sigma_r\left\{\frac{\partial\phi_1}{\partial x_r}\frac{\partial x_r}{\partial q_s} + \frac{\partial\phi_1}{\partial y_r}\frac{\partial y_r}{\partial q_s} + \frac{\partial\phi_1}{\partial z_r}\frac{\partial z_r}{\partial q_s}\right\} = \frac{\partial\phi_1}{\partial q_s}.$$

If there are no relations between the q's, the expression

$$\phi_1(q_1, \ldots q_m) = 0$$

is an *identity*, and all its partial derivatives $\frac{\partial\phi_1}{\partial q_s}$ are equal to zero. Accordingly the terms in $\lambda_1, \lambda_2, \ldots$ disappear.

We have then

$$(11)\qquad \begin{aligned} &\Sigma_r m_r\left\{\frac{d^2x_r}{dt^2}\frac{\partial x_r}{dq_s} + \frac{d^2y_r}{dt^2}\frac{\partial y_r}{\partial q_s} + \frac{d^2z_r}{dt^2}\frac{\partial z_r}{\partial q_s}\right\} \\ &\qquad = \Sigma_r\left\{X_r\frac{\partial x_r}{\partial q_s} + Y_r\frac{\partial y_r}{\partial q_s} + Z_r\frac{\partial z_r}{\partial q_s}\right\}. \end{aligned}$$

Now $$T = \tfrac{1}{2}\Sigma_r m_r(x_r'^2 + y_r'^2 + z_r'^2),$$

$$(12)\qquad \begin{aligned} \frac{\partial T}{\partial q_s} &= \Sigma_r m_r\left(x_r'\frac{\partial x_r'}{\partial q_s} + y_r'\frac{\partial y_r'}{\partial q_s} + z_r'\frac{\partial z_r'}{\partial q_s}\right), \\ \frac{\partial T}{\partial q_s'} &= \Sigma_r m_r\left(x_r'\frac{\partial x_r'}{\partial q_s'} + y_r'\frac{\partial y_r'}{\partial q_s'} + z_r'\frac{\partial z_r'}{\partial q_s'}\right), \end{aligned}$$

but by (1)

$$x_r' = \frac{\partial x_r}{\partial q_1} q_1' + \frac{\partial x_r}{\partial q_2} q_2' + \dots + \frac{\partial x_r}{\partial q_m} q_m',$$

$$y_r' = \frac{\partial y_r}{\partial q_1} q_1' + \frac{\partial y_r}{\partial q_2} q_2' + \dots + \frac{\partial y_r}{\partial q_m} q_m',$$

$$z_r' = \frac{\partial z_r}{\partial q_1} q_1' + \frac{\partial z_r}{\partial q_2} q_2' + \dots + \frac{\partial z_r}{\partial q_m} q_m';$$

hence

$$\frac{\partial x_r'}{\partial q_s'} = \frac{\partial x_r}{\partial q_s}.$$

Differentiating x_r' by q_s

(13)
$$\frac{\partial x_r'}{\partial q_s} = \frac{\partial^2 x_r}{\partial q_1 \partial q_s} q_1' + \frac{\partial^2 x_r}{\partial q_2 \partial q_s} q_2' + \dots = \frac{d\left(\frac{\partial x_r}{\partial q_s}\right)}{dt}.$$

Inserting these values in

$$\frac{\partial T}{\partial q_s'}, \ \frac{\partial T}{\partial q_s} \text{ in } (12),$$

(14)
$$\frac{\partial T}{\partial q_s'} = \Sigma_r m_r \left\{ x_r' \frac{\partial x_r}{\partial q_s} + y_r' \frac{\partial y_r}{\partial q_s} + z_r' \frac{\partial z_r}{\partial q_s} \right\},$$
$$\frac{\partial T}{\partial q_s} = \Sigma_r m_r \left\{ x_r' \frac{d}{dt}\left(\frac{\partial x_r}{\partial q_s}\right) + y_r' \frac{d}{dt}\left(\frac{\partial y_r}{\partial q_s}\right) + z_r' \frac{d}{dt}\left(\frac{\partial z_r}{\partial q_s}\right) \right\};$$

(15)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_s'}\right) - \frac{\partial T}{\partial q_s} = \Sigma_r m_r \left\{ \frac{d}{dt}\left(x_r' \frac{\partial x_r}{\partial q_s}\right) - x_r' \frac{d}{dt}\left(\frac{\partial x_r}{\partial q_s}\right) + \dots \right\}$$
$$= \Sigma_r m_r \left\{ \frac{dx_r'}{dt}\frac{\partial x_r}{\partial q_s} + \frac{dy_r'}{dt}\frac{\partial y_r}{\partial q_s} + \frac{dz_r'}{dt}\frac{\partial z_r}{\partial q_s} \right\},$$

which, since

$$m_r \frac{dx_r'}{dt} = m_r \frac{d^2 x_r}{dt^2} = X_r,$$

is equal to

$$\Sigma_r \left\{ X_r \frac{\partial x_r}{\partial q_s} + Y_r \frac{\partial y_r}{\partial q_s} + Z_r \frac{\partial z_r}{\partial q_s} \right\} = P_s. \qquad (9)$$

Hence we have proved by direct transformation the expression

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_s'}\right) - \frac{\partial T}{\partial q_s}$$

to be equal to P_s.

The derivative $\dfrac{\partial T}{\partial q_s'}$, which is a homogeneous linear function of the q''s, is generally denoted by

$$p_s = \frac{\partial T}{\partial q_s'}.$$

In the case of rectangular coordinates,

$$T = \tfrac{1}{2}\Sigma m_r (x_r'^2 + y_r'^2 + z_r'^2),$$

$$p_s = \frac{\partial T}{\partial x_s'} = m_s x_s',$$

the x-component of the *momentum* of one of the particles. In general, p_s may be called the generalized component of momentum, belonging to the coordinate q_s and velocity q_s'. The equations of motion may be written

$$\text{(16)} \qquad \frac{dp_s}{dt} = \frac{\partial T}{\partial q_s} + P_s, \quad p_s = \frac{\partial T}{\partial q_s'},$$

or if, as we shall in future do, we denote by P_s simply that part of the impressed force which is not derived from the potential energy, under which are included all non-conservative forces,

$$\text{(17)} \qquad \frac{dp_s}{dt} = \frac{\partial (T - W)}{\partial q_s} + P_s.$$

63. Theorem on Reciprocal Functions. The ordinary notation for partial derivatives of functions of several variables sometimes gives rise to a certain confusion, from the lack of indication of what variables are to be considered as constant during the differentiation. For instance, suppose we have a function F of any number of variables, which for convenience we will divide into two classes, denoting them by the letters

$$x_1, x_2 \ldots x_n, \quad \text{and} \quad z_1, z_2 \ldots z_m.$$

Suppose now we have n functions of these variables, given by the equations

$$\text{(1)} \qquad \begin{aligned} y_1 &= f_1(x_1, x_2 \ldots x_n, z_1, z_2 \ldots z_m) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= f_n(x_1, x_2 \ldots x_n, z_1, z_2 \ldots z_m). \end{aligned}$$

Let us now consider the function

$$G(x_1, x_2 \dots x_n, z_1, z_2 \dots z_m) = F(x_1, x_2 \dots x_n, z_1, z_2 \dots z_m) - \sum_1^n x_s y_s.$$

By means of the equations (1) we may insert the values of the y's in terms of the x's and z's, so that G is explicitly given as a function of the variables $x_1 \dots x_n$ and $z_1 \dots z_m$. On the other hand let us solve the equations (1) for the x's obtaining

$$\begin{aligned} x_1 &= \phi_1(y_1, y_2 \dots y_n, z_1, z_2 \dots z_m) \\ &\dots\dots\dots\dots\dots\dots \\ x_n &= \phi_n(y_1, y_2 \dots y_n, z_1, z_2 \dots z_m), \end{aligned} \tag{2}$$

and by means of the latter let us insert in G the values of the x's in terms of the y's and z's. Let the function in this form, that is, explicitly given as a function of $y_1 \dots y_n$ and $z_1 \dots z_m$ be denoted by $\overline{G}$. Then for all values of z's and of x's and y's compatible with the equations (1) or (2), we have identically

$$G(x_1 \dots x_n, z_1 \dots z_m) \equiv \overline{G}(y_1 \dots y_n, z_1 \dots z_m).$$

Differentiating both G and $\overline{G}$ totally by varying all the variables that occur, we have

$$dG = dF - \sum_1^n y_s dx_s - \sum_1^n x_s dy_s$$

$$= \sum_1^n \frac{\partial F}{\partial x_s} dx_s + \sum_1^m \frac{\partial F}{\partial z_s} dz_s - \sum_1^n y_s dx_s - \sum_1^n x_s dy_s,$$

$$d\overline{G} = \sum_1^n \frac{\partial \overline{G}}{\partial y_s} dy_s + \sum_1^m \frac{\partial \overline{G}}{\partial z_s} dz_s,$$

but as these are identically equal, we get by transposing,

$$0 = \sum_1^n \left(\frac{\partial F}{\partial x_s} - y_s\right) dx_s - \sum_1^n \left(\frac{\partial \overline{G}}{\partial y_s} + x_s\right) dy_s + \sum_1^m \left(\frac{\partial F}{\partial z_s} - \frac{\partial \overline{G}}{\partial z_s}\right) \partial z_s. \tag{3}$$

In this equation there appear $2n+m$ differentials, only $n+m$ of which are independent, in virtue of the equations (1), or their equivalents (2). The equation (3) assumes importance when we define the functions y in a particular way, namely as the partial derivatives of the original function F with regard to the variables x,

$$y_s = \frac{\partial F}{\partial x_s}.$$

Then the coefficients of every dx_s vanish, and since we may take the dy's and dz's arbitrarily, in order for the sum to vanish we must have for every dy_s and dz_s,

$$(4) \qquad \frac{\partial \bar{G}}{\partial y_s} = -x_s, \quad \frac{\partial F}{\partial z_s} = \frac{\partial \bar{G}}{\partial z_s}.$$

The function $-\bar{G}$ is called the reciprocal function to the function F with respect to the variables $x_1 \dots x_n$, for we have the reciprocal relations

$$(5) \qquad y_s = \frac{\partial F}{\partial x_s}, \quad x_s = \frac{\partial(-\bar{G})}{\partial y_s}, \quad \frac{\partial F}{\partial z_s} = -\frac{\partial(-\bar{G})}{\partial z_s},$$

or :—

Two reciprocal functions have the property that the partial derivative of either with respect to any variable of reciprocation contained in it is equal to the corresponding variable replacing the original in the other function, whereas the partial derivative of one function with respect to any variable not of reciprocation is the negative *of the derivative of the other function with respect to the same variable.*

In case the function F is homogeneous of degree κ in the variables of reciprocation

$$x_1, x_2, \dots x_n$$

the theorem becomes more striking, for then, by Euler's theorem

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} \dots + x_n \frac{\partial F}{\partial x_n} = \sum_1^n x_s y_s = \kappa F,$$

$$G = (1-\kappa) F,$$

and the reciprocal function is simply a multiple of the original function.

If the original function is of degree *two*, the reciprocal function is identically *equal* to the original function. We have thus a striking example of the remark made at the beginning of this section, for here the derivative of the function when expressed in one form by a variable z is exactly the *negative* of the derivative by the same variable of the function expressed in the other form. In this form the theorem will be frequently used hereafter. By means of it the equations of motion may be transformed from Lagrange's form to that given them by Hamilton.

64. Hamilton's Transformation. We have seen, § 61 (3), that the kinetic energy is a homogeneous quadratic function of the variables q' representing the velocities,

$$T = \tfrac{1}{2}Q_{11}q_1'^2 + \tfrac{1}{2}Q_{22}q_2'^2 \dots + Q_{12}q_1'q_2' + Q_{13}q_1'q_3' \dots,$$

the coefficients being functions of the coordinates q. If we call the reciprocal function with respect to the q''s, $\overline{T}$, by the last section this is also the kinetic energy, expressed not in terms of the velocities, but of the momenta p. Any p_s is a homogeneous linear function of the q''s, so that solving the equations

$$(1)\qquad \begin{aligned} p_1 &= \frac{\partial T}{\partial q_1'} = Q_{11}q_1' + Q_{12}q_2' \dots + Q_{1n}q_n', \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ p_n &= \frac{\partial T}{\partial q_n'} = Q_{n1}q_1' + Q_{n2}q_2' \dots Q_{nn}q_n'; \end{aligned}$$

for the q''s, every q' is a homogeneous linear function of the p's, and $\overline{T}$ is therefore a homogeneous quadratic function of the momenta p. By virtue of the two properties of the reciprocal function we have for every q_s' (variable of reciprocation), and every q_s (not of reciprocation),

$$(2)\qquad q_s' = \frac{\partial \overline{T}}{\partial p_s}, \quad \frac{\partial \overline{T}}{\partial q_s} = -\frac{\partial T}{\partial q_s},$$

so that Lagrange's equations, § 62 (17), are transformed to

$$(3)\qquad \frac{dp_s}{dt} + \frac{\partial \overline{T}}{\partial q_s} = -\frac{\partial W}{\partial q_s} + P_s, \quad q_s' = \frac{dq_s}{dt} = \frac{\partial \overline{T}}{\partial p_s}.$$

If we put $H = \overline{T} + W$, this is the reciprocal function to the Lagrangian function

$$L = T - W,$$

and the equations take the nearly symmetrical form,

$$(4)\qquad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s} + P_s, \quad \frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}.$$

These are Hamilton's equations of motion.

From these equations we may immediately deduce the integral equation of energy. By cross-multiplication of the above equations, after transposing and summing for all the coordinates, we get

$$(5)\qquad \sum_1^n \frac{\partial H}{\partial p_s}\frac{dp_s}{dt} + \sum_1^n \frac{\partial H}{\partial q_s}\frac{dq_s}{dt} = \sum_1^n P_s \frac{dq_s}{dt}.$$

But H is a function only of the p's and q's, so that the left-hand member is $\frac{dH}{dt}$; and since H is equal to $\overline{T}+W$ it represents the total energy. Also $P_s dq_s$ is the work done by the external impressed force component P_s in the displacement dq_s, so that the right-hand side is the time-rate at which the external forces do work on the system, or the *activity* of the external forces. The equation

$$(6) \qquad \frac{dH}{dt}=\sum_1^n P_s \frac{dq_s}{dt}$$

is accordingly sometimes called the Equation of Activity, while if there are no external forces, but only conservative ones, we have the equation of Conservation of Energy,

$$\frac{dH}{dt}=0, \quad H \equiv \overline{T}+W=\text{const.}$$

A case of frequent occurrence is that where there are non-conservative forces proportional to the first powers of the velocities q', so that any $P_s=-\kappa_s q_s'$. We may then form a function F which is also a homogeneous quadratic function of the velocities

$$(7) \qquad F=\sum_1^n \tfrac{1}{2}\kappa_s q_s'^2, \quad P_s=-\frac{\partial F}{\partial q_s'},$$

and since in this case

$$(8) \qquad -\frac{dH}{dt}=-\sum_1^n P_s q_s'=\sum_1^n q_s' \frac{\partial F}{\partial q_s'}=2F,$$

F represents one-half the time-rate of loss, or dissipation of energy. F is called the Dissipation Function. It was introduced by Lord Rayleigh*, and, like the other function used above, is of use in the theory of electric currents.

65. Transformation of Routh and Helmholtz. We shall in general find Lagrange's form of the equations of motion more convenient than those of Hamilton. An intermediate form, introduced by Routh†, and afterwards by Helmholtz‡, is of great importance.

* *Proceedings London Mathematical Society*, June, 1873.

† Routh. *Stability of a given State of Motion*, p. 61. *Rigid Dynamics*, I. p. 318.

‡ Helmholtz. *Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung.* Borchardt's *Jour. für Math.* Bd. 100, 1886. *Wissensch. Abh.* III. p. 203.

Suppose that instead of reciprocating with regard to all the velocities q' as in Hamilton's transformation, we do so with regard to only a number r of them which we will choose so that they shall have the indices from 1 to r, while the q''s with indices $r+1, \dots n$, remain in the reciprocal function, and with all the coordinates q play the part of the variables z in § 63. Then calling the *negative* of the reciprocal function

$$(1) \qquad \overline{T} = T - \sum_1^r q_s' p_s,$$

we have

$$(2) \qquad \begin{aligned} \frac{\partial \overline{T}}{\partial q_s} &= \frac{\partial T}{\partial q_s}, \quad \text{for } s = 1, 2, \dots n, \\ \frac{\partial \overline{T}}{\partial q_s'} &= \frac{\partial T}{\partial q_s'}, \quad \text{for } s = r+1, \dots n, \end{aligned}$$

and

$$(3) \qquad p_s = \frac{\partial T}{\partial q_s'}, \quad -q_s' = \frac{\partial \overline{T}}{\partial p_s}, \quad \text{for } s = 1, 2, \dots r.$$

Replacing T in Lagrange's equations by $\overline{T}$, we obtain

$$(4) \qquad \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial q_s'}\right) - \frac{\partial \overline{T}}{\partial q_s} = -\frac{\partial W}{\partial q_s} + P_s,$$

so that we may use for the suffixes corresponding to the *uneliminated* velocities Lagrange's equations, using the function,

$$\Phi = \overline{T} - W$$

instead of the Lagrangian function

$$L = T - W,$$

and obtaining

$$(5) \qquad \frac{dp_s}{dt} = \frac{\partial \Phi}{\partial q_s} + P_s, \quad p_s = \frac{\partial \Phi}{\partial q_s'}, \quad \text{for } s = r+1, r+2, \dots n.$$

For the suffixes corresponding to the eliminated velocities we must use the Hamiltonian form of the equations

$$(6) \qquad \begin{aligned} \frac{dp_s}{dt} &= \frac{\partial(\overline{T} - W)}{\partial q_s} + P_s \\ -\frac{dq_s}{dt} &= \frac{\partial(\overline{T} - W)}{\partial p_s}, \quad \text{for } s = 1, 2 \dots r. \end{aligned}$$

If $r = n$, $\overline{T}$ becomes $-T$, and we have the complete Hamiltonian form, § 64 (4).

The function Φ is called by Routh the modified Lagrangian function, and on account of its importance has received from Helmholtz the special name of the Kinetic Potential, by which we shall designate it. (Helmholtz calls $-\Phi$ the kinetic potential*.)

It is to be noticed that the equations for the elimination of the velocities, the equations (2) of § 63 are now, instead of § 64 (1)

$$Q_{11}q_1' + Q_{12}q_2' \dots + Q_{1r}q_r' = p_1 - Q_{1r+1}q'_{r+1} \dots - Q_{1n}q_n',$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$Q_{r1}q_1' + Q_{r2}q_2' \dots + Q_{rr}q_r' = p_r - Q_{rr+1}q'_{r+1} \dots - Q_{rn}q_n',$$

so that the q''s become linear functions of the right-hand sides of these equations and hence of

$$p_1, p_2, \dots p_r, q'_{r+1} \dots q_n',$$

thus $\overline{T}$ becomes a homogeneous quadratic function of

$$p_1 \dots p_r \text{ and } q'_{r+1} \dots q_n',$$

but is not *homogeneous* in either the p's or the q''s alone, on account of terms such as $p_i q_j'$ which are *linear* in *either* the p's or q's.

66. Concealed Motions. A system is said to contain concealed masses, when the coordinates which become known to us by observation do not suffice to define the positions of all the masses of the system. The motions of such bodies are called *concealed* motions. It is often possible to solve the problem of the motions of the visible bodies of a system, even when there are concealed motions going on. For it may be possible to form the kinetic potential of the system for the visible motions, not containing the concealed coordinates, and in this case we may use Lagrange's equations, as in the preceding article, for all visible coordinates, while the coordinates of the concealed masses may be ignored. Such problems are incomplete, inasmuch as they tell us nothing of the concealed motions, but very often we are concerned only with the visible motions. Such concealed motions enable us to explain the forces acting between visible systems by means of concealed motions of systems connected with them.

As an example of a concealed motion let us take the case of a closed box containing a gyrostat, or fly-wheel, pivoted on an axis

* Helmholtz's notation is quite different from that here employed.

rigidly fastened to the box. If the box be at rest, there is nothing on the outside to lead one to suspect the presence of the fly-wheel, but if the box be moved about the reactions developed will be very different if the concealed wheel is in rotation or not. In the former case the least experimenting will render us sure of the existence of a concealed motion. To convince oneself of the truth of this statement it is necessary only to take a toy gyroscope in one's hand and turn it about.

67. Cyclic Motions. In certain cases some of the coordinates do not appear in the expression for the kinetic energy, although their velocities may. Such are termed by Helmholtz* *cyclic* coordinates, and we shall distinguish them by a bar. The example just given of the gyrostat is such a case, for the angular coordinate fixing the rotation of the fly-wheel does not appear, but only its derivative, the angular velocity. Further examples are furnished by the case of a heavy belt running over pulleys, or by the case of a fluid circulating in an endless tube. The coordinate expressing how far a point on the belt or in the fluid has travelled does not enter, but its velocity does. The condition for a cyclic coordinate being $\frac{\partial T}{\partial \bar{q}} = 0$ we have for the forces maintaining cyclic motions [§ 61 (7)],

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{q}_s'}\right) = \bar{P}_s.$$

If the forces of the cyclic motions vanish we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{q}_s'}\right) = 0$$

or, integrating,

$$\frac{\partial T}{\partial \bar{q}_s'} = \bar{p}_s = c_s.$$

In this case we may with advantage apply Routh's transformation in the case of the cyclic velocities. The equations for the elimination are now

$$Q_{11}\bar{q}_1' + Q_{12}\bar{q}_2' \ldots + Q_{1r}\bar{q}_r' = c_1 - Q_{1\,r+1}q'_{r+1} \ldots - Q_{1n}q_n',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Q_{r1}\bar{q}_1' + Q_{r2}\bar{q}_2' \ldots + Q_{rr}\bar{q}_r' = c_r - Q_{r\,r+1}q'_{r+1} \ldots - Q_{rn}q_n'.$$

* Helmholtz, *Studien zur Statik monocyklischer Systeme.* Borchardt's *Journ. für Math.*, Bd. 97. *Wiss. Abh.* III. p. 128.

Let the solutions of these for the $\bar{q}'$'s be

$$\bar{q}_1' = R_{11}(c_1 - \dots - Q_{1n}q_n') + R_{12}(c_2 - \dots - Q_{2n}q_n') \dots + R_{1r}(c_r \dots - Q_{rn}q_n'),$$

....

....

$$\bar{q}_r' = R_{r1}(c_1 - \dots - Q_{1n}q_n') + R_{r2}(c_2 - \dots - Q_{2n}q_n') \dots + R_{rr}(c_r \dots - Q_{rn}q_n').$$

The R's being the quotients of the various subdeterminants of the determinant

$$\begin{vmatrix} Q_{11}, & Q_{12}, & \dots & Q_{1r} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ Q_{r1}, & Q_{r2}, & \dots & Q_{rr} \end{vmatrix}$$

by the determinant itself, are functions of the coordinates only, and since by hypothesis the function T did not contain the cyclic coordinates, the R's are functions of only the non-cyclic coordinates. The kinetic potential consequently is a function only of the non-cyclic coordinates and velocities, but on account of the presence of the constants c_s, it is not a *homogeneous* function of the velocities, but contains a linear function of them, as was remarked in § 65. Cases in which the kinetic potential contains a linear function of the velocities may thus be considered as cases with concealed motions. A case of this nature will be found in considering the mutual actions of magnets and electric currents. Physically the difference between the two cases is that while if Φ contains only terms of the second degree in the velocities, if every velocity is reversed the kinetic potential is unchanged, and hence the motion may be reversed without change of circumstances, but if on the other hand there are terms of the first degree in the velocities, the motion cannot be reversed unless the concealed motions are reversed as well.

As an example we will take the case of a gyrostat hung in gimbals. Let the outer ring of the gimbals A, Fig. 27, be pivoted on a vertical axis, and let the angle made by the plane of the ring with a fixed vertical plane be ψ. Let the inner ring B be pivoted on a horizontal axis, and let its plane make an angle θ with the plane of the outer ring. The gyrostat is pivoted on an axis at right angles with the last, and let a fixed radius of the gyrostat make an angle ϕ with the plane of the inner ring. It is shown in the theory of the dynamics of a rigid body that the energy of a body revolving about an axis is one-half the product

of a constant called the moment of inertia of the body multiplied by the square of its angular velocity, and also that if we find the

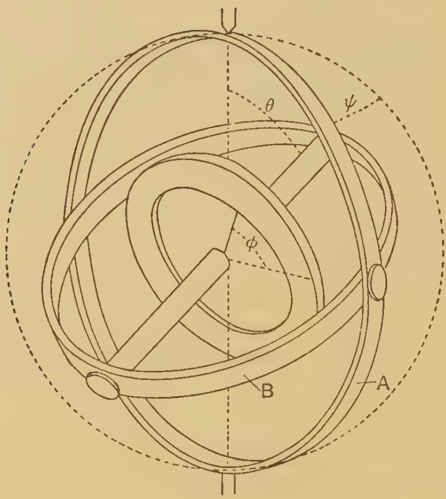

FIG. 27.

angular velocities about three mutually perpendicular axes of symmetry the energy may be found by adding the three parts obtained for the energy of rotation about the three axes. We will resolve the motions of the gyrostat into three angular velocities, about the axis of the top, the axis of the inner ring, and an axis perpendicular to both. About the axis of the gyrostat the angular velocity is $\frac{d\phi}{dt} = \phi'$, but there is also the angular velocity $\frac{d\psi}{dt} = \psi'$ about the vertical axis, which has the component $\psi' \cos\theta$ about the axis of the gyrostat. The velocity about the second axis is $\frac{d\theta}{dt} = \theta'$, and about the third is the other component of the velocity about the vertical, $\psi' \sin\theta$. If A is the moment of inertia of the gyrostat about its own axis, B that about either of the other two, we have for the kinetic energy

$$T = \tfrac{1}{2}\left[A(\phi' + \psi'\cos\theta)^2 + B(\theta'^2 + \psi'^2 \sin^2\theta)\right],$$

so that ϕ and ψ are cyclic coordinates. For the components of the forces tending to increase ψ, θ, ϕ,

$$P_\psi = \frac{d}{dt}\left(\frac{\partial T}{\partial \psi'}\right) = \frac{d}{dt}\left[A\,(\phi' + \psi' \cos\theta)\cos\theta + B\psi' \sin^2\theta\right],$$

$$P_\theta = \frac{d}{dt}\left(\frac{\partial T}{\partial \theta'}\right) - \frac{\partial T}{\partial \theta} = \frac{d}{dt}\,[B\theta']$$

$$+ A\,(\phi' + \psi' \cos\theta)\sin\theta - B\psi'^2 \sin\theta\cos\theta,$$

$$P_\phi = \frac{d}{dt}\left(\frac{\partial T}{\partial \phi'}\right) = \frac{d}{dt}\left[A\,(\phi' + \psi' \cos\theta)\right].$$

If there is no force tending to change the rotation of the gyrostat in its ring

$$P_\phi = 0, \quad A\,(\phi' + \psi' \cos\theta) = c,$$

and eliminating ϕ' by means of this equation,

$$\phi' = \frac{c}{A} - \psi' \cos\theta,$$

$$\Phi = T - c\phi' = -\frac{1}{2}\frac{c^2}{A} + \frac{1}{2}B\,(\theta'^2 + \psi'^2 \sin^2\theta) + c\psi' \cos\theta.$$

the last term containing ψ' in the first power. Using this form of Φ to determine the forces, we obtain

$$P_\psi = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \psi'}\right) = \frac{d}{dt}(B\psi' \sin^2\theta + c\cos\theta)$$

$$P_\theta = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \theta'}\right) - \frac{\partial \Phi}{\partial \theta} = \frac{d}{dt}(B\theta') - B\psi'^2 \sin\theta\cos\theta + c\psi' \sin\theta.$$

The influence of the cyclic motion may be most simply shown if the vertical ring be held fixed. Then $\psi = \text{const.}$, and $\psi' = 0$,

$$P_\psi = -c\sin\theta\,\frac{d\theta}{dt},$$

$$P_\theta = B\,\frac{d^2\theta}{dt^2}.$$

Spinning the inner ring about the horizontal axis requires the same force whether the cyclic motion exists or not, whereas a force is developed tending to make the vertical ring revolve about its axis, which must be balanced by the force $-c\sin\theta\,\frac{d\theta}{dt}$. This force at once shows that there is a concealed motion, even if the disposition of the concealed parts be unknown. This is exemplified in the gyroscopic pendulum, which is simply a pendulum with two degrees of freedom, containing a gyrostat whose axis is

rigidly fixed in the axis of the pendulum. An ordinary pendulum set vibrating in a plane continues to vibrate in a plane, with a periodic reversal of its motion. The gyroscopic pendulum on the other hand describes a curious looped surface, never remaining in a plane nor returning on its course. This example is worked out in Thomson and Tait's *Natural Philosophy*, § 319, Example (D).

68. **Cyclic Systems.** A system in which the kinetic energy is represented with sufficient approximation by a homogeneous quadratic function of its cyclic velocities is called a Cyclic System. Of course the rigid expression of the kinetic energy contains the velocities of every coordinate of the system, cyclic or not, for no mass can be moved without adding a certain amount of kinetic energy. Still if certain of the coordinates change so slowly that their velocities may be neglected in comparison with the velocities of the cyclic coordinates, the approximate condition will be fulfilled. These coordinates define the position of the cyclic systems, and may be called the *positional coordinates* or *parameters* of the system. In the case of the gyrostat the two coordinates of the gimbal rings may be taken for the positional coordinates, while the cyclic coordinate determines the rotation of the gyrostat. In the case of a liquid circulating through an endless rubber tube, the positional co-ordinates would specify the shape and position of the tube. The positional coordinates will be distinguished from the cyclic coordinates by not being marked with a bar. The analytical conditions for a cyclic system will accordingly be, for all coordinates, either

$$(1) \qquad \frac{\partial T}{\partial \bar{q}_s} = 0 \quad \text{or} \quad \frac{\partial T}{\partial q_s'} = p_s = 0,$$

or if we use the Hamiltonian form of T obtained by replacing the velocities by the momenta, which we shall denote by T_p, since the non-cyclic momenta vanish

$$(2) \qquad \frac{\partial T_p}{\partial p_s} = 0, \quad \frac{\partial T_p}{\partial \bar{q}_s} = -\frac{\partial T}{\partial \bar{q}_s} = 0.$$

We accordingly have for the external forces tending to increase the positional coordinates [see § 62, (17)],

$$(3) \qquad P_s = -\frac{\partial (T - W)}{\partial q_s} = \frac{\partial (T_p + W)}{\partial q_s},$$

and for the cyclic coordinates

$$(4) \qquad \overline{P}_s = \frac{d}{dt}\left(\frac{\partial T}{\partial \overline{q}_s'}\right) = \frac{d\overline{p}_s}{dt}.$$

A motion in which there are no forces tending to change the cyclic coordinates is called an *adiabatic* motion, since in it no energy enters or leaves the system through the *cyclic* coordinates. (It may do so through the positional coordinates.) Accordingly in such a motion the cyclic *momenta* remain constant. The case of the gyrostat worked out above was such a motion.

In adiabatic motions the cyclic *velocities* do not generally remain constant. In the above example, for instance, the cyclic velocity ϕ' was given by

$$\phi' = \frac{c}{A} - \psi' \cos\theta.$$

A motion in which the cyclic *velocities* remain constant is called *isocyclic*.

In such a motion the cyclic momenta do not generally remain constant, but forces have to be applied.

If the motion is isocyclic, the only variables appearing in T are the q's, the positional coordinates. The positional forces, (3), are then derivable from a force-function $W - T$*, so that even if the system possessed no potential energy, it would appear to possess an amount of potential energy $-T$. If the motion on the other hand is adiabatic, the energy in the form T_p again contains only the coordinates q_s, and the positional forces are now derivable from the force-function $T_p + W$, so that in this case a system without potential energy would appear to contain the amount of potential energy $+T_p$. In this manner we are enabled to explain potential energy as kinetic energy of concealed cyclic motions, thus adding materially to our conceptions of the nature of force. For it is to be noted that kinetic energy is an entity depending only on the property of inertia, which is possessed by all bodies, while potential energy is a term only employed to disguise our ignorance of the nature of force. Accordingly when we are able to proceed to an explanation of a static force by means of kinetic

* The reason for the appearance of W with the *positive* sign is that, as explained in § 62, end, P_s denotes the *external* impressed forces, which in the case of equilibrium, are equal and *opposite* to the internal forces given by W.

phenomena, we have made a distinct advance in our knowledge of the subject. A striking example is furnished by the kinetic theory of gases, by means of which we are enabled to pass from the bare statement that all gases press against their confining vessels to the statement that this pressure is due to the impact of the molecules of the gas against the walls of the vessel.

69. Properties of Cyclic Systems. Reciprocal Relations. Since by the properties of the kinetic energy we have three different kinds of quantities represented by partial derivatives of one or the other of two functions,

$$(1)\ P_s = -\frac{\partial T}{\partial q_s}, \quad (2)\ \bar{p}_s = \frac{\partial T}{\partial \bar{q}_s'}, \quad (3)\ P_s = \frac{\partial T_p}{\partial q_s}, \quad (4)\ \bar{q}_s' = \frac{\partial T_p}{\partial \bar{p}_s},$$

applying the principle that a derivative by two variables is independent of the order of the differentiations we obtain six reciprocal theorems. We shall throughout suppose that there is no potential energy.

I *a*. In an adiabatic motion if an increase in one positional coordinate q_r causes an increase in the impressed force P_s belonging to another positional coordinate q_s at a certain rate, then an increase in the positional coordinate q_s causes an increase in the impressed force P_r at the same rate. For

$$(5) \qquad \frac{\partial P_s}{\partial q_r} = \frac{\partial^2 T_p}{\partial q_r \partial q_s} = \frac{\partial P_r}{\partial q_s}.$$

I *b*. In an isocyclic motion we have the same property as above. For

$$(6) \qquad \frac{\partial P_s}{\partial q_r} = -\frac{\partial^2 T}{\partial q_r \partial q_s} = \frac{\partial P_r}{\partial q_s}.$$

II *a*. If in any motion an increase of any cyclic momentum $\bar{p}_r$, the positional coordinates being unchanged, causes an increase in a cyclic velocity $\bar{q}_s'$ at a certain rate, then an increase in the momentum $\bar{p}_s$, the positional coordinates being unchanged, causes an increase in the velocity $\bar{q}_r'$ at the same rate. For

$$(7) \qquad \frac{\partial \bar{q}_s'}{\partial \bar{p}_r} = \frac{\partial^2 T_p}{\partial \bar{p}_r \partial \bar{p}_s} = \frac{\partial \bar{q}_r}{\partial \bar{p}_s}.$$

II *b*. If in any motion an increase in any cyclic velocity $\bar{q}_r'$, the positional coordinates being unchanged, causes an increase in a cyclic momentum $\bar{p}_s$, then an increase in the velocity $\bar{q}_s'$ causes

an increase in the momentum $\bar{p}_r$ at the same rate. For

(8) $$\frac{\partial \bar{p}_s}{\partial \bar{q}_r'} = \frac{\partial^2 T}{\partial \bar{q}_r' \partial \bar{q}_s'} = \frac{\partial \bar{p}_r}{\partial \bar{q}_s'}.$$

III *a.* If an increase in one of the cyclic momenta $\bar{p}_r$, the positional coordinates being unchanged, causes an increase in the impressed force P_s necessary to be applied to one of the positional coordinates q_s (in order to prevent its changing), then an adiabatic increase of the positional coordinate q_s will cause the cyclic velocity $\bar{q}_r'$ to increase at the same rate. For

(9) $$\frac{\partial P_s}{\partial \bar{p}_r} = \frac{\partial^2 T_p}{\partial \bar{p}_r \partial q_s} = \frac{\partial \bar{q}_r'}{\partial q_s}.$$

III *b.* If an increase in one of the cyclic velocities $\bar{q}_r'$, the positional coordinates being unchanged, causes an increase in the impressed force P_s necessary to be applied to one of the positional coordinates q_s (in order to prevent its changing), then an isocyclic increase of the positional coordinate q_s will cause the cyclic momentum $\bar{p}_r$ to *decrease* at the same rate. For

(10) $$\frac{\partial P_s}{\partial \bar{q}_r'} = -\frac{\partial^2 T}{\partial \bar{q}_r' \partial q_s} = -\frac{\partial \bar{p}_r}{\partial q_s}.$$

70. Work done by the cyclic and positional forces.

I. In an isocyclic motion, the work done *by* the cyclic forces is double the work done by the system *against* the positional forces. In such motions the energy of the system accordingly increases by one-half the work done by the cyclic forces, the other half being given out against the positional forces. For if we use the energy in the form

$$T = \tfrac{1}{2}\Sigma_s \bar{q}_s' \bar{p}_s,$$

we have in any change

(1) $$\delta T = \tfrac{1}{2}\Sigma_s (\bar{q}_s' \delta \bar{p}_s + \bar{p}_s \delta \bar{q}_s'),$$

and in an isocyclic change, every $\delta \bar{q}_s'$ vanishing,

(2) $$\delta T = \tfrac{1}{2}\Sigma_s \bar{q}_s' \delta p_s.$$

But since

(3) $$\frac{d\bar{p}_s}{dt} = P_s, \quad \delta \bar{p}_s = \bar{P}_s \delta t, \text{ and since } \bar{q}_s' = \frac{d\bar{q}_s}{dt}, \quad \bar{q}_s' \delta t = \delta \bar{q}_s,$$

and the above expression for the gain of energy becomes

(4) $$\delta T = \tfrac{1}{2}\Sigma_s \bar{q}_s' \bar{P}_s \delta t = \tfrac{1}{2}\Sigma \bar{P}_s \delta \bar{q}_s.$$

But the work done by the cyclic forces is

(5) $$\delta\bar{A} = \Sigma_s \bar{P}_s \delta\bar{q}_s = 2\delta T.$$

Hence the last part of the theorem is proved. Again, in any motion

(6) $$\delta T = \Sigma_s \frac{\partial T}{\partial \bar{q}_s'} \delta\bar{q}_s' + \Sigma_s \frac{\partial T}{\partial q_s} \delta q_s,$$

and in an isocyclic motion

(7) $$\delta T = \Sigma_s \frac{\partial T}{\partial q_s} \delta q_s.$$

But since the work of the positional forces is

(8) $$\delta A = \Sigma_s P_s \delta q_s = -\Sigma \frac{\partial T}{\partial q_s} \delta q_s = -\delta T,$$

the first part of the proposition is also proved.

II. In an adiabatic motion, the cyclic velocities will in general be changed.

Then they change in such a way that the positional forces caused by the change of cyclic velocities oppose the motion, that is, do a positive amount of work. For since for any positional force

$$P_s = -\frac{\partial T}{\partial q_s},$$

the change due to the motion is

$$\delta P_s = -\frac{\partial \delta T}{\partial q_s} = -\Sigma_r \frac{\partial^2 T}{\partial q_s \partial q_r} \delta q_r - \Sigma_r \frac{\partial^2 T}{\partial q_s \partial \bar{q}'_r} \delta\bar{q}_r'.$$

Of this the part due to the change in the cyclic velocities is

$$\delta_{\bar{q}'} P_s = -\Sigma_r \frac{\partial^2 T}{\partial q_s \partial \bar{q}_r'} \delta\bar{q}_r' = -\Sigma_r \frac{\partial \bar{p}_r}{\partial q_s} \delta\bar{q}_r',$$

and the work done by these forces is

$$\delta_{\bar{q}'} A = \Sigma_s \delta_{\bar{q}'} P_s \delta q_s = -\Sigma_s \Sigma_r \frac{\partial \bar{p}_r}{\partial q_s} \delta q_s \delta\bar{q}_r'$$

Now we have for any motion

$$\delta\bar{p}_r = \Sigma_s \frac{\partial \bar{p}_r}{\partial q_s} \delta q_s + \Sigma_s \frac{\partial \bar{p}_r}{\partial \bar{q}_s'} \delta\bar{q}_s',$$

and in an adiabatic motion this is zero, so that

$$\Sigma_s \frac{\partial \bar{p}_r}{\partial q_s} \delta q_s = -\Sigma_s \frac{\partial \bar{p}_r}{\partial \bar{q}_s'} \delta\bar{q}_s'.$$

Substituting this in the sum with respect to s in $\delta_{\bar{q}'} A$ we get

$$\delta_{\bar{q}'} A = \Sigma_s \Sigma_r \frac{\partial \bar{p}_r}{\partial \bar{q}_s'} \delta \bar{q}_s' \delta \bar{q}_r' = \Sigma_s \Sigma_r Q_{rs} \delta \bar{q}_s' \delta \bar{q}_r'.$$

But this expression represents (§ 61 (3)) twice the energy of a possible motion in which the velocities would be $\delta \bar{q}_s'$, and must therefore be positive for all values of $\delta \bar{q}_s'$, $\delta \bar{q}_r'$.

Accordingly $$\delta_{\bar{q}'} A > 0.$$

The interpretation of this theorem for electrodynamics is known as Lenz's Law*.

71. Examples of Cyclic Systems. The expression for the kinetic energy of the gyrostat worked out in § 67 shows that the system fulfils the conditions for a cyclic system if the velocity θ' is small enough to be neglected in comparison with the other velocities. The forces acting have been already found, and we can easily verify the theorems of the last two articles for this case.

A very simple case of a cyclic system is that of a mass m sliding on a horizontal rod, revolving about a vertical axis. Let us consider the mass concentrated at a single point m at a distance r from the axis. Let the angle made by the rod with a fixed horizontal line be ϕ, then the velocity perpendicular to the rod is $r\phi'$. The velocity along the rod being r', the kinetic energy of the body m is

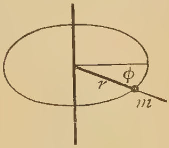

FIG. 28.

$$T = \tfrac{1}{2} m (r^2 \phi'^2 + r'^2).$$

If we suppose the motion along the rod to be so slow that we may neglect r'^2

$$T = \tfrac{1}{2} m r^2 \phi'^2,$$

and the system is cyclic, r is the positional, ϕ the cyclic coordinate.

A system containing a single cyclic coordinate is called by Helmholtz a monocyclic system. We have for the momenta

$$p_r = \frac{\partial T}{\partial r'} = 0, \quad \bar{p}_\phi = \frac{\partial T}{\partial \phi'} = m r^2 \phi',$$

* These Theorems are all given by Hertz, *Principien der Mechanik*, §§ 568–583.

and introducing these instead of the velocities

$$T_p = \frac{1}{2mr^2}\bar{p}_\phi^2.$$

We have for the positional force

$$P_r = -\frac{\partial T}{\partial r} = -mr\phi'^2 = \frac{\partial T_p}{\partial r} = -\frac{1}{mr^3}\bar{p}_\phi^2.$$

This being negative denotes that a force P_r toward the axis must be impressed on the mass m in order to maintain the cyclic state. This may be accomplished by means of a geometrical constraint, or by means of a spring. The force or reaction $-P_r$ which the mass m exerts in the direction from the axis in virtue of the rotation is called the centrifugal force. We see that if the motion is isocyclic, the positional force increases with r, while if it is adiabatic, it decreases when r increases. The verification of the theorems of § 69 is obvious. The cyclic force

$$\bar{P}_\phi = \frac{d\bar{p}_\phi}{dt} = m\frac{d}{dt}(r^2\phi')$$

vanishes when the rotation is uniform, and the radius constant. If, the motion being isocyclic, that is, one of uniform angular velocity, the body moves farther from the axis, $\bar{P}_\phi$, the cyclic force is positive, that is, unless a positive force $\bar{P}_\phi$ is applied, the angular velocity will diminish. In moving out from r_1 to r_2 work will be done against the positional force P_r of amount

$$-A = -\int_{r_1}^{r_2} P_r dr = m\phi'^2\int_{r_1}^{r_2} r\,dr = \frac{m\phi'^2}{2}(r_2^2 - r_1^2),$$

while the energy *increases* by the same amount.

Thus the first theorem of § 70 is verified. If the motion is adiabatic,

$$\bar{p}_\phi = mr^2\phi' = c.$$

If the body move from the axis, ϕ' will accordingly decrease. The change in P_r due to a displacement δr is

$$\delta_r P_r = \frac{3}{mr^4}\bar{p}_\phi^2\,\delta r,$$

which, being of the same sign as δr, does a positive amount of work in the displacement, illustrating the second theorem of § 70.

Dicyclic Systems. The preceding example will suffice as a mechanical model to illustrate the phenomenon of self-induction of an electric current (Chapter XII). To illustrate mutual induction we must have at least two cyclic coordinates. Such models have been proposed by Maxwell, Lord Rayleigh*, Boltzmann, J. J. Thomson†, and the author‡. In the model of J. J. Thomson, there are two carriages of mass m_1 and m_2 sliding on parallel rails, Fig. 29, their distances from a fixed line perpendicular to the rails being x_1 and x_2. Sliding in swivels on the carriages is a bar, on which is a third mass m_3. We shall suppose that this mass is movable along the bar, and is at a distance y from the line midway between the rails, y being positive when m_3 is nearer m_2. Then, if d is the distance between the rails,

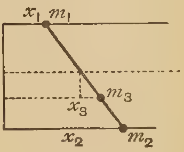

FIG. 29.

$$x_3 = \frac{x_1 + x_2}{2} + (x_2 - x_1)\frac{y}{d},$$

and the kinetic energy is, if we may neglect y' in comparison with x_1', x_2',

$$\begin{aligned} T &= \tfrac{1}{2}\{m_1 x_1'^2 + m_2 x_2'^2 + m_3(x_3'^2 + y'^2)\} \\ &= \frac{1}{2}\left[x_1'^2\left\{m_1 + m_3\left(\frac{1}{4} - \frac{y}{d} + \frac{y^2}{d^2}\right)\right\} + x_2'^2\left\{m_2 + m_3\left(\frac{1}{4} + \frac{y}{d} + \frac{y^2}{d^2}\right)\right\}\right. \\ &\qquad \left. + m_3 x_1' x_2'\left(\frac{1}{2} - \frac{2y^2}{d^2}\right)\right]. \end{aligned}$$

The system is cyclic, y being the positional, x_1 and x_2 the cyclic coordinates. The positional force

$$P_y = -\frac{\partial T}{\partial y} = -m_3\left\{x_1'^2\left(\frac{y}{d^2} - \frac{1}{2d}\right) + x_2'^2\left(\frac{y}{d^2} + \frac{1}{2d}\right) - 2x_1' x_2' \frac{y}{d^2}\right\}$$

vanishes if $x_1' = x_2'$. The cyclic forces are

$$P_{x_1} = \frac{d}{dt}\left[x_1'\left\{m_1 + m_3\left(\frac{1}{4} - \frac{y}{d} + \frac{y^2}{d^2}\right) + x_2' m_3\left(\frac{1}{4} - \frac{y^2}{d^2}\right)\right\}\right],$$

$$P_{x_2} = \frac{d}{dt}\left[x_1' m_3\left(\frac{1}{4} - \frac{y^2}{d^2}\right) + x_2'\left\{m_2 + m_3\left(\frac{1}{4} + \frac{y}{d} + \frac{y^2}{d^2}\right)\right\}\right].$$

* *Phil. Mag.*, July 1890, p. 30.

† *Elements of Mathematical Theory of Electricity and Magnetism*, p. 385.

‡ *Science*, Dec. 13, 1895.

Suppose that the coordinate y and the velocity x_2' are constant. If now x_1' is increased, say by m_1 starting from rest and moving to the right by the application of a positive force P_{x_1}, then P_{x_2} is positive if $|y| < |d/2|$ and m_3 is within the rails,—in other words, unless a force to the right is impressed on m_2 also, x_2' will diminish, and if x_2' was also zero, m_2 will move to the left.

The force P_{x_2} must be greater the smaller $|y|$. This is the analogue of the induction of currents. Similar effects may be produced by moving m_3 along the rod, instead of applying a force to m_1 or m_2.

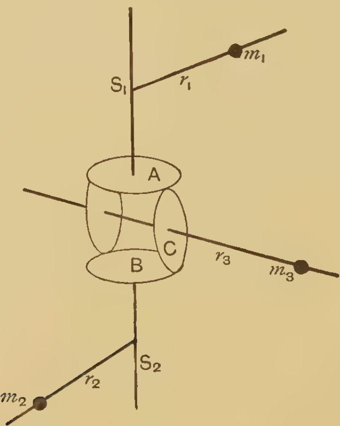

FIG. 30.

Maxwell's model, which undoubtedly suggested Thomson's, differs from it only in having motion of rotation instead of revolution, so that there is no limit to the possible difference in the coordinates x_1, x_2. The independent masses are represented by the moments of inertia of masses m_1, m_2 carried by two shafts S_1, S_2, Fig. 30, each of which carries a bevel-gear wheel A, B at one end. Engaging these is a pair of bevel-gears C running loosely upon a third perpendicular shaft, carrying the intermediate mass, m_3.

If all the bevel-gears are of the same diameter, and ϕ_1, ϕ_2, ϕ_3 are the angles made by the three horizontal rods with a fixed horizontal line, then it is evident, since the velocity of the centre of the wheel C is a mean between the velocities of its highest and lowest points, which have respectively the velocities of the rims of the wheels A and B, that

$$\phi_3' = \tfrac{1}{2}(\phi_1' + \phi_2').$$

Consequently the kinetic energy of the system consisting of the three masses m_1, m_2, m_3 at distances from the axis r_1, r_2, r_3 is

$$T = \tfrac{1}{2}\{m_1 r_1^2 \phi_1'^2 + m_2 r_2^2 \phi_2'^2 + m_3 r_3^2 \phi_3'^2\}$$

$$= \frac{1}{2}\left\{\left(m_1 r_1^2 + \frac{m_3 r_3^2}{4}\right)\phi_1'^2 + \left(m_2 r_2^2 + \frac{m_3 r_3^2}{4}\right) + \frac{m_3 r_3^2}{2}\phi_1'\phi_2'\right\}$$

if the velocities r' can be neglected. The system is cyclic, the r's being positional, the ϕ's being cyclic coordinates. In order to make the model a more complete representation of two electric currents, Boltzmann modified it so to have between the coordinates r_1, r_2, r_3 the relation

$$r_1^2 + r_3^2 = y_1^2, \quad r_2^2 + r_3^2 = y_2^2,$$

where y_1, y_2 are two independent parameters. The two masses m_1, m_2 are chosen equal, being made one-fourth of m_3.

The expression for the energy then becomes

$$T = m\{\tfrac{1}{2}y_1^2\phi_1'^2 + \tfrac{1}{2}y_2^2\phi_2'^2 + r_3^2\phi_1'\phi_2'\},$$

and we may independently change either of the three coefficients.

The Pythagorean theorem suggests a geometrical means of imposing the above constraints. To each of the masses m is attached a string, which runs along the rod to the axis of rotation, where, after passing round a pulley it is carried vertically downward to be attached to the following device (Fig. 31). A

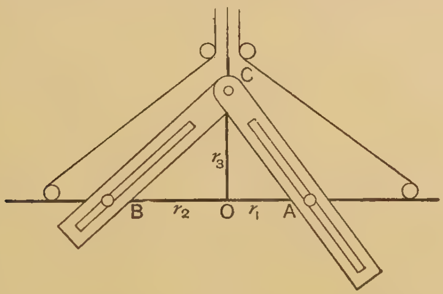

FIG. 31.

pair of rods are articulated at C, the point of articulation being made to slide in a vertical line CO. The string from m_3 is fastened to the point C. Sliding on a horizontal line AB and in slots in the rods AC, BC, are the points of attachment of the strings from m_1 and m_2, which are then carried outward and upward

over pulleys. The lengths of the strings being chosen so that m_3 is at the axis when the rods are horizontal, m_1 and m_2 when the rods are vertical, we must have, if $AC = y_1$, $BC = y_2$,

$$r_1^2 + r_3^2 = y_1^2, \quad r_2^2 + r_3^2 = y_2^2.$$

For the actual construction of the model, the reader is referred to Boltzmann, *Vorlesungen über die Maxwell'sche Theorie der Electricität und des Lichtes.*

By means of these models all the properties of Cyclic Systems may be illustrated, and all the phenomena of induction of currents imitated, as will be described in Chapter XII.

72. Hamilton's Principle the most general dynamical principle. We have seen in this chapter how by means of Hamilton's Principle we may deduce the general equations of motion, and from these the principle of Conservation of Energy. As Hamilton's Principle holds whether the system is conservative or not, it is more general than the principle of Conservation of Energy, which it includes. The principle of energy is not sufficient to deduce the equations of motion. If we know the Lagrangian function we can at once form the equations of motion, and without forming them we may find the energy. For we have

$$L = T - W,$$

$$E = T + W.$$

Accordingly

$$E = 2T - L = \Sigma_s q_s' \frac{\partial T}{\partial q_s'} - L = \Sigma_s q_s' \frac{\partial L}{\partial q_s'} - L,$$

so that the energy is given in terms of L and its partial derivatives. If on the other hand the energy is given as a function of the coordinates and velocities, the Lagrangian function must be found by integrating the above partial differential equation, involving an arbitrary function. In fact if F be a homogeneous linear function of the velocities, the above equation will be satisfied not only by L but also by $L + F$. For, F being homogeneous,

$$F = \Sigma_s q_s' \frac{\partial F}{\partial q_s'}.$$

Consequently a knowledge of the energy is not sufficient to find the motion, while a knowledge of the Lagrangian function or kinetic potential is.

In case we wish to ignore some of the coordinates we may modify the statement of Hamilton's Principle by the use of the modified Lagrangian function and put

$$\delta \int (\Phi + \Sigma_s P_s q_s)\, dt = 0,$$

where we suppose only those coordinates which are not ignored are varied.

CHAPTER IV.

NEWTONIAN POTENTIAL FUNCTION.

73. Definition and fundamental properties of Potential. We have seen in § 59, (29), (31), that if we have any number of material particles m repelling according to the Newtonian Law of the inverse square of the distance, the function

$$U_s = -\left\{\frac{m_1 m_s}{r_1} + \frac{m_2 m_s}{r_2} + \ldots\ldots + \frac{m_n m_s}{r_n}\right\},$$

where $r_1, r_2 \ldots\ldots r_n$ are the distances from the repelling points, is the force-function for all the forces acting upon the particle m_s. If we put the mass m_s equal to unity the function

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2} \ldots\ldots \frac{m_n}{r_n} \tag{1}$$

is called the *potential function* of the field of force due to the repulsions of the particles $m_1, m_2 \ldots\ldots m_n$, and its *negative* vector parameter is the *strength* of the field, that is, the force experienced by unit of mass concentrated at the point in question. Since any term $\frac{m_r}{r_r}$ possesses the same properties as the function $\frac{1}{r}$, § 39, we have for every term, for points where r is not equal to zero, $\Delta\left(\frac{1}{r}\right) = 0$, and consequently

$$\Delta V = m_1 \Delta\left(\frac{1}{r_1}\right) + m_2 \Delta\left(\frac{1}{r_2}\right) \ldots\ldots + m_n \Delta\left(\frac{1}{r_n}\right) = 0. \tag{2}$$

74. Potential of Continuous Distribution. Suppose now that the repelling masses, instead of being in discrete points, form a continuously extended body K.

Let the limit of the ratio of the mass to the volume of any infinitely small part be $\rho = \lim\limits_{\Delta\tau=0} \frac{\Delta m}{\Delta\tau}$, which is called the *density*.

Let the coordinates of a point in the repelling or attracting* body be a, b, c.

The potential at any point P, x, y, z, due to the mass dm at Q, a, b, c, is

$$dV = \frac{dm}{r},$$

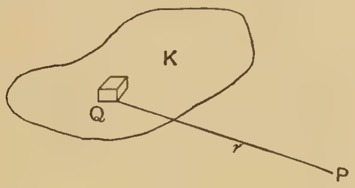

FIG. 32.

where r is the distance of the point x, y, z from the repelling point at a, b, c. The whole potential at x, y, z is the sum of that due to all parts of the attracting body, or the volume integral

$$(3) \qquad V = \iiint_K \frac{dm}{r}.$$

Now we have

$$dm = \rho d\tau,$$

or in rectangular coordinates $d\tau = da\,db\,dc$,

$$dm = \rho\, da\, db\, dc.$$

If the body is not homogeneous, ρ is different in different parts of the body K, and is a function of a, b, c, continuous or discontinuous (e.g. a hole would cause a discontinuity). Since

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2},$$

$$(4) \qquad V = \iiint_K \frac{dm}{r} = \iiint_K \frac{\rho\, da\, db\, dc}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}.$$

For every point x, y, z, V has a single, definite value. It is accordingly a uniform function of the point P, x, y, z.

It may be differentiated in any direction, we may find its level surfaces, its first differential parameter, whose negative is equal to the whole action of K on a point of unit mass, and the *lines of force*, normal to the level, or *equipotential* surfaces.

* In order to save words, and to conform to ordinary usage, we shall say simply *attracting*, for a negative repulsion is an attraction.

If for any point x, y, z outside K, r_1 is the shortest distance to any point of K and r_2 the greatest distance, we have for any point in K

$$r_2 > r > r_1,$$

$$\frac{1}{r_2} < \frac{1}{r} < \frac{1}{r_1},$$

$$\frac{dm}{r_2} < \frac{dm}{r} < \frac{dm}{r_1};$$

(5) $$\iiint_K \frac{dm}{r_2} < \iiint_K \frac{dm}{r} < \iiint_K \frac{dm}{r_1}.$$

Since r_1 and r_2 are constant

$$\frac{1}{r_2}\iiint_K dm < \iiint_K \frac{dm}{r} < \frac{1}{r_1}\iiint_K dm.$$

Now since $\iiint_K dm = M$, the whole mass of the body K, the above is

(6) $$\frac{M}{r_2} < V < \frac{M}{r_1}.$$

Accordingly for an external point V is finite.

If R is the distance of x, y, z from some point in or at a finite distance from K,

$$\frac{RM}{r_2} < RV < \frac{RM}{r_1}.$$

If now we move off x, y, z to an infinite distance we have

$$\lim_{R=\infty} \frac{R}{r_2} = \lim_{R=\infty} \frac{R}{r_1} = 1,$$

and accordingly since RV lies between two quantities having the same limit

(7) $$\lim_{R=\infty} (RV) = M.$$

We say that V vanishes to the *first order* as R becomes infinite.

75. Derivatives. Consider the partial derivatives of V by x, y, z.

The element dm at a, b, c, produces the potential

$$dV = \frac{dm}{r} \text{ at } x, y, z$$

Differentiating by x, (dm and a, b, c being constant), we have

(1) $$\frac{\partial}{\partial x}[dV] = dm\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{dm}{r^2}\frac{\partial r}{\partial x}.$$

By § 39, (7), $$\frac{\partial r}{\partial x} = \frac{x-a}{r},$$

(2) $$\frac{\partial}{\partial x}(dV) = -\frac{dm}{r^2}\frac{x-a}{r}.$$

Now

(3) $$\frac{x-a}{r} = \cos(rx),$$

where the *direction* of r is taken from a, b, c to x, y, z. This being the derivative for that part of the potential due to dm, we have to take the sum of such expressions for all dm's in K, that is, perform a volume integration

(4) $$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\iiint\frac{\rho\, da\, db\, dc}{r} = \iiint\rho\frac{\partial}{\partial x}\left(\frac{1}{r}\right)da\, db\, dc$$

$$= -\iiint\rho\frac{x-a}{r^3}da\, db\, dc = -\iiint\frac{\rho}{r^2}\cos(rx)\, da\, db\, dc.$$

Let the direction cosines of R be $\cos A$, $\cos B$, $\cos C$, and since

$$r_2 > r > r_1,$$

$$\frac{1}{r_2^2} < \frac{1}{r^2} < \frac{1}{r_1^2},$$

$$-\frac{1}{r_2^2} > -\frac{1}{r^2} > -\frac{1}{r_1^2},$$

$$-\frac{\rho}{r_2^2}\cos(rx) > -\frac{\rho}{r^2}\cos(rx) > -\frac{\rho}{r_1^2}\cos(rx).$$

Multiplying and dividing the outside terms by $\cos A$ and integrating,

(5) $$-\frac{\cos A}{r_2^2}\iiint\rho\frac{\cos(rx)}{\cos A}d\tau > \frac{\partial V}{\partial x} > -\frac{\cos A}{r_1^2}\iiint\rho\frac{\cos(rx)}{\cos A}d\tau.$$

Multiplying by R^2 and letting R increase without limit, since

$$\lim_{R=\infty}\frac{R^2}{r_1^2} = \lim_{R=\infty}\frac{R^2}{r_2^2} = \lim_{R=\infty}\frac{\cos(rx)}{\cos A} = 1,$$

$$\lim_{R=\infty}\left[R^2\frac{\partial V}{\partial x}\right] = -M\cos A,$$

(6)
$$\lim_{R=\infty}\left[R^2\frac{\partial V}{\partial y}\right] = -M\cos B,$$
$$\lim_{R=\infty}\left[R^2\frac{\partial V}{\partial z}\right] = -M\cos C.$$

Therefore the first derivatives of V, and hence the parameter, vanish at infinity to the *second* order.

In like manner for the second derivatives

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2}{\partial x^2}\iiint\frac{\rho d\tau}{r} = \iiint\rho\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)d\tau$$
$$= \iiint\rho\left[\frac{3(x-a)^2 - r^2}{r^5}\right]d\tau,$$
$$\frac{\partial^2 V}{\partial x\partial y} = \iiint\rho\frac{3(x-a)(y-b)}{r^5}d\tau.$$

Every element in all the integrals discussed is finite, unless $r = 0$, hence all the integrals are finite. We might proceed in this manner, and should thus find that:

At points not in the attracting masses, V and all its derivatives are finite and (since their derivatives are finite) continuous, as well as uniform.

Also since

(7)
$$\frac{\partial^2 V}{\partial x^2} = \iiint\rho\left[\frac{3(x-a)^2 - r^2}{r^5}\right]d\tau,$$
$$\frac{\partial^2 V}{\partial y^2} = \iiint\rho\left[\frac{3(y-b)^2 - r^2}{r^5}\right]d\tau,$$
$$\frac{\partial^2 V}{\partial z^2} = \iiint\rho\left[\frac{3(z-c)^2 - r^2}{r^5}\right]d\tau,$$

we have by addition

(8)
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0;$$

that is, V satisfies Laplace's equation.

This is also proved by applying Gauss's theorem (§ 39 (4)) to each element $\dfrac{dm}{r}$.

76. Points in the Attracting Mass. Let us now examine the potential and its derivatives at points in the substance of the attracting mass.

If P is within the mass, the element $\dfrac{dm}{r}$ at which the point Q, where dm is placed, coincides with P, becomes infinite. It does not therefore follow that the integral becomes infinite (§ 25).

FIG. 33.

Let us separate from the mass K a small sphere of radius ϵ with the centre at P. Call the part of the body within this sphere K' and the rest K''. Call the part of the integral due to K', V', and that due to K'', V''. Now since P is not in the mass K'', V'' and its derivatives are finite at P, and we have only to examine V' and its derivatives.

Let us insert polar coordinates

$$(1) \qquad V' = \iiint_{K'} \frac{\rho d\tau}{r} = \int_0^{\epsilon}\int_0^{\pi}\int_0^{2\pi} \frac{\rho r^2 dr \sin\theta \, d\theta \, d\phi}{r},$$

so that, by § 23 (5),

$$|V'| < 4\pi\rho_m \int_0^{\epsilon} r dr = 4\pi\rho_m \frac{\epsilon^2}{2},$$

if ρ_m is the greatest value of ρ in K'.

As we make the radius ϵ diminish indefinitely, this vanishes, hence the limit

$$\lim_{\epsilon=0} (V' + V'')$$

is finite.

In like manner for the derivative

$$\frac{\partial V}{\partial x} = -\iiint \rho \frac{x-a}{r^3} d\tau = -\iiint \frac{\rho}{r^2} \cos(rx) \, d\tau.$$

Separate off K' from K''. The part of the integral from K'' is finite. In the other K' introduce polar coordinates, putting $\theta = (rx)$,

(2) $$-\frac{\partial V'}{\partial x} = \int_0^\epsilon \int_0^\pi \int_0^{2\pi} \frac{\rho \cos\theta}{r^2} r^2 \sin\theta \, d\theta d\phi dr,$$

$$\left|\frac{\partial V'}{\partial x}\right| < \rho_m \int_0^\epsilon dr \int_0^\pi \int_0^{2\pi} |\sin\theta \cos\theta| \, d\theta d\phi,$$

$$< 2\pi^2 \rho_m \epsilon,$$

which also vanishes with ϵ. Hence $\frac{\partial V}{\partial x}$ is everywhere finite, and in like manner $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$.

If we attempt this process for the second derivatives $\frac{\partial^2 V}{\partial x^2}$, ... it fails on account of $\frac{dr}{r}$, which gives a logarithm becoming ∞ in the limit.

We will give another proof of the finiteness of $\frac{\partial V}{\partial x}$.

We have

(3) $$\frac{\partial V}{\partial x} = \iiint \rho \frac{a-x}{r^3} da db dc$$

$$= \iiint \rho \frac{\partial}{\partial a}\left(-\frac{1}{r}\right) da db dc,$$

which by Green's theorem is equal to

$$\iint \frac{\rho}{r} \cos(nx) \, dS + \iiint \frac{1}{r} \frac{\partial \rho}{\partial a} d\tau.$$

This is however only to be applied in case the function $\frac{\rho}{r}$ is everywhere finite and continuous. This ceases to be the case when P is in the attracting mass, hence we must exclude P by drawing a small sphere about it. Applying Green's theorem to the rest of the space K'', we have to add to the surface-integral the integral over the surface of the small sphere.

Since $\cos(nx) \leqq 1$, this is not greater than $\rho_m \iint \frac{dS}{r} = 4\pi\epsilon\rho_m$, which vanishes with ϵ. Hence the infinite element contributes nothing to the integral.

In the same way that $\frac{\partial V}{\partial x}$ was proved finite, it may be proved continuous. Dividing it into two parts $\frac{\partial V'}{\partial x}$ and $\frac{\partial V''}{\partial x}$, of which the

second is continuous, we may make, as shown, $\frac{\partial V'}{\partial x}$ as small as we please by making the sphere at P small enough. At a neighbouring point P_1 draw a small sphere, and let the corresponding parts be $\frac{\partial V_1'}{\partial x}$ and $\frac{\partial V_1''}{\partial x}$. Then we can make $\frac{\partial V_1'}{\partial x}$ as small as we please, and hence also the difference $\frac{\partial V'}{\partial x} - \frac{\partial V_1'}{\partial x}$. Hence by taking P and P_1 near enough together, we can make the increment of $\frac{\partial V}{\partial x}$ as small as we please, or $\frac{\partial V}{\partial x}$ is continuous.

77. Poisson's Equation. By Gauss's theorem (§ 39 (5)), we have

$$\iint_S \frac{\cos(nr)\,dS}{r^2} = -4\pi,$$

when r is drawn from O, a point within S. Multiplying by m, a mass concentrated at O,

$$(1) \qquad \iint \frac{m}{r^2}\cos(nr)\,dS = -\iint \frac{\partial V}{\partial n}\,dS = -4\pi m.$$

The integral

$$-\iint \frac{\partial V}{\partial n}\,dS = -\iint P\cos(Pn)\,dS$$

is the surface integral of the *outward* normal component of the parameter P, or of the *inward* component of the force.

The surface integral of the normal component of force in the inward direction through S is called the flux of force into S, and we see that it is equal to -4π times the element of mass within S. Masses without contribute nothing to the integral. Every mass dm situated within S contributes $\frac{dm}{r}$ to the potential at any point and $-4\pi dm$ to the flux through the surface S. Hence the whole mass, when potential is $V = \iiint_K \frac{dm}{r}$, contributes to the flux

$$-4\pi M = -4\pi \iiint_K \rho\,d\tau,$$

and

$$(2) \qquad -\iint_S \frac{\partial V}{\partial n}\,dS = -4\pi \iiint_K \rho\,d\tau.$$

Now the surface integral is, by the divergence theorem, equal to

$$(3) \qquad \iiint_\tau \Delta V d\tau = -4\pi \iiint_K \rho d\tau.$$

The surface S may be drawn inside the attracting mass, providing that we take for the potential only that due to matter in the space τ within S.

Accordingly for τ we may take any part whatever of the attracting mass, and

$$\iiint_\tau \Delta V d\tau = -4\pi \iiint_\tau \rho d\tau;$$

$$(4) \qquad \iiint (\Delta V + 4\pi\rho)\, d\tau = 0.$$

As the above theorem applies to any field of integration whatever, we must have everywhere (by § 23)

$$(5) \qquad \Delta V + 4\pi\rho = 0.$$

This is Poisson's extension of Laplace's equation, and says that at any point the second differential parameter of V is equal to -4π times the density at that point. Outside the attracting bodies, where $\rho = 0$, this becomes Laplace's equation.

In our nomenclature, the *concentration* of the potential at any point is proportional to the density at that point.

A more elementary proof of the same theorem may be given as follows. At a point x, y, z construct a small rectangular parallelopiped whose faces have the coordinates

$$x, \ x + \xi, \ y, \ y + \eta, \ z, \ z + \zeta,$$

and find the flux of force through its six faces. At the face normal to the X-axis whose x coordinate is x let the mean value of the force be $-\frac{\partial V}{\partial x} = -P_x$.

The area of the face is $\eta\zeta$, so that this face contributes to the integral $-\iint P \cos (Pn)\, dS$ the amount $-\frac{\partial V}{\partial x}\eta\zeta$.

At the opposite face, since $\frac{\partial V}{\partial x}$ is continuous, we have for its value

$$\frac{\partial V}{\partial x} + \xi \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \text{terms of higher order in } \xi,$$

and hence, the normal being directed the other way, this side contributes to the integral the amount

$$\left\{\frac{\partial V}{\partial x} + \xi\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \ldots\right\}\eta\zeta,$$

and the two together

$$\xi\eta\zeta\frac{\partial^2 V}{\partial x^2} + \text{terms of higher order.}$$

Similarly the faces perpendicular to Y-axis contribute $\xi\eta\zeta\frac{\partial^2 V}{\partial y^2}$, and the others $\xi\eta\zeta\frac{\partial^2 V}{\partial z^2}$.

Hence the surface integral is

$$\xi\eta\zeta\left\{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right\},$$

and by Gauss's theorem this is equal to

$$-4\pi m = -4\pi\rho\xi\eta\zeta,$$

where ρ is the mean density in the parallelopiped. Now making the parallelopiped infinitely small, and dividing by $\xi\eta\zeta$, we get

$$\Delta V = -4\pi\rho.$$

78. Abbreviations for Operators. If ρ is any point function, the potential function at any point due to a distribution through all space of matter whose density at any point is ρ has been denoted by Gibbs and Heaviside by the abbreviation Pot ρ, standing for the definite integral

$$\text{Pot}\,\rho = \iiint_\infty \frac{\rho}{r}\,d\tau.$$

(The suffix ∞ denotes integration through all space.)

We may thus abbreviate Poisson's equation

$$-\Delta\,\text{Pot}\,\rho = \nabla^2\,\text{Pot}\,\rho = 4\pi\rho,$$

so that the operation Pot followed by the operation $-\Delta = \nabla^2$, performed on any scalar function, has the effect only of multiplying it by 4π, or the operations Δ and $-\frac{\text{Pot}}{4\pi}$ are the inverses of each other.

79. Characteristics of Potential Function. We have now found the following properties of the potential function.

1st. It is everywhere holomorphic, that is, uniform, finite, continuous.

2nd. Its first partial derivatives are everywhere holomorphic.

3rd. Its second derivatives are finite.

4th. V vanishes at infinity to the first order,

$$\lim_{R=\infty} (RV) = M;$$

$\frac{\partial V}{\partial x}, \ldots$ vanish to second order,

$$\lim_{R=\infty} \left(R^2 \frac{\partial V}{\partial x}\right) = -M \cos A.$$

5th. V satisfies everywhere Poisson's differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho,$$

and outside of attracting matter, Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Any function having all these properties is a Newtonian potential function.

The force X, Y, Z is a solenoidal vector at all points outside of the attracting bodies, and hence if we construct tubes of force, the flux of force is constant through any cross-section of a given tube. A tube for which the flux is unity will be called a *unit tube.* The conception of lines of force and of the solenoidal property is due to Faraday.

Since V is a harmonic function outside of the attracting bodies, it has neither maximum nor minimum in free space, but its maximum and minimum must lie within the attracting bodies or at infinity.

In the attracting bodies the equation $-\Delta V = 4\pi\rho$ says that the concentration of the potential at, or the divergence of the force from any point is proportional to the density at that point.

80. Examples. Potential of a homogeneous Sphere. Let the radius of the sphere be R, h the distance of P from its center,

$$V=\iiint\frac{\rho d\tau}{r}.$$

Let us put s instead of r, using the latter symbol for the polar coordinate,

$$V=\iiint\frac{\rho r^2}{s}\sin\theta\, d\theta\, d\phi\, dr.$$

Now $$s^2=h^2+r^2-2hr\cos\theta.$$

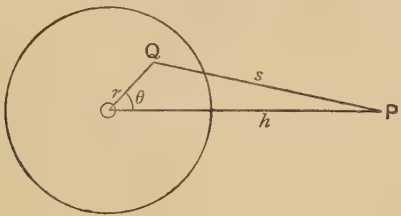

FIG. 34.

Differentiating, keeping r constant,

$$sds=hr\sin\theta\, d\theta,$$

and introducing s as variable instead of θ,

$$V=\iiint\frac{\rho r}{h}\, ds\, d\phi\, dr.$$

We must integrate first with respect to s from $h-r$ to $h+r$, if P is external;

$$V=\frac{2\pi\rho}{h}\int_0^R\int_{h-r}^{h+r} r\,dr\,ds=\frac{4\pi\rho}{h}\int_0^R r^2dr$$

$$=\frac{4\pi\rho R^3}{3h}=\frac{M}{h}.$$

Hence the attraction of a sphere upon an external point is the same as if the whole mass were concentrated at the center.

A body having the property that the line of direction of its resultant attraction on a point passes always through a fixed point in the body is called *centrobaric*.

If instead of a whole sphere we consider a spherical shell of internal radius R_1 and outer R_2, the limits for r being R_1, R_2,

$$V=\frac{4\pi\rho}{h}\int_{R_1}^{R_2} r^2dr=\frac{4\pi\rho}{3h}(R_2{}^3-R_1{}^3)=\frac{M}{h}.$$

We have
$$\frac{dV}{dh} = -\frac{M}{h^2},$$
$$\frac{d^2V}{dh^2} = \frac{2M}{h^3}.$$

If, on the other hand, P is in the spherical cavity, $h < R_1$, the limits for s are $r - h$, $r + h$

$$V = \frac{2\pi\rho}{h}\int_{R_1}^{R_2}\int_{r-h}^{r+h} r\,dr\,ds = 4\pi\rho\int_{R_1}^{R_2} r\,dr$$
$$= 2\pi\rho\,(R_2^2 - R_1^2),$$

which is independent of h, that is, is constant in the whole cavity. Hence $\frac{\partial V}{\partial h} = 0$, and we get the theorem that a homogeneous spherical shell exercises no force on a body within. (On account of symmetry the force can be only radial.)

If P is in the substance of the shell, we divide the shell into two by a spherical surface passing through P, find the potential due to the part within P, and add it to that without, getting

$$V = \frac{4\pi\rho}{3h}(h^3 - R_1^3) + 2\pi\rho\,(R_2^2 - h^2)$$
$$= 2\pi\rho\left\{R_2^2 - \frac{h^2}{3}\right\} - \frac{4\pi\rho R_1^3}{3h},$$
$$\frac{dV}{dh} = \frac{4\pi\rho}{3}\left\{\frac{R_1^3}{h^2} - h\right\},$$
$$\frac{d^2V}{dh^2} = -\frac{4\pi\rho}{3}\left\{\frac{2R_1^3}{h^3} + 1\right\}.$$

Tabulating these results

	$h<R_1$	$R_1<h<R_2$	$h>R_2$
V	$2\pi\rho\,(R_2^2 - R_1^2)$	$2\pi\rho\left\{R_2^2 - \frac{h^2}{3}\right\} - \frac{4\pi\rho R_1^3}{3h}$	$\frac{4\pi\rho}{3h}(R_2^3 - R_1^3)$
$\frac{dV}{dh}$	0	$\frac{4\pi\rho}{3}\left\{\frac{R_1^3}{h^2} - h\right\}$	$-\frac{4\pi\rho}{3h^2}(R_2^3 - R_1^3)$
$\frac{d^2V}{dh^2}$	0	$-\frac{4\pi\rho}{3}\left\{\frac{2R_1^3}{h^2} + 1\right\}$	$\frac{8\pi\rho}{3h^3}(R_2^3 - R_1^3)$

Plotting the above results (Fig. 35) shows the continuity of

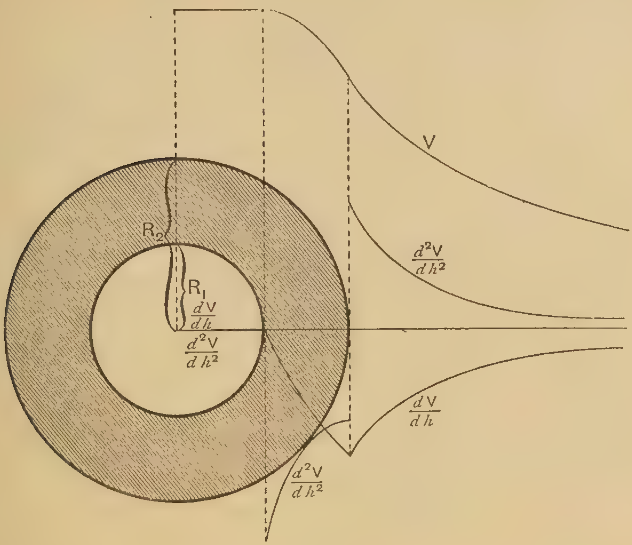

FIG. 35.

V and its first derivatives and the discontinuity of the second derivatives at the surfaces of the attracting mass.

We see that the attraction of a solid sphere at a point within it is proportional to the distance from the center, for if $R_1 = 0$,

$$\frac{dV}{dh} = -\frac{4\pi\rho h}{3},$$

and is independent of the radius of the sphere. Hence experiments on the decrease of the force of gravity in mines at known depths might give us the dimensions of the earth.

81. Disc, Cylinder, Cone. Let us find the attraction of a circular disc of infinitesimal thickness at a point on a line normal

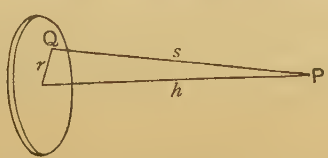

FIG. 36.

to the disc at its center. Let the radius be R_1, thickness ϵ, distance of P from the center h

$$V = \int_0^R \int_0^{2\pi} \rho \frac{\epsilon r dr d\phi}{\sqrt{h^2 + r^2}}$$

$$= 2\pi\epsilon\rho \int_0^R \frac{rdr}{\sqrt{h^2 + r^2}} = 2\pi\epsilon\rho \left[\sqrt{h^2 + r^2}\right]_0^R$$

$$= 2\pi\epsilon\rho \{\sqrt{h^2 + R^2} - h\},$$

$$\frac{dV}{dh} = 2\pi\epsilon\rho \left\{\frac{h}{\sqrt{h^2 + R^2}} - 1\right\}.$$

Attraction of circular cylinder on point in its axis. Let the length be l and let the point be external, at a distance h from the center.

By the above, a disc of thickness dx at a distance x from the center produces a potential at P

$$dV = 2\pi\rho dx \{\sqrt{R^2 + (h-x)^2} - (h-x)\}.$$

Hence the whole is

$$V = 2\pi\rho \int_{-l/2}^{l/2} \{\sqrt{R^2 + (h-x)^2} - (h-x)\}\, dx$$

$$= 2\pi\rho \left\{\frac{x-h}{2} \sqrt{R^2 + (h-x)^2} + \frac{R^2}{2} \log\left(x - h + \sqrt{R^2 + (h-x)^2}\right)\right\}_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \pi\rho \left\{\left[\left(\frac{l}{2} - h\right)\sqrt{R^2 + \left(\frac{l}{2} - h\right)^2} + R^2 \log\left\{\frac{l}{2} - h + \sqrt{R^2 + \left(\frac{l}{2} - h\right)^2}\right\}\right]\right.$$
$$\left. - \left[\left(-\frac{l}{2} - h\right)\sqrt{R^2 + \left(-\frac{l}{2} - h\right)^2} + R^2 \log\left\{-\frac{l}{2} - h + \sqrt{R^2 + \left(-\frac{l}{2} - h\right)^2}\right\}\right]\right\}.$$

Circular cone on point in axis.

Let R be the radius of base, a the altitude, h the height of P above the vertex.

A disc at distance x below vertex and radius r causes potential at P,

$$dV = 2\pi\rho dx \{\sqrt{(h+x)^2 + r^2} - (h+x)\};$$

and
$$\frac{r}{x} = \frac{R}{a}, \quad r = \frac{R}{a} x,$$

$$V = 2\pi\rho \int_0^a dx \left\{ \sqrt{(h+x)^2 + \frac{R^2}{a^2} x^2} - (h+x) \right\}.$$

If we have a conical mountain of uniform density on the earth, and determine the force of gravity at its summit and at the sea level, this gives us the ratio of the attraction of the sphere and cone to that of the sphere alone, and from this we get the ratio of the mass of the earth to the mass of the mountain. Such a determination was carried out by Mendenhall, on Fujiyama, Japan, in 1880, giving 5·77 for the earth's density.

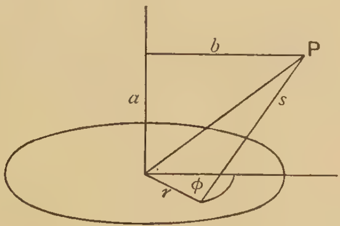

FIG. 37.

Circular disc on point not on axis. Let the coordinates of P with respect to the center be a, b, 0. Then

$$s^2 = a^2 + (b - r\cos\phi)^2 + r^2 \sin^2\phi,$$

$$V = \int_0^R \int_0^{2\pi} \frac{\epsilon r\, dr\, d\phi}{\sqrt{a^2 + (b - r\cos\phi)^2 + r^2\sin^2\phi}},$$

an elliptic integral. The development in an infinite series will be given in § 102.

82. Surface Distributions. In the case of the circular disc of thickness ϵ, $\epsilon\rho$ is the amount of matter per unit of surface of the disc. It is often convenient to consider distributions of matter over surfaces, in such a manner that though ϵ be considered infinitesimal ρ increases so that the product $\epsilon\rho$ remains finite. The product $\epsilon\rho = \sigma$ is called the surface density, and the distribution is called a surface distribution.

We have

$$dm = \sigma dS, \quad V = \iint \frac{\sigma dS}{r}.$$

In the case of the disc, we had

$$\frac{\partial V}{\partial h} = 2\pi\epsilon\rho\left\{\frac{h}{\sqrt{h^2+R^2}} - 1\right\}.$$

When $h = 0$ we have

$$\left(\frac{\partial V}{\partial h}\right)_{h=0} = -\,2\pi\sigma.$$

The repulsion of a disc upon a particle in contact with it at its center is independent of the radius of the disc, and is equal to 2π times the surface density.

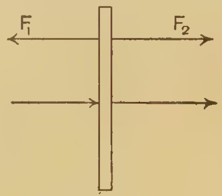

FIG. 38.

If the force on a particle in contact on the right be called F_2, positive if to the right, we have

$$F_2 = +\,2\pi\sigma.$$

By symmetry, the force on a particle at the left in contact with the disc is

$$F_1 = -\,2\pi\sigma,$$

$$F_1 - F_2 = -\,4\pi\sigma.$$

Now if x denote the direction of the normal to the right,

$$F_1 = -\left(\frac{\partial V}{\partial x}\right)_1,$$

$$F_2 = -\left(\frac{\partial V}{\partial x}\right)_2,$$

and we see that on passing through the surface there is a discontinuity in the value of $\frac{\partial V}{\partial x}$ of the magnitude $4\pi\sigma$.

Consider a thin spherical shell. We have for an external point

$$V = \frac{4\pi\rho}{3h}(R_2^3 - R_1^3) = \frac{4\pi\rho}{3h}(R_2 - R_1)(R_2^2 + R_2R_1 + R_1^2),$$

and making $R_2 - R_1 = \epsilon$, $\lim R_1 = \lim R_2 = R$,

$$V = \frac{4\pi\sigma}{3h} \cdot 3R^2,$$

$$\frac{dV}{dh} = -\frac{4\pi\sigma}{h^2} R^2,$$

and on the outside for $h = R$,

$$\frac{dV}{dh} = -4\pi\sigma.$$

Within we have everywhere

$$V = \text{const.}, \quad \frac{dV}{dh} = 0.$$

Hence there is in like manner a discontinuity in the first derivative of the potential in the direction of the normal, on passing through the attracting surface, of the amount $4\pi\sigma$.

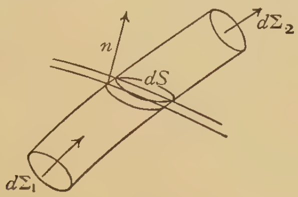

FIG. 39.

Consider now any surface distribution of surface density σ. Apply Gauss's theorem to a small tube of force bounded by portions of two equipotential surfaces $d\Sigma_1$ and $d\Sigma_2$ on opposite sides of the element of the attracting surface dS (Fig. 39). The flux out from the tubes is

$$F_2 d\Sigma_2 - F_1 d\Sigma_1,$$

and this must be equal to $+4\pi$ times the matter contained in the tube, which is σdS. Therefore

$$F_2 d\Sigma_2 - F_1 d\Sigma_1 = 4\pi\sigma dS.$$

But if the length and diameter of the tube are infinitesimal $d\Sigma_1$ and $d\Sigma_2$ are the projections of dS,

$$d\Sigma_1 = d\Sigma_2 = dS \cos(Fn),$$

where n is the normal to the attracting surface. Accordingly

$$F_2 \cos(Fn)\, dS - F_1 \cos(Fn)\, dS = 4\pi\sigma dS,$$

and since
$$F_2 \cos (Fn) = -\left(\frac{\partial V}{\partial n}\right)_2,$$
$$F_1 \cos (Fn) = -\left(\frac{\partial V}{\partial n}\right)_1,$$
$$\left(\frac{\partial V}{\partial n}\right)_1 - \left(\frac{\partial V}{\partial n}\right)_2 = 4\pi\sigma.$$

The normal to S is here drawn toward the side 2. We find then that in general, on traversing a repelling surface distribution, the normal force has a discontinuity equal to $4\pi\sigma$.

This is Poisson's equation for a surface distribution. If we draw the normal *away* from the surface on each side, we may write
$$\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2} = -4\pi\sigma,$$
or
$$F_1 \cos (F_1 n_1) + F_2 \cos (F_2 n_2) = F_{1n_1} + F_{2n_2} = 4\pi\sigma.$$

83. Green's formulae. Let us apply Green's theorem to two functions, of which one, V, is the potential function due to any distribution of matter, and the other, $U = \frac{1}{r}$, where r is the distance from a fixed point P, lying in the space τ over which we take the integral. Let the space τ concerned be that bounded by a closed surface S, a small sphere Σ of radius ϵ about P, and, if P is without S, a sphere of infinite radius with center P.

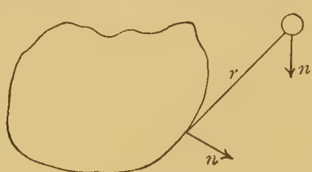

FIG. 40.

Now the theorem was stated in § 33 (2) for the normal drawn in toward τ, which means outward from S and Σ, and inward from the infinite sphere, as

(1) $$\iint\left(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\right) dS = \iiint (U\Delta V - V\Delta U)\, d\tau,$$

and since
$$U = \frac{1}{r}, \quad \Delta U = 0,$$

in the whole space τ, so that (1) becomes

$$(2)\qquad \iint\left(V\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{r}\frac{\partial V}{\partial n}\right)dS=\iiint\frac{1}{r}\Delta V d\tau.$$

The surface integrals are to be taken over S, over the small sphere, and over the infinite sphere. For a sphere with center at P,

$$\frac{\partial\frac{1}{r}}{\partial n}=\pm\frac{\partial\frac{1}{r}}{\partial r}=\mp\frac{1}{r^2},\quad dS=r^2d\omega,$$

the upper or lower sign being taken according as the sphere is the inner or outer boundary of τ;

$$\iint V\frac{\partial\frac{1}{r}}{\partial n}dS=\mp\iint Vd\omega,$$

and for

$$r=\infty$$

V vanishes, hence this integral vanishes. Also

$$(3)\qquad -\iint\frac{1}{r}\frac{\partial V}{\partial n}dS=-\iint\frac{1}{r}\frac{\partial V}{\partial n}r^2d\omega=-r\iint\frac{\partial V}{\partial n}d\omega.$$

Now at infinity, $\dfrac{\partial V}{\partial n}$ is of order $\dfrac{1}{r^2}$, and being multiplied by r, still vanishes. Accordingly the infinite sphere contributes nothing. For the small sphere the case is different. The first integral

$$-\iint Vd\omega$$

becomes, as the radius ϵ of the sphere diminishes,

$$(4)\qquad -V_P\iint d\omega=-4\pi V_P.$$

The second part

$$-\epsilon\iint\frac{\partial V}{\partial n}d\omega,$$

however, since $\dfrac{\partial V}{\partial n}$ is finite in the sphere, vanishes with ϵ. Hence there remain on the left side of the equation only $-4\pi V_P$ and the integral over S. We obtain therefore

$$-4\pi V_P+\iint_S\left(V\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{r}\frac{\partial V}{\partial n}\right)dS=\iiint\frac{\Delta V}{r}d\tau,$$

$$(5)\qquad V_P=\frac{1}{4\pi}\iint_S\left\{V\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{r}\frac{\partial V}{\partial n}\right\}dS-\frac{1}{4\pi}\iiint\frac{\Delta V}{r}d\tau\ \ (P \text{ outside } S),$$

the normal being drawn *outward* from S. This formula is due to Green.

Hence we see that any function which is uniform and continuous everywhere outside of a certain closed surface, vanishes at infinity to the first order, and whose parameter vanishes at infinity to the second order, is determined at every point of space considered if we know at every point of that space the value of the second differential parameter, and in addition the values on the surface S of the function and its vector parameter resolved in the direction of the outer normal.

In particular, if V is *harmonic* in all the space considered, we have

$$(6) \qquad V_P = \frac{1}{4\pi}\iint\left(V\frac{\partial\frac{1}{r}}{\partial n} - \frac{1}{r}\frac{\partial V}{\partial n}\right)dS,$$

and a harmonic function is determined everywhere by its values and those of its normal component of parameter at all points of the surface S.

Since

$$\frac{\partial\frac{1}{r}}{\partial n} = -\frac{1}{r^2}\frac{\partial r}{\partial n}$$

$$= -\frac{1}{r^2}\left\{\cos(nx)\frac{\partial r}{\partial x} + \cos(ny)\frac{\partial r}{\partial y} + \cos(nz)\frac{\partial r}{\partial z}\right\} = -\frac{\cos(nr)}{r^2},$$

we may write (6)

$$(7) \qquad V_P = -\frac{1}{4\pi}\iint\left(\frac{V}{r^2}\cos(nr) + \frac{1}{r}\frac{\partial V}{\partial n}\right)dS.$$

Consequently, we may produce at all points outside of a closed surface S the same field of force as is produced by any distribution of masses lying *inside* of S, whose potential is V, if we distribute over the surface S a *surface* distribution of surface-density,

$$(8) \qquad \sigma = -\frac{1}{4\pi}\left\{\frac{V\cos(nr)}{r} + \frac{\partial V}{\partial n}\right\}.$$

In the general expression (5), the surface integral representing the potential due to the masses *within* S, the volume integral

$$-\frac{1}{4\pi}\iiint\frac{\Delta V}{r}d\tau$$

represents, since everywhere

$$-\frac{1}{4\pi}\Delta V = \rho,$$

$$\iiint \frac{\rho d\tau}{r},$$

that is, the potential due to all the masses in the region τ, viz., outside S.

84. Equipotential Layers. As a still more particular case of (7), if the surface S is taken as one of the equipotential surfaces of the internal distribution, we have all over the surface $V = V_S =$ const., and the constant may be taken out from the first integral,

(9) $$V_P = -\frac{V_S}{4\pi}\iint \frac{\cos(nr)}{r^2}\,dS - \frac{1}{4\pi}\iint \frac{1}{r}\frac{\partial V}{\partial n}\,dS.$$

Now by Gauss's theorem $\iint \frac{\cos(nr)}{r^2}\,dS = 0$; accordingly,

(10) $$V_P = -\frac{1}{4\pi}\iint \frac{1}{r}\frac{\partial V}{\partial n}\,dS = \iint \frac{\sigma}{r}\,dS,$$

so that V_P is represented as the potential of a surface distribution of surface-density

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n} = \frac{1}{4\pi}F\cos(Fn) = \pm\frac{F}{4\pi}.$$

The whole mass of the equivalent surface distribution is

(11) $$\iint \sigma\,dS = -\frac{1}{4\pi}\iint \frac{\partial V}{\partial n}\,dS$$

$$= \frac{1}{4\pi}\iint F\cos(Fn)\,dS,$$

which, being the flux of force outward from S, is by Gauss's theorem, § 77 (1), equal to M, the mass within S.

Accordingly we may enunciate the theorem, due to Chasles and Gauss*:—

We may produce outside any equipotential surface of a distribution M the same effect as the distribution itself produces, by

* Chasles, "Sur l'attraction d'une couche ellipsoidale infiniment mince." *Journ. Éc. Polytec.*, Cahier 25, p. 266, 1837; Gauss, *Allgemeine Lehrsätze*, § 36.

distributing over that surface a layer of surface-density equal to $\frac{1}{4\pi}$ times the outward force at every point of the surface. The mass of the whole layer will be precisely that of the original internal distribution. Such a layer is called an *equipotential layer.* (Definition—A superficial layer which coincides with one of its own equipotential surfaces.) Reversing the sign of this density will give us a layer which will, *outside*, neutralize the effect of the bodies within.

Let us now suppose the point P is *within* S. In this case, we apply Green's theorem to the space within S, and we do not have the integrals over the infinite sphere. The normal in the above formulae is now drawn inward from S, or if we still wish to use the outward normal, we change the sign of the surface integral in (5),

$$(12)\quad V_p = -\frac{1}{4\pi}\iint_S \left(V\frac{\partial\frac{1}{r}}{\partial n_e} - \frac{1}{r}\frac{\partial V}{\partial n_e}\right) dS - \frac{1}{4\pi}\iiint \frac{\Delta V}{r}\, d\tau,$$

(P inside S).

Note that both formulae (5) and (12) are identical if the normal is drawn into the space in which P lies.

Hence within a closed surface a holomorphic function is determined at every point solely by its values and those of its normally resolved parameter at all points of the *surface*, and by the values of its *second* parameter at all points in the space within the surface.

A harmonic function may be represented by a potential function of a surface distribution.

Now if the surface S is equipotential, the function V cannot be harmonic everywhere within unless it is constant. For since two equipotential surfaces cannot cut each other, the potential being a one-valued function, successive equipotential surfaces must surround each other, and the innermost one, which is reduced to a point, will be a point of maximum or minimum. But we have seen (§ 34) that this is impossible for a harmonic function. Accordingly a function constant on a closed surface and harmonic within must be a constant.

If however there be matter within and without S, the volume integral, as before, denotes the potential due to the matter in the

space τ (within S), and the surface integral that due to the matter without. If the surface is equipotential, the surface integral is

$$-\frac{V_S}{4\pi}\iint\frac{\partial\frac{1}{r}}{\partial n_e}+\frac{1}{4\pi}\iint\frac{1}{r}\frac{\partial V}{\partial n_e}dS$$

$$=\frac{V_S}{4\pi}\iint\frac{\cos(n_e r)}{r^2}dS+\frac{1}{4\pi}\iint\frac{1}{r}\frac{\partial V}{\partial n_e}dS.$$

The first integral is now equal to 4π, so that

(13) $$V_P=V_S+\frac{1}{4\pi}\iint\frac{1}{r}\frac{\partial V}{\partial n_e}dS-\iiint\frac{\Delta V}{r}d\tau,$$

V_S being constant contributes nothing to the derivatives of V, so that the outside bodies may be replaced by a surface layer of density

(14) $$\sigma=\frac{1}{4\pi}\frac{\partial V}{\partial n_e}=-\frac{1}{4\pi}F\cos(Fn_e)=\pm\frac{F}{4\pi}.$$

The mass of the surface distribution

(15) $$\iint\sigma dS=\frac{1}{4\pi}\iint\frac{\partial V}{\partial n_e}dS=-\frac{1}{4\pi}\iint F\cos(Fn_e)\,dS,$$

n_e being the outward normal, is the inward flux of force through S, which is equal to minus the mass of the *interior* matter, and is not, as in the former case, equal to the mass which it replaces.

85. Potential completely determined by its characteristic properties. We have proved that the potential function due to any volume distribution has the following properties:

1. It is, together with its first differential parameter, uniform, finite, and continuous.

2. It vanishes to the first order at ∞, and its parameter to the second order.

3. It is harmonic outside the attracting bodies, and in them satisfies

$$\Delta V=-4\pi\rho.$$

The preceding investigation shows that a function having these properties is a potential function, and is completely determined.

For we may apply the above formula (5) to all space, and then the only surface integral being that due to the infinite sphere, which vanishes, we have

$$(16)\qquad V = -\frac{1}{4\pi}\iiint_\infty \frac{\Delta V}{r}\,d\tau = \iiint_\infty \frac{\rho}{r}\,d\tau.$$

If however, the above conditions are fulfilled by a function V, except that at certain surfaces S its first parameter is discon-

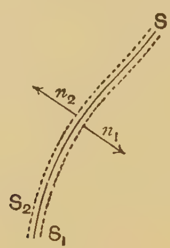

FIG. 41.

tinuous, let us draw on each side of the surface S surfaces at distances equal to ϵ from S, and exclude that portion of space lying between these, which we will call S_1 and S_2.

If the normals are drawn *into* τ we have

$$(5)\qquad V = \frac{1}{4\pi}\iint\left(V\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial V}{\partial n}\right)dS - \iiint\frac{\Delta V}{r}\,d\tau.$$

The surface integrals are to be taken over both surfaces S_1 and S_2 and the volume integrals over all space except the thin layer between S_1 and S_2. This is the only region where there is discontinuity, hence in τ the theorem applies, and

$$(17)\qquad \begin{aligned} 4\pi V = &\iint_{S_1} V\frac{\partial}{\partial n_1}\left(\frac{1}{r}\right)dS_1 + \iint_{S_2} V\frac{\partial}{\partial n_2}\left(\frac{1}{r}\right)dS_2 \\ &- \iint_{S_1}\frac{1}{r}\frac{\partial V}{\partial n_1}\,dS_1 - \iint_{S_2}\frac{1}{r}\frac{\partial V}{\partial n_2}\,dS_2 - \iiint_\infty \frac{\Delta V}{r}\,d\tau. \end{aligned}$$

Now let us make ϵ infinitesimal, then the surfaces S_1, S_2 approach each other and S. V is continuous at S, that is, is the same on both sides, hence, since $\frac{\partial}{\partial n_1}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial n_2}\left(\frac{1}{r}\right)$, in the limit the first two terms destroy each other. This is not so for the next two, for $\frac{\partial V}{\partial n_1}$ is *not* equal to $\frac{\partial V}{\partial n_2}$ because of the discontinuity.

In the limit, then

$$(18) \qquad V_P = -\frac{1}{4\pi}\iint \frac{1}{r}\left(\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2}\right) dS - \iiint_\infty \frac{\Delta V}{r}\, d\tau.$$

The volume integral, as before, denotes the potential $\iiint \frac{\rho}{r}\, d\tau$ due to the volume distribution, while the surface integral denotes the potential of a surface distribution $\iint \frac{\sigma dS}{r}$,

where

$$(19) \qquad \sigma = -\frac{1}{4\pi}\left\{\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2}\right\}.$$

Hence we get a new proof of Poisson's surface condition, § 82.

86. Kelvin and Dirichlet's Principle. We shall now consider a question known on the continent of Europe as *Dirichlet's Problem.*

Given the values of a function at all points of a closed surface S—is it possible to find a function which, assuming these values on the surface, is, with its parameters, uniform, finite, continuous, and is itself *harmonic* at all points within S?

This is the internal problem—the external may be stated in like manner, specifying the conditions as to vanishing at infinity.

Consider the integral

$$(1) \qquad J(u) = \iiint \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} d\tau,$$

of a function u throughout the space τ within S.

J must be positive, for every element is a sum of squares. It cannot vanish, unless everywhere $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$, that is $u = \text{constant}$. But since u is continuous, unless it is constant on S, this will not be the case.

Accordingly $\qquad J(u) > 0.$

Now of the infinite variety of functions u there must be, according to Dirichlet, at least one which makes J less than for any of the others. Call this function v, and call the difference between this and any other hs, so that

$$u = v + hs,$$

h being constant.

The condition for a minimum is that

$$J(v) < J(v + hs),$$

for all values of h.

Now
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + h\frac{\partial s}{\partial x}, \text{ etc.}$$

(2) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$
$$= \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + h^2\left\{\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2\right\}$$
$$+ 2h\left\{\frac{\partial v}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial s}{\partial z}\right\}.$$

Integrating

(3) $$J(u) = J(v) + h^2 J(s) + 2h\iiint\left(\frac{\partial v}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial s}{\partial z}\right)d\tau.$$

Now in order that $J(v)$ may be a minimum, we must have

(4) $$h^2 J(s) + 2h\iiint\left(\frac{\partial v}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial s}{\partial z}\right)d\tau > 0,$$

for *all* values of h, positive or negative. But as s is as yet unlimited, we may take h so small that the absolute value of the term in h is greater than that of the term in h^2, and if we choose the sign of h opposite to that of the integral, making the product negative, the whole will be negative.

The only way to leave the sum always positive is to have the integral vanish. (It will be observed that the above is exactly the process of the calculus of variations. We might put δv instead of hs.)

The condition for a minimum is then

(5) $$\iiint\left\{\frac{\partial v}{\partial x}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial s}{\partial z}\right\}d\tau = 0.$$

But by Green's theorem, this is equal to

(6) $$-\iint_S s\frac{\partial v}{\partial n}dS - \iiint s\Delta v d\tau.$$

Now at the surface the function is given, hence u and v must have the same values, and $s = 0$.

Consequently the surface integral vanishes, and

$$\iiint s\Delta v d\tau = 0.$$

But since s is arbitrary, the integral can vanish only if everywhere in τ, $\Delta v = 0$, v is therefore the function which solves the problem. The proof that there is such a function depends on the assumption that there *is* a function which makes the integral J a minimum. This assumption has been declared by Weierstrass, Kronecker, and others, to be faulty. The principle of Lord Kelvin and Dirichlet, which declares that there is a function v, has been rigidly proved for a number of special cases, but the above general proof is no longer admitted. It is given here on account of its historical interest*.

We can however prove that *if* there is a function v, satisfying the conditions, it is unique. For, if there is another, v', put

$$u = v - v'.$$

$$(7) \quad J(u) = \iiint \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} d\tau$$
$$= -\iint u\frac{\partial u}{\partial n} dS - \iiint u\Delta u d\tau.$$

On the surface, since $v = v'$, $u = 0$. In τ, since Δv and $\Delta v'$ are zero, $\Delta u = 0$. Accordingly $J(u) = 0$. But, as we have shown, this can only be if $u =$ const. But on S, $u = 0$, hence, throughout τ, $u = 0$ and $v = v'$.

87. Green's Theorem in Curvilinear Orthogonal Coordinates. We shall now consider Green's theorem in terms of any orthogonal coordinates, limiting ourselves to the special case $U =$ const., or the divergence theorem, § 35,

$$\iint \frac{\partial V}{\partial n_s} dS = \iiint \Delta V d\tau,$$

where n_s is the *outward* normal to S.

* Thomson, "Theorems with reference to the solution of certain Partial Differential Equations," *Cambridge and Dublin Math. Journ.*, Jan. 1878; Reprint of "Papers in Electrostatics and Magnetism," XIII. The name *Dirichlet's Princip* was given by Riemann (*Werke*, p. 90). For a historical and critical discussion of this matter the student may consult Bacharach, *Abriss der Geschichte der Potentialtheorie*, as well as Harkness and Morley, *Theory of Functions*, Chap. IX., Picard, *Traité d'Analyse*, Tom. II., p. 38.

Let the coordinates be q_1, q_2, q_3.

The parameter P is the resultant of the derivatives of V in any three perpendicular directions. Let these be in the directions of the normals to the level surfaces q_1, q_2, q_3.

Then, calling these P_{q_1}, P_{q_2}, P_{q_3}.

$$\text{(1)} \qquad \frac{\partial V}{\partial n_s} = P \cos(Pn_s)$$
$$= P_{q_1} \cos(n_1 n_s) + P_{q_2} \cos(n_2 n_s) + P_{q_3} \cos(n_3 n_s).$$

Now P_{q_1}, the partial parameter with respect to q_1, is (§ 16) $h_1 \dfrac{\partial V}{\partial q_1}$.

Hence

$$\text{(2)} \qquad \frac{\partial V}{\partial n_s} = h_1 \frac{\partial V}{\partial q_1} \cos(n_1 n_s) + h_2 \frac{\partial V}{\partial q_2} \cos(n_2 n_s) + h_3 \frac{\partial V}{\partial q_3} \cos(n_3 n_s).$$

If we divide the volume τ up into elementary curved prisms bounded by level surfaces of q_2 and q_3, as in the case of rectangular

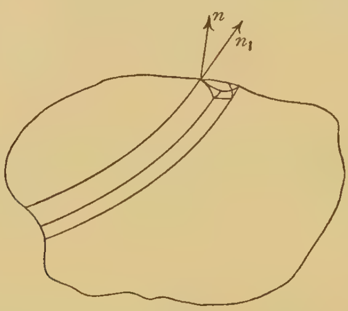

FIG. 42.

coordinates, we have, at each case of cutting into or out of S, $\pm\, dS \cos(n_s n_1) = dS_1$, where dS_1 is the area of the part cut by the prism from the level surface q_1.

By § 20,

$$dS_1 = \frac{dq_2}{h_2} \frac{dq_3}{h_3},$$

accordingly

$$\text{(3)} \qquad \iint h_1 \frac{\partial V}{\partial q_1} \cos(n_1 n_s)\, dS = \iint h_1 \frac{\partial V}{\partial q_1} \frac{dq_2}{h_2} \frac{dq_3}{h_3}$$
$$= \iiint \frac{\partial}{\partial q_1} \left\{ \frac{h_1}{h_2 h_3} \frac{\partial V}{\partial q_1} \right\} dq_1 dq_2 dq_3.$$

Transforming the other two integrals in like manner,

$$(4)\quad \iint \frac{\partial V}{\partial n_s}\,dS = \iint \left\{ h_1 \frac{\partial V}{\partial q_1}\cos(n_1 n_s) + h_2 \frac{\partial V}{\partial q_2}\cos(n_2 n_s) + h_3 \frac{\partial V}{\partial q_3}\cos(n_3 n_s)\right\} dS$$

$$= \iiint \left[\frac{\partial}{\partial q_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_2}{h_3 h_1}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_3}{h_1 h_2}\frac{\partial V}{\partial q_3}\right)\right] dq_1 dq_2 dq_3.$$

Now this is equal to $\iiint \Delta V d\tau.$

But $$d\tau = \frac{dq_1 dq_2 dq_3}{h_1 h_2 h_3}.$$

Multiplying and dividing in the last integral of (4) by $h_1 h_2 h_3$, we find that, since the integrals are equal for any volume, the integrands must be equal, or

$$(5)\quad \Delta V = h_1 h_2 h_3 \left\{\frac{\partial}{\partial q_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_2}{h_3 h_1}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_3}{h_1 h_2}\frac{\partial V}{\partial q_3}\right)\right\}.$$

This result was given by Lamé, by means of a laborious direct transformation. The method here used is similar to one used by Jacobi, and is given by Somoff*.

88. Laplace's Equation in Spherical and Cylindrical Coordinates. Applying this to spherical coordinates

$$h_r = 1,\ h_\theta = \frac{1}{r},\ h_\phi = \frac{1}{r\sin\theta},$$

$$(6)\quad \Delta V = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right)\right\}$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

We may apply this equation to determine the attraction of a sphere. For external points $\Delta V = 0$, and since by symmetry V is independent of θ and ϕ,

* Lamé, *Journal de l'École Polytechnique*, Cahier 23, p. 215, 1833; *Leçons sur les Coordonnées curvilignes*, II. Jacobi, "Ueber eine particuläre Lösung der partiellen Differentialgleichung $\Delta V = 0$," *Crelle's Journal*, Bd. 36, p. 113. Somoff, *Theoretische Mechanik*, II. Theil, § 51—2.

(7) $$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=0 \text{ or } \frac{d}{dr}\left(r^2\frac{dV}{dr}\right)=0,$$

$$r^2\frac{dV}{dr}=c, \quad \frac{dV}{dr}=\frac{c}{r^2},$$

$$V=-\frac{c}{r}+c'.$$

But since
$$\operatorname*{Lim}_{r=\infty}(rV)=M,$$

$$\operatorname*{Lim}_{r=\infty}[-c+c'r]=M,$$

we must have $c'=0$, $-c=M$.

Apply the above transformation to cylindrical coordinates

$$h_z=1, \quad h_\rho=1, \quad h_\omega=\frac{1}{\rho},$$

(8) $$\Delta V=\frac{1}{\rho}\left\{\frac{\partial}{\partial z}\left(\rho\frac{\partial V}{\partial z}\right)+\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{\partial}{\partial\omega}\left(\frac{1}{\rho}\frac{\partial V}{\partial\omega}\right)\right.$$
$$=\frac{\partial^2V}{\partial z^2}+\frac{\partial^2V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2V}{\partial\omega^2}.$$

If we apply this to calculate the potential due to a cylindrical homogeneous body with generators parallel to the axis of z and of infinite length, the potential is independent of z and satisfies at external points,

(9) $$0=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}$$
$$=\frac{\partial^2V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2V}{\partial\omega^2}.$$

If the cylinder is circular, V is independent of ω, and we have the ordinary differential equation

$$\frac{d^2V}{d\rho^2}+\frac{1}{\rho}\frac{dV}{d\rho}=0,$$

$$\frac{d^2V}{d\rho^2}\Big/\frac{dV}{d\rho}=-\frac{1}{\rho},$$

$$\frac{d}{d\rho}\left(\log\frac{dV}{d\rho}\right)=-\frac{1}{\rho},$$

$$\log\frac{dV}{d\rho}=-\log\rho+\text{const.}$$

$$\frac{dV}{d\rho}=\frac{C}{\rho},$$

$$V=C\log\rho+C'.$$

The force in the direction of ρ is inversely proportional to the *first* power of ρ.

We may verify this by direct calculation. Let us consider the

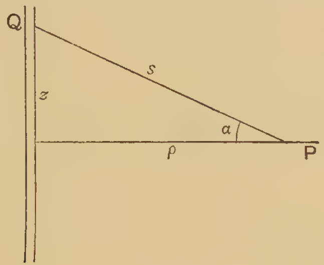

FIG. 43.

cylinder as infinitely thin, with cross-section ϖ. We will find the component of *force* in the direction of ρ.

The action of dm at z on P at distance ρ (Fig. 43) is

$$\frac{dm}{s^2} = \frac{dm}{\sqrt{\rho^2 + z^2}}.$$

The component parallel to ρ is

$$\frac{dm}{s^2} \cos(\rho s) = \frac{\rho\, dm}{s^3}.$$

Now since, calling the density δ, $dm = \delta\varpi\, dz$, we have for the total force in direction ρ

$$F = 2\int_0^\infty \frac{\delta\varpi\rho\, dz}{\sqrt{(\rho^2 + z^2)^3}}.$$

Put

$$z = \rho \tan\theta,$$
$$dz = \rho \sec^2\theta\, d\theta.$$

$$F = 2\varpi\delta \int_0^{\frac{\pi}{2}} \frac{\rho^2 \sec^2\theta\, d\theta}{\rho^3 \sec^3\theta} = \frac{2\varpi\delta}{\rho}\Big[\sin\theta\Big]_0^{\frac{\pi}{2}}$$
$$= \frac{2\varpi\delta}{\rho} = \frac{C}{\rho}, \text{ as before.}$$

89. Logarithmic Potential. We may state the above result in terms of the following two-dimensional problem. Suppose that on a plane there be distributed a layer of mass in such a way that a point of mass m repels a point of unit mass in

the plane with a force $\frac{m}{r}$ where r is their distance apart. The potential due to m is $V = -m \log r$ and it satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Similarly, in the case of any mass distributed in the plane, with surface-density μ, an element $dm = \mu dS$ produces the potential $-dm \log r$, and the whole the potential

$$V = -\iint dm \log r = -\iint \mu \log r dS,$$

where r is the distance from the repelling dm at a, b to the repelled point x, y, so that

$$r^2 = (x-a)^2 + (y-b)^2.$$

We may verify by direct differentiation that, at external points, this V satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

$$\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\iint \mu \log r da db = -\iint \mu \frac{\partial}{\partial x}(\log r)\, da db$$

$$= -\iint \frac{\mu}{r}\frac{\partial r}{\partial x} da db = -\iint \frac{\mu(x-a)}{r^2} da db,$$

$$\frac{\partial^2 V}{\partial x^2} = -\iint \mu \frac{\partial}{\partial x}\left\{\frac{x-a}{r^2}\right\} da db = -\iint \mu \left\{\frac{1}{r^2} - \frac{2(x-a)^2}{r^4}\right\} da db,$$

$$\frac{\partial^2 V}{\partial y^2} = -\iint \mu \left\{\frac{1}{r^2} - \frac{2(y-b)^2}{r^4}\right\} da db,$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\iint \mu \left\{\frac{2}{r^2} - \frac{2[(x-a)^2 + (y-b)^2]}{r^4}\right\} da db = 0.$$

This potential is called the *logarithmic potential* and is of great importance in the theory of functions of a complex variable.

90. Green's Theorem for a Plane. In exactly the same manner that we proved Green's Theorem for three dimensions, we may prove it when the integral is the double integral in a plane

(1) $$I = \iint_A \left\{\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\right\} dx dy,$$

over an area A bounded by any closed contour C. Since we have for a continuous function W

$$(2)\qquad \iint \frac{\partial W}{\partial x}\,dx\,dy = \int \left[W_2 - W_1 + \ldots + W_{2n} - W_{2n-1} \right] dy$$

$$= -\int_C W \cos(nx)\,ds,$$

where n is the inward normal, ds the element of arc of the contour.

Applying this to $W = U\dfrac{\partial V}{\partial x}$, we obtain

$$(3)\qquad \iint \frac{\partial}{\partial x}\left\{U\frac{\partial V}{\partial x}\right\} = -\int U\frac{\partial V}{\partial x}\cos(nx)\,ds.$$

Treating the other term in like manner, we obtain

$$(4)\qquad I = -\int_C U\frac{\partial V}{\partial n}\,ds - \iint_A U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) dx\,dy.$$

Interchanging U and V we obtain the second form

$$(5)\qquad \int_C \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) ds = \iint_A (V\Delta U - U\Delta V)\,ds,$$

where we write

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}.$$

91. Application to Logarithmic Potential. If in the second form above we put $U = 1$, we obtain

$$(6)\qquad \int_C \frac{\partial V}{\partial n}\,ds = -\iint_A \Delta V\,dx\,dy,$$

which is the divergence theorem in two dimensions. If the function V is harmonic everywhere within the contour, we have

$$\int_C \frac{\partial V}{\partial n}\,ds = 0.$$

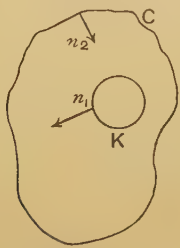

FIG. 44.

Applying this to the harmonic function $\log r$, where P, the fixed pole from which r is measured, is outside the contour,

(7) $$\int_C \frac{\partial \log r}{\partial n} ds = \int_C \frac{1}{r}\frac{\partial r}{\partial n} ds = \int_C \frac{\cos(rn)}{r} ds = 0.$$

If the pole P is within the contour, we draw a circle K of any radius about the pole, and apply the theorem to the area outside of this circle and within the contour, obtaining

(8) $$\int_C \frac{\partial \log r}{\partial n} ds = -\int_K \frac{\cos(rn)}{r} ds = -\int_0^{2\pi} d\theta = -2\pi.$$

These two results are Gauss's theorem for two dimensions. They may of course be deduced geometrically. We may now deduce Poisson's equation for the logarithmic potential as in § 77 for the Newtonian Potential. The logarithmic potential due to a mass dm being $-dm \log r$ gives rise to the flux of force $2\pi dm$ *outward* through any closed contour surrounding it, and a total mass m causes the flux

$$2\pi m = 2\pi \iint \mu dx dy.$$

Put in terms of the potential this is

(9) $$\int_C \frac{\partial V}{\partial n_i} ds = -\iint_A \Delta V dx dy = 2\pi \iint_A \mu dx dy,$$

and since this is true for any area of the plane, we must have

(10) $$\Delta V = -2\pi\mu.$$

This is Poisson's equation for the logarithmic potential.

92. Green's formula for Logarithmic Potential. Applying Green's Theorem (5) to the functions $-\log r$ and any harmonic function V, supposing the pole of P to be within the contour, and extending the integral to the area within the contour and without a circle K of radius ϵ about the pole,

(11) $$\int_C \left(\log r \frac{\partial V}{\partial n} - V\frac{\partial \log r}{\partial n}\right) ds$$
$$+\int_K \left(\log r \frac{\partial V}{\partial n} - V\frac{\partial \log r}{\partial n}\right) ds = 0.$$

The third term is

$$\int_K \log r \frac{\partial V}{\partial n} ds = \log \epsilon \int \frac{\partial V}{\partial n} ds = 0,$$

(since V is harmonic in K) and the fourth

$$-\int_K V\frac{\partial \log r}{\partial n}\,ds = -\int_K \frac{V}{r}\,r d\theta = -\int_K V d\theta,$$

which, when we make ϵ decrease indefinitely, becomes

$$-2\pi V_P.$$

Accordingly we obtain the equation

$$(12) \qquad V_P = \frac{1}{2\pi}\int_C \left(\log r\frac{\partial V}{\partial n} - V\frac{\partial \log r}{\partial n}\right) ds,$$

which is the analogue of equation (6), § 83. In a similar way we may find for nearly every theorem on the Newtonian Potential a corresponding theorem for the Logarithmic Potential. A comparison of the corresponding theorems will be found in C. Neumann's work, *Untersuchungen über das logarithmische und das Newton'sche Potential**.

The Kelvin-Dirichlet Problem and Principle may be stated and demonstrated for the logarithmic potential precisely as in § 86.

93. Dirichlet's Problem for a Circle. Trigonometric Series. We shall call a homogeneous harmonic function of order n of the coordinates x, y of a point in a plane a Circular Harmonic, since it is equal to ρ^n multiplied by a homogeneous function of $\cos\omega$ and $\sin\omega$, and consequently on the circumference of a circle about the origin is simply a trigonometric function of the angular coordinate ω. Any homogeneous function V_n of degree n satisfies the differential equation

$$(1) \qquad x\frac{\partial V_n}{\partial x} + y\frac{\partial V_n}{\partial y} = nV_n,$$

so that a circular harmonic is a solution of this and Laplace's Equation simultaneously. The homogeneous function of degree n

$$a_n x^n + a_{n-1}x^{n-1}y + \ldots\ldots a_1 xy^{n-1} + a_0 y^n$$

contains $n+1$ terms, the sum of its second derivatives is a homogeneous function of degree $n-2$ containing $n-1$ terms, and if this is to vanish identically each of its $n-1$ coefficients must vanish, consequently there are $n-1$ relations between the $n+1$ coefficients of V_n, or only two are arbitrary. Accordingly all har-

* See also Harnack, *Die Grundlagen der Theorie des logarithmischen Potentiales*; Picard, *Traité d'Analyse*, tom. II.

monics of degree n can be expressed in terms of two independent ones. We have found in § 44 that the real and imaginary parts of the function $(x+iy)^n$ are harmonic functions of x, y, being respectively equal to

$$\rho^n \cos n\omega \text{ and } \rho^n \sin n\omega.$$

Accordingly the general harmonic of degree n is

(2) $$V_n = \rho^n \{A_n \cos n\omega + B_n \sin n\omega\} = \rho^n T_n.$$

We may call the trigonometric factor T_n, which is the value of the harmonic on the circumference of a circle of radius unity, the peripheral harmonic of degree n.

If a function which is harmonic in a circular area can be developed in an infinite trigonometric series

(3) $$V(x, y) = \sum_{n=0}^{n=\infty} \{A_n \cos n\omega + B_n \sin n\omega\} = \sum_0^\infty T_n$$

on the circumference of the circle of radius R, the solution of Dirichlet's Problem for the interior of the circle is given by the series

(4) $$V = T_0 + \frac{\rho}{R} T_1 + \frac{\rho^2}{R^2} T_2 + \ldots .$$

For every term is harmonic, and therefore the series, if convergent, is harmonic. At the circumference $\rho = R$, and the series takes the given values of V. The absolute value of every term is less than the absolute value of the corresponding term in the series (3), in virtue of the factor ρ^n/R^n, therefore if the series (3) converges, the series (4) does as well. Since the series fulfils all conditions, by Dirichlet's principle it is the only function satisfying them.

We may fulfil the outer problem by means of harmonics of negative degree. Taking n negative, the series

(5) $$V = T_0 + \frac{R}{\rho} T_1 + \frac{R^2}{\rho^2} T_2 + \ldots$$

is convergent, takes the required values on the circumference, and vanishes at infinity except the constant term. For a ring-shaped area between two concentric circles, we may satisfy the conditions by a series in both positive and negative harmonics,

(6) $$V = \sum_0^\infty \rho^n \{A_n \cos n\omega + B_n \sin n\omega\}$$
$$+ \sum_1^\infty \rho^{-n} \{A_n' \cos n\omega + B_n' \sin n\omega\}.$$

94. Development in Circular Harmonics. We may use the formula (12), § 92, to obtain the development of a function in a trigonometric series on the circumference of a circle. Let the polar coordinates of a point on the circumference of the circle be R, ω and of a point P within the circumference ρ, ϕ. Then we have for the distance between the two points

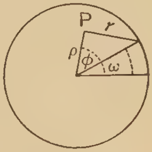

FIG. 45.

$$r = [R^2 + \rho^2 - 2R\rho \cos(\omega - \phi)]^{\frac{1}{2}}.$$

Removing the factor R^2, inserting for $\cos(\omega - \rho)$ its value in exponentials, and separating into factors we obtain

$$(7) \qquad r = R\left[1 + \frac{\rho^2}{R^2} - \frac{\rho}{R}\left(e^{i(\omega-\phi)} + e^{-i(\omega-\phi)}\right)\right]^{\frac{1}{2}}$$

$$= R\left[\left(1 - \frac{\rho}{R}e^{i(\omega-\phi)}\right)\left(1 - \frac{\rho}{R}e^{-i(\omega-\phi)}\right)\right]^{\frac{1}{2}}.$$

Taking the logarithm we may develop

$$\log\left(1 - \frac{\rho}{R}e^{i(\omega-\phi)}\right)$$

and

$$\log\left(1 - \frac{\rho}{R}e^{-i(\omega-\phi)}\right)$$

by Taylor's Theorem, obtaining

$$(8) \quad \log r = \log R - \frac{1}{2}\sum_1^\infty \frac{1}{n}\frac{\rho^n}{R^n}\left(e^{ni(\omega-\phi)} + e^{-ni(\omega-\phi)}\right)$$

$$= \log R - \sum_1^\infty \frac{\rho^n}{nR^n}\cos n(\omega - \phi).$$

This series is convergent if $\rho < R$, and also if $\rho = R$, unless $\omega = \phi$.

Inserting this value of $\log r$ in (12), differentiation with respect to the normal being according to $-R$, we have

$$(9) \quad \begin{aligned} \frac{\partial(\log r)}{\partial n} &= -\left\{\frac{1}{R} + \sum_1^\infty \frac{\rho^n}{R^{n+1}}\cos n(\omega - \phi)\right\}, \\ V_P &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial V}{\partial n}\left(\log R - \sum_1^\infty \frac{\rho^n}{nR^n}\cos n(\omega-\phi)\right)\right. \\ &\qquad \left. + V\left(\sum_0^\infty \frac{\rho^n}{R^{n+1}}\cos n(\omega - \phi)\right)\right\} R\,d\omega. \end{aligned}$$

Expanding the cosines, we may take out from each term of the integral, except the first, a factor $\rho^n \cos n\phi$ or $\rho^n \sin n\phi$, so that V_P is developed as a function of its coordinates ρ, ϕ, in an infinite series of circular harmonics, the coefficients of which are definite integrals around the circumference, involving the peripheral values of V and $\partial V/\partial n$. This does not establish the convergence of the series *on* the circumference. Admitting the possibility of the development, we may proceed to find it in a more convenient form. In order to do this let us apply the last equation to a function V_m, which is a circular harmonic of degree m. Then at the circumference we have

$$V_m = R^m T_m \quad \frac{\partial V_m}{\partial n} = -mR^{m-1} T_m,$$

and

$$(10) \qquad V_m(P) = \frac{R^m(1 - m \log R)}{2\pi} \int_0^{2\pi} T_m d\omega$$
$$+ \frac{1}{2\pi} \sum_{n=1}^{n=\infty} \rho^n R^{m-n} \left(\frac{m}{n} + 1\right) \int_0^{2\pi} T_m \cos n(\omega - \phi)\, d\omega.$$

The expression on the right is an infinite series in powers of ρ, while $V_m(P)$ is simply $\rho^m T_m$. As this equality must hold for all values of ρ less than R, the coefficient of every power of ρ except the mth must vanish, and we have the important equations

$$(11) \qquad \int_0^{2\pi} T_m \cos n(\omega - \phi)\, d\omega = 0, \qquad m \neq n,$$

$$(12) \qquad T_m(\phi) = \frac{1}{\pi} \int_0^{2\pi} T_m(\omega) \cos m(\omega - \phi)\, d\omega,$$

for all values of n, and for all values of m except 0. Since T_0 is a constant, we evidently have

$$T_0 = \frac{1}{2\pi} \int_0^{2\pi} T_0\, d\omega.$$

These two important results can be very simply deduced by direct integration, inserting the value of $T_m(\omega)$, but we have preferred to deduce them as a consequence of Green's formula (12), § 92, in order to show the analogy with Spherical Harmonics. Let us now suppose that the function $V(\omega)$ can be developed in the convergent infinite trigonometric series

$$V(\omega) = \sum_0^{\infty} (A_n \cos n\omega + B_n \sin n\omega) = \sum_0^{\infty} T_n(\omega).$$

Multiply both sides by $\cos m(\omega-\phi)\,d\omega$ and integrate from 0 to 2π.

$$(13)\quad \int_0^{2\pi} V(\omega)\cos m(\omega-\phi)\,d\omega = \sum_0^{\infty}\int_0^{2\pi} T_n(\omega)\cos m(\omega-\phi)\,d\omega.$$

Every term on the right vanishes except the mth which is equal to $\pi T_m(\phi)$. Accordingly we find for the circular harmonic T_m the definite integral

$$(14)\qquad T_m(\phi) = \frac{1}{\pi}\int_0^{2\pi} V(\omega)\cos m(\omega-\phi)\,d\omega.$$

For $m=0$, we must divide by 2.

Writing for $T_m(\phi)$ its value

$$A_m \cos m\phi + B_m \sin m\phi,$$

expanding the cosine in the integral, and writing the two terms separately, we obtain the coefficients

$$(15)\quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} V(\omega)\,d\omega,\quad A_m = \frac{1}{\pi}\int_0^{2\pi} V(\omega)\cos m\omega\,d\omega,$$

$$B_m = \frac{1}{\pi}\int_0^{2\pi} V(\omega)\sin m\omega\,d\omega.$$

This form for the coefficients was given by Fourier*, who *assuming* that the development was possible, was able to determine the coefficients. The question of proving that the development thus found actually represents the function, and the determination of the conditions that the development shall be possible, formed one of the most important mathematical questions of this century, which was first satisfactorily treated by Dirichlet†. For the full and rigid treatment of this important subject, the student should consult Riemann, *Partielle Differentialgleichungen;* Picard, *Traité d'Analyse*, tom. 1, chap. IX.‡

95. Spherical Harmonics. A Spherical Harmonic of degree n is defined as a homogeneous harmonic function of the coordinates x, y, z of a point in space, that is as a solution of the simultaneous equations

* Fourier, *Théorie analytique de la Chaleur*, Chap. IX., 1822.

† Dirichlet, "Sur la Convergence des Séries Trigonométriques," *Crelle's Journal*, Bd. 4, 1829.

‡ A resumé of the literature is given by Sachse, *Bulletin des Sciences Mathématiques*, 1880.

(1) $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(2) $$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV.$$

The general homogeneous function of degree n

$$\begin{aligned} a_{n0}x^n + a_{n-1,0}\,x^{n-1}y + a_{n-2,0}\,x^{n-2}y^2 \ldots\ldots + a_{0,0}y^n \\ + a_{n-1,1}\,x^{n-1}z + a_{n-2,1}\,x^{n-2}yz \ldots\ldots + a_{0,1}y^{n-1}z \\ + a_{n-2,2}\,x^{n-2}z^2 \ldots\ldots + a_{0,2}y^{n-2}z^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + a_{0,n}\,z^n \end{aligned}$$

contains $1 + 2 + 3 \ldots\ldots + n + 1 = (n+1)(n+2)/2$ terms. The sum of its second derivatives is a homogeneous function of degree $n-2$ and accordingly contains $(n-1)\,n/2$ terms. If the function is to vanish identically, these $(n-1)\,n/2$ coefficients must all vanish, so that there are $(n-1)\,n/2$ relations among the $(n+1)(n+2)/2$ coefficients of a harmonic of the nth degree, leaving $2n+1$ *arbitrary* coefficients. The general harmonic of degree n can accordingly be expressed as a linear function of $2n+1$ independent harmonics.

EXAMPLES. Differentiating the arbitrary homogeneous function, and determining the coefficients, we find for $n = 0, 1, 2, 3$, the following independent harmonics:

$n=0$ constant

$n=1$ x, y, z

$n=2$ $x^2 - y^2$, $y^2 - z^2$, xy, yz, zx

$n=3$ $3x^2y - y^3$, $3x^2z - z^3$, $3y^2x - x^3$, $3y^2z - z^3$, $3z^2x - x^3$, $3z^2y - y^3$, xyz.

If we insert spherical coordinates r, θ, ϕ,

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta$$

the harmonic V_n becomes

$$V_n = r^n Y_n(\theta, \phi)$$

where Y_n is a homogeneous function of the trigonometric functions $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$. Y_n being the value of V_n on the surface of a sphere of unit radius, is called a surface harmonic. The equation $Y_n=0$ represents a cone of order n, whose intersection with the sphere gives a geometrical representation of the harmonic V_n.

If u and v be any two continuous functions of x, y, z,

$$\frac{\partial^2(uv)}{\partial x^2}=u\frac{\partial^2 v}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+v\frac{\partial^2 u}{\partial x^2}.$$

(3) $$\Delta(uv)=u\Delta v+v\Delta u+2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right).$$

Put $u=r^m$, and since

$$\frac{\partial(r^m)}{\partial x}=mr^{m-1}\frac{\partial r}{\partial x}=mr^{m-2}x,$$

$$\frac{\partial^2(r^m)}{\partial x^2}=mr^{m-2}+m(m-2)r^{m-4}x^2,$$

we get

(4) $$\Delta(r^m)=3mr^{m-2}+m(m-2)r^{m-4}(x^2+y^2+z^2)$$
$$=m(m+1)r^{m-2}.$$

If V_n is a harmonic of degree n,

(5) $$\Delta(r^mV_n)=r^m\Delta V_n+m(m+1)r^{m-2}V_n$$
$$+2mr^{m-2}\left(x\frac{\partial V_n}{\partial x}+y\frac{\partial V_n}{\partial y}+z\frac{\partial V_n}{\partial z}\right)$$
$$=[m(m+1)+2mn]\,r^{m-2}V_n,$$

by virtue of equations (1) and (2).

Consequently if $m=-(2n+1)$, the product r^mV_n is a harmonic. Since V_n is of degree n, and r is of degree unity in the coordinates, $r^{-(2n+1)}V_n$ is of degree $-(n+1)$. Accordingly to any spherical harmonic $V_n=r^nY_n$ of degree n there corresponds another,

$$V_{-(n+1)}=\frac{V_n}{r^{n+1}},$$

of degree $-(n+1)$. Compare this with the corresponding property of circular harmonics, where the degrees of the two corresponding harmonics are *equal* and opposite.

96. Dirichlet's Problem for Sphere. By means of these harmonics we may solve Dirichlet's problem for the sphere. If a function harmonic within a sphere of radius R can be developed at the surface in an infinite series of surface harmonics,

$$V = Y_0 + Y_1 + Y_2 \ldots\ldots, \tag{6}$$

the internal problem is solved by the series

$$V = Y_0 + \frac{r}{R} Y_1 + \frac{r^2}{R^2} Y_2 + \ldots\ldots. \tag{7}$$

For each term is harmonic, and therefore the series (7), if convergent, is harmonic. At the surface the series takes the given values of V. Every term of the series (7) is less than the corresponding term of the series (6) in virtue of the factor r^n/R^n, therefore if the series (6) converges, the series (7) does as well. Since the series fulfils all the conditions it is the only solution.

We may in like manner fulfil the outer problem by the series of harmonics of negative degree, which vanish at infinity.

$$V = \frac{R}{r} Y_0 + \frac{R^2}{r^2} Y_1 + \frac{R^3}{r^3} Y_2 + \ldots. \tag{8}$$

For the space bounded by two concentric spheres, we must use the series in positive and negative degrees, as will be illustrated by an example in § 198.

97. Forms of Spherical Harmonics. Before considering the question of development in spherical harmonics, we will briefly consider some convenient forms. Since if

$$\Delta V_n = 0,$$

we have

$$\frac{\partial}{\partial x} \Delta V_n = \Delta \frac{\partial V_n}{\partial x} = 0,$$

and any derivative of a harmonic is itself a harmonic, so that

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial^\gamma}{\partial z^\gamma} V_n$$

is a harmonic of degree $n - (\alpha + \beta + \gamma)$. Since to $V_0 = c$ corresponds the harmonic $V_{-1} = c/r$, we have

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \frac{\partial^\gamma}{\partial z^\gamma} \left(\frac{1}{r}\right) = V_{-(1+\alpha+\beta+\gamma)}. \tag{9}$$

If h_1 be any constant direction whose direction cosines are

$$\cos(h_1x) = l_1, \quad \cos(h_1y) = m_1, \quad \cos(h_1z) = n_1,$$

$$\frac{\partial}{\partial h_1} = l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial z},$$

and $\frac{\partial}{\partial h_1}\left(\frac{1}{r}\right)$ is a harmonic of degree -2, and to it corresponds the harmonic,

(10) $$V_1 = r^3\frac{\partial}{\partial h_1}\left(\frac{1}{r}\right),$$

which is of the first degree. Since $l_1^2 + m_1^2 + n_1^2 = 1$, the harmonic contains *two* arbitrary constants, and multiplying by a third, A, we have the general harmonic of degree 1, in the form

(11) $$V_1 = Ar^3\frac{\partial}{\partial h_1}\left(\frac{1}{r}\right).$$

If in like manner $h_2, h_3 \ldots\ldots h_n$, denote vectors with direction cosines $l_2, m_2, n_2 \ldots\ldots l_n, m_n, n_n$,

$$\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\cdots\cdots\frac{\partial}{\partial h_n}\left(\frac{1}{r}\right)$$

is a spherical harmonic of degree $-(n+1)$ and to it corresponds

(12) $$V_n = r^{2n+1}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\cdots\cdots\frac{\partial}{\partial h_n}\left(\frac{1}{r}\right),$$

a harmonic of degree n, and since every h introduces two arbitrary constants, multiplying by another, A, gives us $2n+1$, and we have the general harmonic of degree n in the form,

(13) $$V_n = A\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\cdots\cdots\frac{\partial}{\partial h_n}\left(\frac{1}{r}\right).$$

The directions $h_1, h_2, \ldots\ldots h_n$ are called the *axes* of the harmonic. To illustrate the method of deriving the harmonics we shall find the first two.

$$V_1 = Ar^3\frac{\partial}{\partial h}\left(\frac{1}{r}\right) = Ar^3\left(-\frac{lx}{r^3} - \frac{my}{r^3} - \frac{nz}{r^3}\right) = -A\,(lx + my + nz),$$

$$V_2 = Ar^5\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\left(\frac{1}{r}\right)$$

$$= Ar^5\left(l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial z}\right)\left(l_2\frac{\partial}{\partial x} + m_2\frac{\partial}{\partial y} + n_2\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= -Ar^5 \left(l_1 \frac{\partial}{\partial x} + m_1 \frac{\partial}{\partial y} + n_1 \frac{\partial}{\partial z}\right) \left(\frac{l_2 x + m_2 y + n_2 z}{r^3}\right)$$

$$= -Ar^5 \left\{ l_1 \left(\frac{l_2}{r^3} - \frac{3l_2 x^2}{r^5} - \frac{3m_2 xy}{r^5} - \frac{3n_2 xz}{r^5}\right)\right.$$
$$+ m_1 \left(\frac{m_2}{r^3} - \frac{3m_2 y^2}{r^5} - \frac{3l_2 xy}{r^5} - \frac{3n_2 yz}{r^5}\right)$$
$$\left. + n_1 \left(\frac{n_2}{r^3} - \frac{3n_2 z^2}{r^5} - \frac{3l_2 xz}{r^5} - \frac{3m_2 yz}{r^5}\right)\right\},$$

$$V_2 = A \{-(l_1 l_2 + m_1 m_2 + n_1 n_2)(x^2 + y^2 + z^2)$$
$$+ 3 (l_1 l_2 x^2 + m_1 m_2 y^2 + n_1 n_2 z^2 + (l_1 m_2 + l_2 m_1) xy$$
$$+ (m_1 n_2 + m_2 n_1) yz + (n_1 l_2 + n_2 l_1) zx)\}.$$

The coefficients are of course subject to the relations

$$l_1^2 + m_1^2 + n_1^2 = 1, \quad l_2^2 + m_2^2 + n_2^2 = 1.$$

98. Zonal Harmonics. If all the axes of the harmonic coincide, we may conveniently take the axis for one of the coordinate axes, and write

(14) $$V_n = Ar^{2n+1} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r}\right).$$

It is evident that this will contain only powers of z and r, so that the *surface* harmonic will be simply a polynomial in

$$z/r = \cos(rz).$$

The equation $Y_n(\cos(rz)) = 0$ may be shown to have n real roots lying between 1 and -1, and hence represents n circular cones of angles whose cosines are these roots, intersecting the surface of a sphere in n parallels of latitude which divide the surface into zones. The harmonics are therefore called Zonal Harmonics. The polynomial in $\cos(rz)$ which constitutes the zonal surface harmonic, when the value of the constant A is determined in the manner to be shortly given, is called a Legendre's Polynomial, and denoted by

$$P_n(\cos(rz)) = Ar^{n+1} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r}\right).$$

99. Harmonics in Spherical Coordinates. We have transformed Laplace's Operator into spherical coordinates in § 88, and $\Delta V = 0$ becomes

(15) $$\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta} \frac{\partial V}{\partial \phi}\right) = 0.$$

If we put in this $V_n = r^n Y_n$ we obtain

(16) $$\sin\theta \,.\, n(n+1)\, Y_n + \frac{\partial}{\partial\theta}\left\{\sin\theta\,\frac{\partial Y_n}{\partial\theta}\right\} + \frac{1}{\sin\theta}\,\frac{\partial^2 Y_n}{\partial\phi^2} = 0,$$

as the differential equation satisfied by a surface harmonic $Y_n(\theta, \phi)$. This is the form of Laplace's equation originally given by Laplace*. If Y_n is the zonal harmonic P_n, which is independent of ϕ, we have

(17) $$n(n+1)\, P_n + \frac{1}{\sin\theta}\,\frac{d}{d\theta}\left\{\sin\theta\,\frac{dP_n}{d\theta}\right\} = 0,$$

or putting
$$\cos\theta = \mu,$$

(18) $$\frac{d}{d\mu}\left\{(1-\mu^2)\,\frac{dP_n}{d\mu}\right\} + n(n+1)\, P_n = 0.$$

This is known as Legendre's Differential Equation. We shall now, without considering more in detail the general surface harmonic, find the general expression for the zonal harmonic. It may be at once shown, by inserting for $P_n(\mu)$ a power-series in μ and determining the coefficients, that for integral values of n the differential equation is satisfied by a polynomial in μ. The form of these polynomials we shall find from one of their important properties.

100. Development of Reciprocal Distance. We know that $1/r$, the reciprocal of the distance of the point x, y, z from any fixed point P, is a harmonic function of the coordinates x, y, z, and although it is not a homogeneous function except when the fixed point is the origin, it may always be developed in a series of homogeneous functions, that is, in a series of spherical harmonics. We shall now use the letter d for the distance from any fixed point, reserving r for the distance from the origin. Let us for convenience take the axis of z as passing through the fixed point P, which lies at a distance r' from the origin, and put $\cos(rz) = \mu$. Then we have

(19) $$\frac{1}{d} = [r^2 + r'^2 - 2rr'\mu]^{-\frac{1}{2}} = [x^2 + y^2 + (z - r')^2]^{-\frac{1}{2}}.$$

Considering this as a function of z let us develop by Taylor's Theorem,

(20) $$\frac{1}{d} = f(z - r') = f(z) + (-r')\left(\frac{\partial f}{\partial z}\right)_{r'=0} + \frac{1}{2\,!}(-r')^2\left(\frac{\partial^2 f}{\partial z^2}\right)_{r'=0} + \ldots\ldots$$

* Laplace, "Théorie des attractions des sphéroïdes et de la figure des planètes." *Mém. de l'Acad. de Paris.* Année 1782 (pub. 1785).

and since for $r'=0$, $\dfrac{1}{d}=\dfrac{1}{r}$, $\dfrac{\partial^n f}{\partial z^n}=\dfrac{\partial^n}{\partial z^n}\left(\dfrac{1}{r}\right)$,

(21) $$\frac{1}{d}=\frac{1}{r}+(-r')\frac{\partial}{\partial z}\left(\frac{1}{r}\right)+\frac{1}{2\,!}(-r')^2\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)+\ldots.$$

Now multiplying and dividing each term by r^{n+1}, we find

(22) $$\frac{1}{d}=\frac{1}{r}\left\{P_0+\frac{r'}{r}P_1+\frac{r'^2}{r^2}P_2+\ldots+\frac{r'^n}{r^n}P_n+\ldots\ldots\right\},$$

where

$$P_0=1,\ P_n=\frac{(-1)^n}{n\,!}r^{n+1}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

This is the determination of the constant A adopted by Legendre, for the reason that, since by the binomial theorem, for $r'<r$, and $\mu=1$,

$$\frac{1}{d}=\frac{1}{r}\left\{1-\frac{r'}{r}\right\}^{-1}=\frac{1}{r}\left\{1+\frac{r'}{r}+\frac{r'^2}{r^2}+\ldots\ldots+\frac{r'^n}{r^n}+\ldots\ldots\right\}$$

it makes for every n,

(23) $$P_n(1)=1.$$

The term P_n/r^{n+1} is a spherical harmonic of degree $-(n+1)$, and the series (22) is convergent for $r'<r$. In like manner if $r'>r$ we find

(24) $$\frac{1}{d}=\frac{1}{r'}\left\{P_0+\frac{r}{r'}P_1+\frac{r^2}{r'^2}P_2+\ldots\ldots\frac{r^n}{r'^n}P_n+\ldots\ldots\right\}.$$

In order to find P_n as a polynomial in μ we may write r/d as

$$\frac{r}{d}=\left[1-2\frac{r'}{r}\left(\mu-\frac{r'}{2r}\right)\right]^{-\frac{1}{2}}$$

and develop by the binomial theorem.

(25) $$\frac{r}{d}=\sum_{s=0}^{s=\infty}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots\ldots(\frac{1}{2}+s-1)}{s\,!}\left(2\frac{r'}{r}\right)^s\left(\mu-\frac{r'}{2r}\right)^s.$$

Developing the last factor

$$\left(\mu-\frac{r'}{2r}\right)^s=\sum_{r=0}^{r=s}(-1)^r\frac{s(s-1)\ldots\ldots(s-r+1)}{r\,!}\mu^{s-r}\left(\frac{r'}{2r}\right)^r.$$

(26) $$\frac{r}{d}=\sum_{s=0}^{s=\infty}\sum_{r=0}^{r=s}(-1)^r2^{(s-r)}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots\ldots(\frac{1}{2}+s-1)}{r\,!\,(s-r)\,!}\left(\frac{r'}{r}\right)^{s+r}\mu^{s-r}.$$

Picking out all the terms for which $s+r=n$ we get for the coefficient of $\left(\frac{r'}{r}\right)^n$

$$(27)\qquad P_n=\frac{1.3.5\ldots\ldots(2n-1)}{n!}\left[\mu^n-\frac{n(n-1)}{1.2(2n-1)}\mu^{n-2}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\mu^{n-4}\ldots\ldots\right].$$

The first polynomials have the values

$$P_0(\mu)=1,$$
$$P_1(\mu)=\mu,$$
$$P_2(\mu)=\tfrac{1}{2}(3\mu^2-1),$$
$$P_3(\mu)=\tfrac{1}{2}(5\mu^3-3\mu),$$
$$P_5(\mu)=\tfrac{1}{8}(63\mu^5-70\mu^3+15\mu).$$

101. Development in Spherical Harmonics. We may use the formula (6) § 83 for an internal point, to obtain the development of a function of θ, ϕ, on the surface of a sphere in the same manner as in § 94 for the case of a circle. Since the polynomials in the development of the reciprocal distance involve only the cosine of the angle between the radii to the fixed and variable points, we have if $r'<r$,

$$(22)\qquad \frac{1}{d}=\frac{1}{r}\sum_0^\infty\left(\frac{r'}{r}\right)^s P_s(\mu)\qquad \mu=\cos(r'r),$$

and differentiating this with respect to $-r$, the internal normal,

$$(28)\qquad \frac{\partial\left(\frac{1}{d}\right)}{\partial n}=-\frac{\partial\left(\frac{1}{d}\right)}{\partial r}=\sum_0^\infty(s+1)\frac{r'^s}{r^{s+2}}P_s(\mu).$$

Inserting these values in (6), § 83, namely

$$V_{(P)}=\frac{1}{4\pi}\iint\left\{V\frac{\partial\left(\frac{1}{d}\right)}{\partial n}-\frac{1}{d}\frac{\partial V}{\partial n}\right\}dS,$$

and applying it to the case that V is a spherical harmonic

$$V_m=r^m Y_m,$$

we obtain, since

$$dS=r^2\sin\theta\, d\theta\, d\phi,$$

$$(29)\quad V_m(P)=\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\left\{r^m Y_m\sum_0^\infty(s+1)\frac{r'^s}{r^{s+2}}P_s(\mu)\right.$$
$$\left.+mr^{m-2}Y_m\sum_0^\infty\left(\frac{r'}{r}\right)^s P_s(\mu)\right\}r^2\sin\theta\, d\theta\, d\phi.$$

If the coordinates of P be r', θ', ϕ' we have,

$$V_m(P) = r'^m\, Y_m(\theta', \phi'),$$

while on the right we have an infinite series in powers of r', with definite integrals as coefficients. Since the equality must hold for all values of r' less than r, we must have, collecting in terms in r'^s

(30)
$$\int_0^\pi \int_0^{2\pi} Y_m(\theta, \phi)\, P_s(\mu) \sin\theta\, d\theta d\phi = 0, \qquad s \neq m,$$

$$Y_m(\theta', \phi') = \frac{m+s+1}{4\pi} \int_0^\pi \int_0^{2\pi} Y_m(\theta, \phi)\, P_s(\mu) \sin\theta d\theta d\phi, \quad s = m,$$

so that we have for the values of the integral

(31)
$$\int_0^\pi \int_0^{2\pi} Y_m(\theta, \phi)\, P_m(\mu) \sin\theta\, d\theta d\phi = \frac{4\pi}{2m+1} Y_m(\theta', \phi').$$

In performing the integration, we must put for μ the value obtained by spherical trigonometry,

$$\mu = \cos(rr') = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

By means of the above integral expressions (30) and (31) we may find the development of a function of θ, ϕ, *assuming that the development is possible.* Suppose we are to find the development

(32)
$$f(\theta, \phi) = Y_0 + Y_1 + Y_2 + \ldots\ldots$$

Multiply both sides by $P_n(\mu) \sin\theta d\theta d\phi$, and integrate over the surface of the sphere, and since every term vanishes except the nth we obtain

(33)
$$\int_0^\pi \int_0^{2\pi} f(\theta, \phi)\, P_n(\mu) \sin\theta\, d\theta d\phi = \frac{4\pi}{2n+1} Y_n(\theta', \phi'),$$

(34)
$$Y_n(\theta', \phi') = \frac{2n+1}{4\pi} \int_0^\pi \int_0^{2\pi} f(\theta, \phi)\, P_n(\mu) \sin\theta d\theta d\phi.$$

Accordingly to find the value of any term Y_n at any point P, (θ', ϕ') we find the zonal surface harmonic whose axis passes through the point P, multiply its value at every point of the sphere by the value of f for that point, and integrate the product over the surface. It remains to show that the development is possible, that is that the sum of the series

$$\frac{1}{4\pi} \sum_0^\infty (2n+1) \int_0^\pi \int_0^{2\pi} f(\theta, \phi)\, P_n(\mu) \sin\theta d\theta d\phi,$$

actually represents the function $f(\theta', \phi')$. This theorem was demonstrated by Laplace, but without sufficient rigor, afterwards

by Poisson, and finally in a rigorous manner by Dirichlet. A proof due to Darboux is given by Jordan, *Traité d'Analyse*, Tom. II. p. 249 (2me éd.).

102. Potential of Circular Disc at points not on axis. We have found in § 81 the potential of a disc of surface density σ, radius R, at a point situated at a distance r from the center on the axis to be

$$(1)\qquad V = 2\pi\sigma\{\sqrt{r^2+R^2} - r\}.$$

Developing by the binomial theorem for the two cases $r < R$, $r > R$,

$$(2)\qquad V = 2\pi\sigma\left\{R\left(1+\frac{r^2}{R^2}\right)^{\frac{1}{2}} - r\right\}$$

$$= 2\pi\sigma\left\{-r + R + \frac{1}{2}\frac{r^2}{R} - \frac{1}{2}\cdot\frac{1}{4}\frac{r^4}{R^3} + \frac{1.1.3}{2.4.6}\frac{r^6}{R^5}\ldots\ldots\right\} \quad r < R.$$

$$(3)\qquad V = 2\pi\sigma\left\{r\left(1+\frac{R^2}{r^2}\right)^{\frac{1}{2}} - r\right\}$$

$$= 2\pi\sigma\left\{\frac{1}{2}\frac{R^2}{r} - \frac{1.1}{2.4}\frac{R^4}{r^3} + \frac{1.1.3}{2.4.6}\frac{R^6}{r^5}\ldots\ldots\right\} \qquad r > R.$$

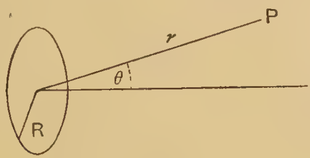

FIG. 46.

If now the point be not on the axis, but on a line through the center making an angle $\theta < \frac{\pi}{2}$ with the axis, and at a distance r from the center, we may put

$$(4)\qquad V = 2\pi\sigma\left\{R - rP_1(\cos\theta) + \frac{1}{2}\frac{r^2}{R}P_2(\cos\theta)\right.$$

$$\left. - \frac{1}{2}\cdot\frac{1}{4}\frac{r^4}{R^3}P_4(\cos\theta)\ldots\ldots\right\} \qquad r < R,$$

$$(5)\qquad V = 2\pi\sigma\left\{\frac{1}{2}\frac{R^2}{r} - \frac{1.1}{2.4}\frac{R^4}{r^3}P_2(\cos\theta)\right.$$

$$\left. + \frac{1.1.3}{2.4.6}\frac{R^6}{r^5}P_4(\cos\theta)\ldots\ldots\right\} \qquad r > R.$$

For both of these series are convergent under the assumptions made, both are harmonic, for $r^n P_n$ and P_n/r^{n+1} are zonal harmonics, and both take the given values when $\theta = 0$.

103. Equations of Lines of Force. In § 36, we have considered the integration of the differential equations of the lines of any solenoidal vector-function and have found that the lines may be represented as the intersections of two families of surfaces. We shall now consider the same subject in terms of generalized orthogonal curvilinear coordinates. Let us call the components of the force at any point q_1, q_2, q_3 in the directions of the coordinate axes at the point, R_1, R_2, R_3, which by § 16 are

$$(1) \qquad R_1 = -h_1\frac{\partial V}{\partial q_1}, \quad R_2 = -h_2\frac{\partial V}{\partial q_2}, \quad R_3 = -h_3\frac{\partial V}{\partial q_3}.$$

If now ds be an element of a line of force, its projections on the three axes being

$$ds_1 = \frac{dq_1}{h_1}, \quad ds_2 = \frac{dq_2}{h_2}, \quad ds_3 = \frac{dq_3}{h_3},$$

we have

$$(2) \qquad ds_1 : ds_2 : ds_3 = R_1 : R_2 : R_3,$$

or

$$dq_1 : dq_2 : dq_3 = h_1R_1 : h_2R_2 : h_3R_3,$$

so that the differential equations of the line of force are

$$(3) \qquad dq_1 : dq_2 : dq_3 = h_1^2\frac{\partial V}{\partial q_1} : h_2^2\frac{\partial V}{\partial q_2} : h_3^2\frac{\partial V}{\partial q_3},$$

or, dividing by $h_1h_2h_3$,

$$(4) \quad dq_1 : dq_2 : dq_3 = \frac{h_1}{h_2h_3}\frac{\partial V}{\partial q_1} : \frac{h_2}{h_3h_1}\frac{\partial V}{\partial q_2} : \frac{h_3}{h_1h_2}\frac{\partial V}{\partial q_3} = Q_1 : Q_2 : Q_3,$$

while we have by Laplace's equation the relation, (§ 87 (5))

$$\frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2h_3}\frac{\partial V}{\partial q_1}\right\} + \frac{\partial}{\partial q_2}\left\{\frac{h_2}{h_3h_1}\frac{\partial V}{\partial q_2}\right\} + \frac{\partial}{\partial q_3}\left\{\frac{h_3}{h_1h_2}\frac{\partial V}{\partial q_3}\right\} = 0,$$

that is

$$(5) \qquad \frac{\partial Q_1}{\partial q_1} + \frac{\partial Q_2}{\partial q_2} + \frac{\partial Q_3}{\partial q_3} = 0.$$

We may now use the principle of the last multiplier demonstrated in § 37, replacing x, y, z, by q_1, q_2, q_3 and X, Y, Z by Q_1, Q_2, Q_3. That is to say, if we have found an integral

$$\lambda(q_1, q_2, q_3) = \text{const.},$$

we may obtain the other at once by a quadrature as

$$(6) \qquad \mu = \int \frac{1}{\dfrac{\partial\lambda}{\partial q_3}}(Q_2dq_1 - Q_1dq_2) = \text{const.}$$

and inserting the values of Q_2, Q_1,

(7) $$\mu = \int \frac{1}{\frac{\partial \lambda}{\partial q_3} h_3} \left(\frac{h_2}{h_1} \frac{\partial V}{\partial q_2} dq_1 - \frac{h_1}{h_2} \frac{\partial V}{\partial q_1} dq_2 \right) = \text{const.},$$

where of course all the functions under the integral are to be expressed in terms of q_1, q_2, λ. This principle will be made use of in the treatment of the flow of electric currents in thin curved surfaces.

The theorem becomes very simple in two particular applications. First let q_1, q_2, q_3 be rectangular coordinates x, y, z, and let V be independent of z, that is, the problem is uniplanar, or the lines of force lie in planes all parallel to the Z-plane. Then $\lambda = z =$ const. is one integral and the other is

(8) $$\mu = \int \left(\frac{\partial V}{\partial y} dx - \frac{\partial V}{\partial x} dy \right) = \text{const.}$$

From this we obtain

(9) $$d\mu = \frac{\partial \mu}{\partial x} dx + \frac{\partial \mu}{\partial y} dy = \frac{\partial V}{\partial y} dx - \frac{\partial V}{\partial x} dy,$$
$$\frac{\partial \mu}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial \mu}{\partial y} = -\frac{\partial V}{\partial x},$$

and the function μ is the function *conjugate* to the potential function V, as found in § 42. Since by § 36 the flux of the vector R across any cross-section of a vector tube defined by four surfaces λ, $\lambda + d\lambda$, μ, $\mu + d\mu$ is $d\lambda d\mu$, the function μ represents the flux through a tube bounded by two parallel planes $z = 0$, $z = 1$, by the surface $\mu = 0$, and by the surface $\mu =$ const. If the vector R represent the velocity of a fluid motion, μ is called Earnshaw's current function, and the amount of fluid crossing unit height perpendicular to the Z-plane of any cylindrical surface projected into a curve on the Z-plane is given by the difference in the values of μ at the two ends of the curve. We may call the function μ for any vector the *flux-function.*

In the second case let q_1, q_2, q_3 be cylindrical coordinates ρ, ω, z, and let V be independent of ω, so that the lines of force are in planes intersecting in the Z-axis. The figure is then symmetrical around this axis, and we have a problem of revolution. We then have an integral $\lambda = \omega =$ const. and for the second,

(10) $$\mu = \int \rho \left\{ \frac{\partial V}{\partial z} d\rho - \frac{\partial V}{\partial \rho} dz \right\} = \text{const.}$$

The function μ represents the flux for any tube bounded by the surfaces $\mu=0$, $\mu=\text{const.}$ and two planes through the Z-axis making a dihedral angle with each other equal to unity, and μ is then called Stokes's current- or flux-function.

104. Functions of Complex Variable on Surface. Both of the cases just considered are cases of a class of problems of considerable generality. If the vector lines lie in one of the coordinate surfaces itself, we have the particular integral $\lambda=q_3=\text{const.}$, and accordingly

(1) $$\mu=\int\left\{\frac{h_2}{h_3h_1}\frac{\partial V}{\partial q_2}dq_1-\frac{h_1}{h_2h_3}\frac{\partial V}{\partial q_1}dq_2\right\}=\text{const.}$$

or the differential

(2) $$d\mu=\frac{\partial\mu}{\partial q_1}dq_1+\frac{\partial\mu}{\partial q_2}dq_2=\frac{h_2}{h_3h_1}\frac{\partial V}{\partial q_2}dq_1-\frac{h_1}{h_2h_3}\frac{\partial V}{\partial q_1}dq_2.$$

From this we must have

(3) $$\frac{\partial\mu}{\partial q_1}=\frac{h_2}{h_3h_1}\frac{\partial V}{\partial q_2},\quad -\frac{\partial\mu}{\partial q_2}=\frac{h_1}{h_2h_3}\frac{\partial V}{\partial q_1}.$$

Differentiating the first of these equations by q_2, the second by q_1, and adding,

(4) $$\frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_3h_2}\frac{\partial V}{\partial q_1}\right\}+\frac{\partial}{\partial q_2}\left\{\frac{h_2}{h_1h_3}\frac{\partial V}{\partial q_2}\right\}=0.$$

Expressing the derivatives of V in terms of those of μ

(5) $$\frac{\partial V}{\partial q_2}=\frac{h_3h_1}{h_2}\frac{\partial\mu}{\partial q_1},\quad -\frac{\partial V}{\partial q_1}=\frac{h_2h_3}{h_1}\frac{\partial\mu}{\partial q_2}.$$

Differentiating the first by q_1, the second by q_2, and adding,

(6) $$\frac{\partial}{\partial q_1}\left\{\frac{h_3h_1}{h_2}\frac{\partial\mu}{\partial q_1}\right\}+\frac{\partial}{\partial q_2}\left\{\frac{h_2h_3}{h_1}\frac{\partial\mu}{\partial q_1}\right\}=0.$$

Now if h_3 is independent of q_1 and q_2, which will be the case if two consecutive surfaces q_3, q_3+dq_3 are parallel, or everywhere the same distance apart, namely,

$$dn_3=\frac{dq_3}{h_3},$$

then h_3 comes out as a factor of both differential equations, and we find that V and μ satisfy the same differential equation

(7) $$\frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2}\frac{\partial f}{\partial q_1}\right\}+\frac{\partial}{\partial q_2}\left\{\frac{h_2}{h_1}\frac{\partial f}{\partial q_2}\right\}=0.$$

In this case from the solution of any problem for the surface q_3 we may by interchanging the functions V and μ obtain the solution of a new problem, as in the case of uniplanar problems. But this is not the full extent of the analogy. We have for the length of the arc of any curve on the surface q_3, by § 20,

$$ds^2 = \frac{dq_1^2}{h_1^2} + \frac{dq_2^2}{h_2^2}.$$

If we can find any two functions $u(q_1, q_2)$, $v(q_1, q_2)$ such that

$$du^2 + dv^2 = M ds^2,$$

where M is a function of the position of the point, and does not involve the differentials dq_1, dq_2, we have

$$du^2 + dv^2 = M\left(\frac{dq_1^2}{h_1^2} + \frac{dq_2^2}{h_2^2}\right). \tag{8}$$

Now each member of the last equation may be factored into complex factors linear in dq_1, dq_2,

$$(du + idv)(du - idv) = M\left(\frac{dq_1}{h_1} + i\frac{dq_2}{h_2}\right)\left(\frac{dq_1}{h_1} - i\frac{dq_2}{h_2}\right). \tag{9}$$

Each of the four factors in this equation is linear in dq_1 and dq_2, the first, for instance, being

$$\left(\frac{\partial u}{\partial q_1} + i\frac{\partial v}{\partial q_1}\right)dq_1 + \left(\frac{\partial u}{\partial q_2} + i\frac{\partial v}{\partial q_2}\right)dq_2.$$

Now if a product of two linear forms is identically equal to a product of two others, each factor on one side of the equation must be a multiple of one of those on the other, so that in this case we must have either

$$du + idv = \left(\frac{dq_1}{h_1} + i\frac{dq_2}{h_2}\right)\phi,$$

or

$$du + idv = \left(\frac{dq_1}{h_1} - i\frac{dq_2}{h_2}\right)\psi,$$

where ϕ and ψ are independent of the differentials dq_1, dq_2.

If we put

$$du + idv = \left(\frac{dq_1}{h_1} + i\frac{dq_2}{h_2}\right)\phi, \tag{10}$$

that is

$$\frac{\partial u}{\partial q_1}dq_1 + \frac{\partial u}{\partial q_2}dq_2 + i\left(\frac{\partial v}{\partial q_1}dq_1 + \frac{\partial v}{\partial q_2}dq_2\right) = \left(\frac{dq_1}{h_1} + i\frac{dq_2}{h_2}\right)\phi, \tag{11}$$

equating the coefficients of dq_1, dq_2, we obtain

$$(12)\qquad \begin{aligned} \frac{\partial u}{\partial q_1} + i\frac{\partial v}{\partial q_1} &= \frac{\phi}{h_1}, \\ \frac{\partial u}{\partial q_2} + i\frac{\partial v}{\partial q_2} &= i\frac{\phi}{h_2}. \end{aligned}$$

Now eliminating ϕ,

$$(13)\qquad h_2\left\{\frac{\partial u}{\partial q_2} + i\frac{\partial v}{\partial q_2}\right\} = ih_1\left\{\frac{\partial u}{\partial q_1} + i\frac{\partial v}{\partial q_1}\right\},$$

and equating the real parts on each side, and the imaginary parts in like manner we obtain

$$(14)\qquad h_2\frac{\partial u}{\partial q_2} = -h_1\frac{\partial v}{\partial q_1}, \quad h_2\frac{\partial v}{\partial q_2} = h_1\frac{\partial u}{\partial q_1}.$$

Solving for the derivatives of v

$$(15)\qquad -\frac{\partial v}{\partial q_1} = \frac{h_2}{h_1}\frac{\partial u}{\partial q_2}, \quad \frac{\partial v}{\partial q_2} = \frac{h_1}{h_2}\frac{\partial u}{\partial q_1},$$

differentiating respectively by q_2 and q_1, and adding

$$(16)\qquad \frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2}\frac{\partial u}{\partial q_2}\right\} + \frac{\partial}{\partial q_2}\left\{\frac{h_2}{h_1}\frac{\partial u}{\partial q_2}\right\} = 0.$$

Solving for the derivatives of u

$$(17)\qquad -\frac{\partial u}{\partial q_2} = \frac{h_1}{h_2}\frac{\partial v}{\partial q_1}, \quad \frac{\partial u}{\partial q_1} = \frac{h_2}{h_1}\frac{\partial v}{\partial q_2},$$

differentiating and adding

$$(18)\qquad \frac{\partial}{\partial q_1}\left(\frac{h_1}{h_2}\frac{\partial v}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_2}{h_1}\frac{\partial v}{\partial q_2}\right) = 0,$$

that is, the functions u and v satisfy the same equation as the potential and flux-functions V and μ. Such a pair of functions, forming a set of orthogonal coordinate lines on the surface q_3, may accordingly be taken for the potential and flux-function. If we have a second pair of functions u', v' such that

$$du'^2 + dv'^2 = M'ds^2,$$

we have as before

$$du' + idv' = \left(\frac{dq_1}{h_1} + i\frac{dq_2}{h_2}\right)\phi',$$

$$\frac{du' + idv'}{du + idv} = \frac{\phi'}{\phi},$$

which is the condition, § 42, that the complex variable $u'+iv'$ is an analytic function of $u+iv$. Thus from the solution of one problem for the surface q_3 may be deduced the solution of any number of others for the same surface.

If now the quantities u, v be taken as rectangular coordinates in a plane, the arc of any curve is expressed in the form,

$$d\sigma^2 = du^2 + dv^2.$$

To any point u, v in the plane corresponds a point with the same values of u, v, on the surface q_3. In virtue of the relation

$$d\sigma^2 = M ds^2$$

between corresponding arcs on the plane and on the surface, we see, as in § 43, that corresponding infinitesimal triangles are similar, or the surface q_3 is conformally represented upon the plane. If the UV-plane is conformally transformed to another plane XY, we have seen that we have $u+iv$ an analytic function of the complex variable $x+iy$ and the real functions u, v are potential and flux-functions in the XY-plane.

As we have just proved that they retain this property on the surface q_3, we see that the method of the functions of a complex variable will give us the solution of any number of cases upon a surface, and that the surface may be conformally represented on the plane in an infinite number of ways. Such a representation of a surface on a plane constitutes a map. Surfaces which may be conformally represented on a plane may be conformally represented on each other. The theory of such transformations is the subject of an important memoir by Gauss*. The method here given is due to Beltrami†, and may be applied even when the coordinates q_1, q_2 are not orthogonal. The method is particularly applicable to the case of electrical currents flowing in thin conducting surfaces, and the conformal transformations may be found by experiment. A thin space bounded by two surfaces q_3 in which is distributed a solenoidal vector which may be represented by a potential or by a flux-function as here described, is termed a *vector-sheet.*

* Gauss, "Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird." *Werke*, Bd. iv., p. 189.

† Beltrami, "Delle variabili complesse sopra una superficie qualunque." *Annali di Matematica*, ser. 2, t. I., p. 329.

105. Example. Conformal Representation of Sphere on Plane. Let the surface q_3 be a sphere of radius R, and take for the coordinates q_1 and q_2, the co-latitude θ and the longitude ϕ. Then by § 17, we have

$$h_1 = \frac{1}{R}, \quad h_2 = \frac{1}{R\sin\theta}, \quad h_3 = 1,$$

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) = (du^2 + dv^2)M,$$

and the differential equation satisfied by u and v is

$$\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial u}{\partial\theta}\right\} + \frac{\partial}{\partial\phi}\left\{\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}\right\} = 0.$$

If we take

$$\sqrt{M}\,du = Rd\theta, \quad \sqrt{M}\,dv = R\sin\theta d\phi,$$

and if we choose $\sqrt{M} = R\sin\theta$, then

$$dv = d\phi, \quad du = \frac{d\theta}{\sin\theta}.$$

Integrating we obtain

$$v = \phi, \quad u = \log\tan\frac{\theta}{2}.$$

If now we take u and v for rectangular coordinates in a plane, the surface of the sphere is conformally represented upon the plane by means of the above transformation. This particular representation is known as Mercator's Projection. The meridians $\phi = \text{const.}$ correspond to the straight lines $v = \text{const.}$, and the parallels $\theta = \text{const.}$ correspond to the lines $u = \text{const.}$*

Since the whole sphere is covered by a variation of ϕ between the limits 0, 2π, the projection on the plane has the finite width 2π, but the length of the projection is infinite, the poles $\theta = 0$, $\theta = \pi$ corresponding to $u = -\infty$, $u = \infty$. If we make a conformal transformation of the UV-plane by means of the function

$$u + iv = \log(x + iy),$$

we obtain the formulae,

$$u = \log r = \log\sqrt{x^2 + y^2}, \quad v = \tan^{-1}\frac{y}{x},$$

$$\sqrt{x^2 + y^2} = r = \tan\frac{\theta}{2}, \quad \phi = \tan^{-1}\frac{y}{x},$$

* For an example see Fig. 71, § 177.

which give a new conformal representation of the sphere on the plane, the meridians corresponding to radial lines $y/x = \tan\phi$, and the parallels to concentric circles. This is the stereographic projection, obtained by projecting points on the sphere upon a plane tangent at one pole from the other pole as a center of projection. Figure 23 projected upon the sphere by this transformation is shown in perspective in Fig. 47.

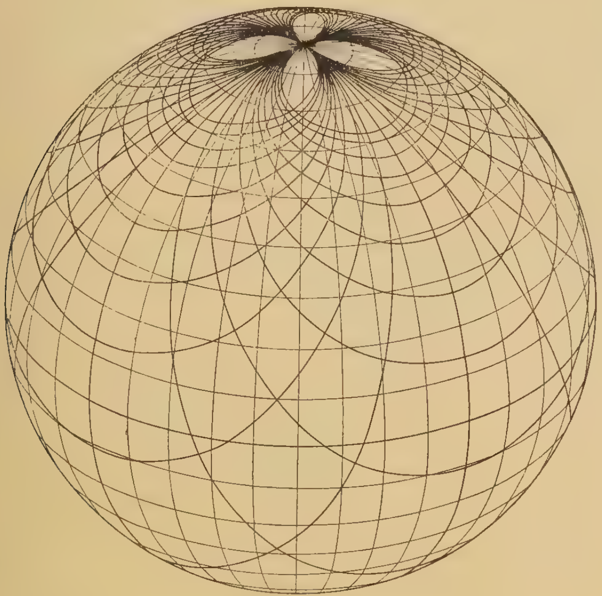

Fig. 47.

106. Diagrams. If we have a diagram representing a plane section of a set of equipotential surfaces, corresponding to equal increments of potential, and we superpose upon this a second diagram representing a second set of equipotential surfaces, drawn for the same differences of potential, we may draw the curves representing the equipotentials due to a distribution which is the sum or difference of the other two by simply drawing lines connecting opposite corners of the curvilinear quadrilaterals into which the diagram is divided by the two equipotential systems. For as we go from vertex to vertex, the increase of potential due to one system is just counterbalanced by the decrease due to the other. Fig. 49 represents a combination of Fig. 23 with a straight field in this manner.

In like manner if we have diagrams of the flux-function of any two systems superposed, we may draw diagrams of the flux-function of the sum or difference of the two systems, for if we consider two flux tubes bounded by the lines AB, CD, and $A'B'$, $C'D'$, Fig. 48, the line PQ has the flux $\Delta\mu$ through it in opposite directions from the two systems, so that the total flux through it is zero, or it is a flux-line. In this manner the Figures 49*, 72, 74, 75, 76, 77, 78 have been drawn.

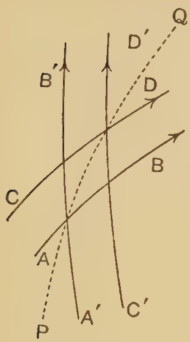

FIG. 48.

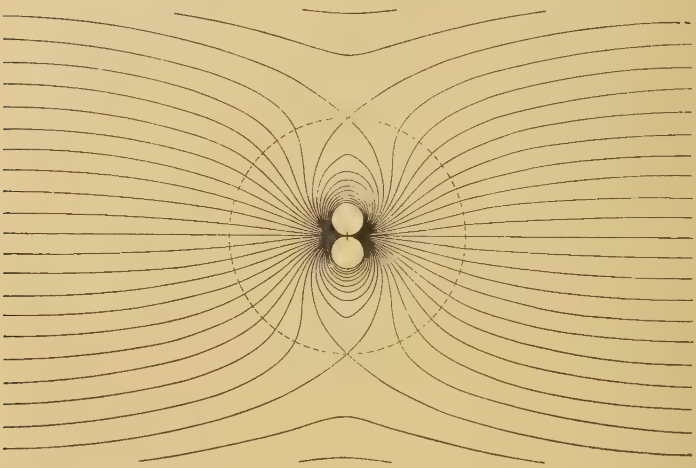

FIG. 49.

* Fig. 49 is to be considered a diagram of lines of flow or of equipotentials according as the directions of the component vectors at the origin are the same or opposite. The analogous cases of the rotational problem are represented in Figs. 74, 75.

CHAPTER V.

ATTRACTION OF ELLIPSOIDS. ENERGY. POLARIZED DISTRIBUTIONS.

107. Ellipsoidal Homœoids. Newton's Theorem. If we transform Laplace's equation to elliptic coordinates and attempt to apply the methods of § 88 to the problem of finding the potential of a homogeneous ellipsoid, we are at once confronted with a difficulty. It is not evident, nor is it true, that the potential is independent of two of the coordinates, and that the equipotential surfaces are ellipsoids.

The following theorem was proved geometrically by Newton. A shell of homogeneous matter bounded by two similar and similarly placed ellipsoids exerts no force on a point placed anywhere within the cavity. Such a shell will be called an *ellipsoidal homœoid.*

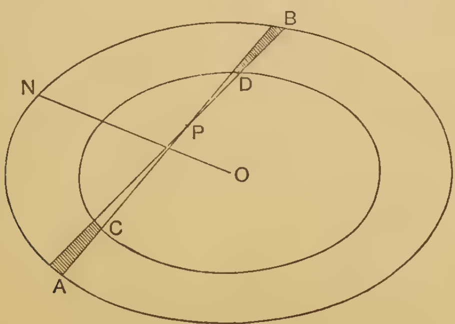

Fig. 49 *a*.

Let P, Fig. 49 a, be the attracted point inside. Since the attraction of a cone of solid angle $d\omega$ on a point of unit mass at

its vertex is

$$\int_0^r \frac{dm}{r^2} = \int_0^r \frac{r^2\, d\omega\, dr}{r^2} = r d\omega,$$

we have for an element of the homœoid the attraction

$$d\omega\,(BP - DP),$$

in one direction, and

$$d\omega\,(AP - CP)$$

in the other, or in the direction PB,

$$d\omega\,(BD - AC).$$

Draw a plane through ABO, and let ON be the chord of the elliptical section conjugate to AB. Since the ellipsoids are similar and similarly placed, the same diameter is conjugate to the chord CD in both. But CD and AB being bisected in the same point,

$$AC = BD,$$

and the attraction of every part is counterbalanced by that of the opposite part.

108. Condition for Infinite Family of Equipotentials. Although the equipotentials of an ellipsoid are not in general ellipsoids, we may inquire whether there is any distribution of mass that will have ellipsoids as equipotential surfaces.

Let us examine, in general, whether any singly infinite system of surfaces

$$F(x, y, z, q) = 0$$

can be equipotential surfaces. If so, for any particular value of the parameter q, V must be constant, in other words $V = f(q)$. If x, y, z are given, q is found from $F(x, y, z, q) = 0$ and from that V from the preceding equation.

Now in free space, V satisfies the equation $\Delta V = 0$. But, since V is a function of q only,

$$\frac{\partial V}{\partial x} = \frac{dV}{dq}\frac{\partial q}{\partial x},$$

$$(1) \qquad \frac{\partial^2 V}{\partial x^2} = \frac{dV}{dq}\frac{\partial^2 q}{\partial x^2} + \frac{\partial q}{\partial x}\frac{\partial}{\partial x}\left(\frac{dV}{dq}\right)$$

$$= \frac{dV}{dq}\frac{\partial^2 q}{\partial x^2} + \left(\frac{\partial q}{\partial x}\right)^2 \frac{d^2 V}{dq^2}.$$

In like manner

$$\frac{\partial^2 V}{\partial y^2} = \frac{dV}{dq}\frac{\partial^2 q}{\partial y^2} + \left(\frac{\partial q}{\partial y}\right)^2 \frac{d^2 V}{dq^2},$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{dV}{dq}\frac{\partial^2 q}{\partial z^2} + \left(\frac{\partial q}{\partial z}\right)^2 \frac{d^2 V}{dq^2},$$

(2)
$$\Delta V = \frac{dV}{dq}\Delta q + \left\{\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 + \left(\frac{\partial q}{\partial z}\right)^2\right\}\frac{d^2 V}{dq^2}$$

$$= \frac{dV}{dq}\Delta q + \frac{d^2 V}{dq^2} h_q^2 = 0.$$

Accordingly

(3)
$$\frac{\Delta q}{h_q^2} = -\frac{\dfrac{d^2 V}{dq^2}}{\dfrac{dV}{dq}} = -\frac{d}{dq}\left(\log\frac{dV}{dq}\right).$$

Now since V is a function of q only, the expression on the right must be a function of q only, say $\phi(q)$. Consequently, that

$$F(x, y, z, q) = 0$$

may represent a set of equipotential surfaces, the parameter q must be such that the ratio of its second to the square of its first differential parameter is a function only of q,

$$\frac{\Delta q}{h_q^2} = \phi(q).$$

If this is satisfied, we have

(4)
$$-\phi(q) = \frac{d}{dq}\left(\log\frac{dV}{dq}\right),$$

$$\log\frac{dV}{dq} = -\int\phi(q)\,dq + C,$$

$$\frac{dV}{dq} = Ae^{-\int\phi(q)\,dq},$$

(5)
$$V = A\int e^{-\int\phi(q)\,dq}\,dq + B.$$

There must be one value q such that the level surface is a sphere of infinite radius, and for this V must vanish.

These conditions are satisfied by the polar coordinate r, for by § 95, (4)

$$\Delta r = \frac{2}{r},$$

$$h_r = 1, \quad \frac{\Delta r}{h_r^2} = \frac{2}{r} = \phi(r),$$

$$V = A\int e^{-\int \frac{2}{r} dr} + B$$

$$= A\int \frac{dr}{r^2} + B$$

$$= -\frac{A}{r} + B.$$

For $r = \infty$, we must have $V = 0$, accordingly we must put $B = 0$.

We may get a convenient expression for $\frac{\Delta q}{h_q^2}$ by transforming Δq into terms of three orthogonal coordinates, of which it is itself one. Put $q = q_1$, and since it is independent of q_2 and q_3

$$\Delta q_1 = h_1 h_2 h_3 \frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2 h_3}\frac{\partial q_1}{\partial q_1}\right\},$$

(6) $$\frac{\Delta q_1}{h_1^2} = \frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2 h_3}\right\}$$

$$= \frac{\partial}{\partial q_1}\log\left(\frac{h_1}{h_2 h_3}\right)$$

$$= \frac{1}{h_1}\frac{\partial h_1}{\partial q_1} - \frac{1}{h_2}\frac{\partial h_2}{\partial q_1} - \frac{1}{h_3}\frac{\partial h_3}{\partial q_1}.$$

109. Application to Elliptic Coordinates. Applying this to elliptic coordinates gives

(7) $$\frac{\Delta\lambda}{h_\lambda^2} = \frac{\partial}{\partial\lambda}$$

$$\left\{\log\frac{1}{2}\sqrt{\frac{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)(\mu-\nu)(\mu-\lambda)(\nu-\mu)(\nu-\lambda)}{(\lambda-\mu)(\lambda-\nu)(a^2+\mu)(b^2+\mu)(c^2+\mu)(a^2+\nu)(b^2+\nu)(c^2+\nu)}}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda}\right\} = \phi(\lambda),$$

which is independent of μ and ν, and hence the system of ellipsoids λ *can* represent a family of equipotential surfaces. We have

$$\int\phi(\lambda)\,d\lambda = \frac{1}{2}\int\left\{\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right\}d\lambda \tag{8}$$

$$= \log\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},$$

$$e^{-\int\phi(\lambda)\,d\lambda} = \frac{1}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

$$V = A\int\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} + B. \tag{9}$$

The constant B must be such that for $\lambda = \infty$, which gives the infinite sphere, $V = 0$. This is obtained by taking the definite integral between λ and ∞,

$$V = A\int_\lambda^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \tag{10}$$

λ being taken for the *lower* limit, so that A may be positive, making V decrease as λ increases. V is an elliptic integral in terms of λ, or λ is an elliptic function of V. For

$$\frac{dV}{d\lambda} = -\frac{A}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

$$A^2\left(\frac{d\lambda}{dV}\right)^2 = (a^2+\lambda)(b^2+\lambda)(c^2+\lambda), \tag{11}$$

a differential equation which is satisfied by an elliptic function.

We may determine the constant A by the property that

$$\lim_{x=\infty}(rV) = M,$$

or that
$$\lim_{r=\infty}\left(r^2\frac{\partial V}{\partial x}\right) = -M\cos(rx).$$

We have

$$\frac{\partial V}{\partial x} = \frac{dV}{d\lambda}\frac{\partial\lambda}{\partial x}$$

$$= -\frac{A}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\,\frac{2x\delta_\lambda^2}{(a^2+\lambda)}, \qquad \text{[by § 19, (5)]}$$

$$r^2\frac{\partial V}{\partial x} = -\frac{2Ax\delta_\lambda^2 r^2}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

From the geometrical definition of λ,

$$\lim_{r=\infty}\left(\frac{\lambda}{r^2}\right)=1.$$

Now consider, for simplicity, a point on the X-axis, where $\delta_\lambda=x=r$. The denominator becomes infinite in $\lambda^{\frac{5}{2}}$, that is, r^5, and so does the numerator. Hence

$$\lim_{r=\infty}\left\{r^2\frac{\partial V}{\partial x}\right\}=-2A=-M,$$

so that

(12) $$V=\frac{M}{2}\int_\lambda^\infty\frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}.$$

110. Chasles's Theorem. We have now found the potential due to a mass M of such nature that its equipotential surfaces are confocal ellipsoids, but it remains to determine the nature of the mass. This may be varied in an infinite number of ways; we will attempt to find an equipotential surface layer. By Chasles's theorem, § 84 (11), this will have the same mass as that of a body within it which would have the same potentials outside.

If we find the required layer on an equipotential surface S, since the potential is constant on S, it must be constant at all points within, or the layer does not affect internal bodies.

The surface density must be given by § 84 (10),

$$\sigma=-\frac{1}{4\pi}\frac{\partial V}{\partial n_\lambda},\text{ where } n_\lambda \text{ is the } \textit{outward} \text{ normal to } \lambda,$$

and

$$\frac{\partial V}{\partial n_\lambda}=\frac{dV}{d\lambda}\frac{\partial\lambda}{\partial n_\lambda}=h_\lambda\frac{dV}{d\lambda}.$$

Now since $$h_\lambda=2\delta_\lambda,$$

$$\sigma=-\frac{1}{2\pi}\delta_\lambda\frac{dV}{d\lambda}.$$

Since V is a function of λ alone, the same is true of $\frac{dV}{d\lambda}$, which for a constant value of λ is constant. Hence σ varies on the ellipsoid S as δ_λ. Therefore if we distribute on the given ellipsoid S a surface layer with surface density proportional at every point to the perpendicular from the origin on the tangent plane at the point, this layer is equipotential, and all its equipotential surfaces

are ellipsoids confocal with it. Consequently if we distribute on *any* one of a set of confocal ellipsoids a layer of given mass whose surface density is proportional to δ, the attraction of such various layers at given external points is the same, or if the masses differ, is proportional simply to the masses of the layers. For it depends only on λ, which depends only on the position of the point where we calculate the potential.

Since by the definition of a homœoid, the normal thickness of an infinitely thin homœoid is proportional at any point to the perpendicular on the tangent plane, we may replace the words *surface layer*, etc., above by the words *homogeneous infinitely thin homœoid.* The theorem was given in this form by Chasles.*

111. Maclaurin's Theorem. Consider two confocal ellipsoids, 1, Fig. 50, with semi-axes α_1, β_1, γ_1, and 2, with semi-axes

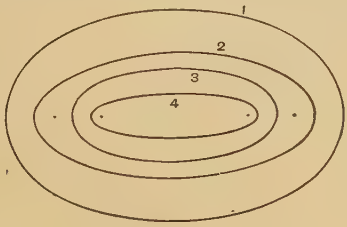

FIG. 50.

α_2, β_2, γ_2. The condition of confocality is

$$\alpha_2^2 - \alpha_1^2 = \beta_2^2 - \beta_1^2 = \gamma_2^2 - \gamma_1^2 = s, \text{ say.}$$

If we now construct two ellipsoids 3 and 4 *similar* respectively to 1 and 2, and whose axes are in the same ratio θ to those of 1 and 3, these two ellipsoids 3 and 4 are confocal (with each other, though not with 1 and 2). For the semi-axes of 3 are $\theta\alpha_1$, $\theta\beta_1$, $\theta\gamma_1$, and of 4 are $\theta\alpha_2$, $\theta\beta_2$, $\theta\gamma_2$, and hence the condition of confocality,

$$\theta^2\alpha_2^2 - \theta^2\alpha_1^2 = \theta^2\beta_2^2 - \theta^2\beta_1^2 = \theta^2\gamma_2^2 - \theta^2\gamma_1^2 = \theta^2 s$$

is satisfied. Now if on 3 we distribute one infinitely thin homœoidal layer between 3 and another ellipsoid for which θ is increased by $d\theta$, and on 4 a homœoidal layer given by the same values of θ and $d\theta$, and furthermore choose the densities such that these two homœoidal layers have the same mass, then (since these homœoids are confocal) their attractions at external points will be identical.

* Chasles, "Nouvelle solution du problème de l'attraction d'un ellipsoide hétérogène sur un point extérieur." *Journal de Liouville*, t. v. 1840.

Now the volume of an ellipsoid with axes a, b, c, is $\frac{4}{3}\pi abc$, that of the inner ellipsoid of the shell 3 is accordingly

$$\tfrac{4}{3}\pi\theta^3\alpha_1\beta_1\gamma_1,$$

and that of the shell is the increment of this on increasing θ by $d\theta$, or

$$(\text{vol. } 3) = 4\pi\theta^2 d\theta\alpha_1\beta_1\gamma_1.$$

Similarly $$(\text{vol. } 4) = 4\pi\theta^2 d\theta\alpha_2\beta_2\gamma_2.$$

Consequently, if we suppose the ellipsoids 1 and 2 filled with matter of uniform density ρ_1 and ρ_2 the condition of equal masses of the thin layers 3 and 4,

$$4\pi\rho_1\theta^2 d\theta\alpha_1\beta_1\gamma_1 = 4\pi\rho_2\theta^2 d\theta\alpha_2\beta_2\gamma_2,$$

is simply $$\tfrac{4}{3}\pi\rho_1\alpha_1\beta_1\gamma_1 = \tfrac{4}{3}\pi\rho_2\alpha_2\beta_2\gamma_2,$$

that is, equality of masses of the two ellipsoids. And since for any two corresponding homœoids such as 3 and 4 (θ and $\theta + d\theta$) the attraction on outside points is the same, the attraction of the entire ellipsoids on external points is the same.

This is Maclaurin's celebrated theorem: Confocal homogeneous solid ellipsoids of equal masses attract external points identically, or the attractions of confocal homogeneous ellipsoids at external points are proportional to their masses.*

112. Potential of Ellipsoid. The potential due to any homœoidal layer of semi-axes α, β, γ, is to be found from our preceding expression for V, § 109 (12),

$$V = \frac{M}{2}\int_\lambda^\infty \frac{ds}{\sqrt{(\alpha^2+s)(\beta^2+s)(\gamma^2+s)}},$$

where λ is the greatest root of

$$\frac{x^2}{\alpha^2+\lambda} + \frac{y^2}{\beta^2+\lambda} + \frac{z^2}{\gamma^2+\lambda} = 1.$$

Now if the semi-axes of the solid ellipsoid are a, b, c, those of the shell $\alpha = \theta a$, $\beta = \theta b$, $\gamma = \theta c$, we have $M = 4\pi\theta^2 d\theta abc$, if the density is unity, and

$$(1) \qquad d_\theta V = 2\pi\theta^2 d\theta abc \int_\lambda^\infty \frac{ds}{\sqrt{(a^2\theta^2+s)(b^2\theta^2+s)(c^2\theta^2+s)}},$$

* Maclaurin, *A Treatise on Fluxions*, 1742.

where λ is defined by

$$(2) \qquad \frac{x^2}{a^2\theta^2+\lambda}+\frac{y^2}{b^2\theta^2+\lambda}+\frac{z^2}{c^2\theta^2+\lambda}=1.$$

To get the potential of the whole ellipsoid, we must integrate for all the shells, and

$$(3) \qquad V=2\pi abc\int_0^1 \theta^2 d\theta \int_\lambda^\infty \frac{ds}{\sqrt{(a^2\theta^2+s)(b^2\theta^2+s)(c^2\theta^2+s)}}.$$

For every value of θ there is *one* value of λ, given by the cubic (2), we may say $\lambda=\phi(\theta)$.

Let us now change the variable s to t, where, θ being constant, $s=\theta^2 t$, $ds=\theta^2 dt$; and put $\lambda=\theta^2 u$.

Then

$$(4) \qquad V=2\pi abc\int_0^1 \theta d\theta \int_u^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}},$$

where u is defined by

$$(5) \qquad \frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=\theta^2.$$

Since θ^2 is thus given as a uniform function of u, we will now change the variable from θ to u.

Differentiating (5) by θ,

$$(6) \qquad 2\theta d\theta=-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}du.$$

When $\theta=0$, $u=\infty$, and when $\theta=1$, u has a value which we will call σ, defined by

$$(7) \qquad \frac{x^2}{a^2+\sigma}+\frac{y^2}{b^2+\sigma}+\frac{z^2}{c^2+\sigma}=1.$$

Accordingly, changing the variable,

$$(8) \quad V=\pi abc\int_\sigma^\infty \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}du\int_u^\infty \frac{dt}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

The three double integrals above are of the form

$$(9) \qquad J=\int_\sigma^\infty \frac{du}{(C+u)^2}\int_u^\infty f(t)\,dt,$$

where $$f(t)=\frac{1}{\sqrt{(a^2+t)(b^2+t)(c^2+t)}}.$$

This may be integrated by parts.

Call
$$\int_u^\infty f(t)\,dt = \phi(u),$$

(10)
$$J = \int_\sigma^\infty \frac{\phi(u)\,du}{(C+u)^2} = -\left[\frac{\phi(u)}{C+u}\right]_\sigma^\infty + \int_\sigma^\infty \frac{\phi'(u)\,du}{C+u}.$$

Now
$$\phi(\infty) = \int_\infty^\infty f(t)\,dt = 0, \quad (\text{since } f(\infty) = 0),$$
$$\phi(\sigma) = \int_\sigma^\infty f(t)\,dt,$$
$$\phi'(u) = -f(u).$$

Inserting these values

(11)
$$J = \frac{1}{C+\sigma}\int_\sigma^\infty f(t)\,dt - \int_\sigma^\infty \frac{f(u)\,du}{C+u},$$

or the variable of integration being indifferent, we may put u for t in the first integral.

Applying this to our integral, by putting C successively equal to a^2, b^2, c^2, multiplying by x^2, y^2, z^2, and adding,

(12)
$$V = \pi abc \int_\sigma^\infty \left\{\frac{x^2}{a^2+\sigma} + \frac{y^2}{b^2+\sigma} + \frac{z^2}{c^2+\sigma} - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\} f(u)\,du.$$

Now the first three terms of the integrand are, by definition, equal to 1, so that

(13)
$$V = \pi abc \int_\sigma^\infty \left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

This form was given by Dirichlet*.

If the point x, y, z lies on the surface of the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then $\sigma = 0$ and

(14)
$$V = \pi abc \int_0^\infty \left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

* Dirichlet, "Ueber eine neue Methode zur Bestimmung vielfacher Integrale." *Abh. der Berliner Akad.*, 1839. Translated in *Journ. de Liouville*, t. iv., 1839.

We find for the derivatives of V

$$\frac{\partial V}{\partial x} = -2\pi abcx \int_\sigma^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$-\pi abc \frac{\partial \sigma}{\partial x}\left\{1 - \frac{x^2}{a^2+\sigma} - \frac{y^2}{b^2+\sigma} - \frac{z^2}{c^2+\sigma}\right\}\frac{1}{\sqrt{(a^2+\sigma)(b^2+\sigma)(c^2+\sigma)}}.$$

By definition of σ, the parenthesis in the last term vanishes, and

$$\frac{\partial V}{\partial x} = -2\pi abcx \int_\sigma^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

(15) $$\frac{\partial V}{\partial y} = -2\pi abcy \int_\sigma^\infty \frac{du}{(b^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$\frac{\partial V}{\partial z} = -2\pi abcz \int_\sigma^\infty \frac{du}{(c^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

113. Internal Point. In the case of an internal point, we pass through it an ellipsoid similar to the given ellipsoid, then by Newton's theorem it is unattracted by the homœoidal shell without, and we may use the above formulae for the attraction, putting for a, b, c, the values for the ellipsoid through x, y, z, say θa, θb, θc. Since the point is on the surface of this, $\sigma = 0$.

$$\frac{\partial V}{\partial x} = -2\pi\theta^3 abcx \int_0^\infty \frac{du}{(\theta^2a^2+u)\sqrt{(\theta^2a^2+u)(\theta^2b^2+u)(\theta^2c^2+u)}}.$$

Now let us insert a variable u' proportional to u, $u = \theta^2 u'$,

$$\frac{\partial V}{\partial x} = -2\pi\theta^3 abcx \int_0^\infty \frac{\theta^2 du'}{\theta^2(a^2+u')\,\theta^3\sqrt{(a^2+u')(b^2+u')(c^2+u')}}.$$

The θ divides out, and writing u for the variable of integration

$$\frac{\partial V}{\partial x} = -2\pi abcx \int_0^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

So that for *any* internal point, we put $\sigma = 0$ in the general formula. Integrating with respect to x, y, z, we have

$$V = \pi abc \int_0^\infty \left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

The constant term must be taken as above in order that at the surface V may be continuous.

In the case of an internal point the above four integrals may be made to depend on the first. Calling

$$\Phi = \pi abc \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$\frac{\partial \Phi}{\partial (a^2)} = \pi abc \int_0^\infty -\frac{1}{2} \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

and accordingly,

$$V = \Phi + 2\frac{\partial \Phi}{\partial (a^2)}x^2 + 2\frac{\partial \Phi}{\partial (b^2)}y^2 + 2\frac{\partial \Phi}{\partial (c^2)}z^2.$$

The integral Φ is an elliptic integral independent of x, y, z, and so are its derivatives with respect to a^2, b^2, c^2. Calling these respectively $-\frac{L}{4}$, $-\frac{M}{4}$, $-\frac{N}{4}$, we have

$$V = \Phi - \tfrac{1}{2}\{Lx^2 + My^2 + Nz^2\},$$

a symmetrical function of the second order, and since L, M, N are of the same sign, the equipotential surfaces are ellipsoids, similar to each other. Their relation to the given ellipsoid is however transcendental, their semi-axes being

$$\sqrt{\frac{V-\Phi}{2\frac{\partial \Phi}{\partial (a^2)}}},\quad \sqrt{\frac{V-\Phi}{2\frac{\partial \Phi}{\partial (b^2)}}},\quad \sqrt{\frac{V-\Phi}{2\frac{\partial \Phi}{\partial (c^2)}}}.$$

We have for the force

$$-\frac{\partial V}{\partial x} = X = Lx,\quad -\frac{\partial V}{\partial y} = My,\quad -\frac{\partial V}{\partial z} = Nz.$$

Hence, since for two points on the same radius-vector,

$$\frac{x_2}{x_1} = \frac{y_2}{y_1} = \frac{z_2}{z_1} = \frac{r_2}{r_1}, \text{ we have } \frac{X_2}{X_1} = \frac{Y_2}{Y_1} = \frac{Z_2}{Z_1} = \frac{r_2}{r_1}.$$

The forces are *parallel* and proportional to the distance from the center.

114. Verification by Differentiation. For an outside point, we have

$$V = \pi abc \int_\sigma^\infty \left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right\} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$\frac{\partial V}{\partial x} = -2\pi abcx \int_\sigma^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$\frac{\partial^2 V}{\partial x^2} = -2\pi abc \int_\sigma^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$+ 2\pi abcx \frac{\partial \sigma}{\partial x}\left\{\frac{1}{(a^2+\sigma)\sqrt{(a^2+\sigma)(b^2+\sigma)(c^2+\sigma)}}\right\}.$$

Now by § 19, (5),

$$\frac{\partial \sigma}{\partial x} = \frac{2x}{a^2+\sigma} \Big/ \left\{\frac{x^2}{(a^2+\sigma)^2} + \frac{y^2}{(b^2+\sigma)^2} + \frac{z^2}{(c^2+\sigma)^2}\right\}.$$

Forming $\dfrac{\partial^2 V}{\partial y^2}$ and $\dfrac{\partial^2 V}{\partial z^2}$ and adding,

$$\Delta V = -2\pi abc \int_\sigma^\infty \left\{\frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u}\right\} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$+ \frac{4\pi abc}{\sqrt{(a^2+\sigma)(b^2+\sigma)(c^2+\sigma)}}.$$

The integration may be at once effected.

Since $$d(uvw) = uvw\left\{\frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}\right\},$$

we have

$$d\left\{\frac{1}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\right\}$$

$$= \frac{1}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\left\{-\frac{\sqrt{a^2+u}}{2\sqrt{(a^2+u)^3}} - \ldots\ldots\right\} du$$

$$= -\frac{1}{2}\left\{\frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u}\right\} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

The integral becomes then

$$-\frac{4\pi abc}{\sqrt{(a^2+\sigma)(b^2+\sigma)(c^2+\sigma)}},$$

which cancels the second term, and $\Delta V = 0$.

For an internal point

$$\frac{\partial^2 V}{\partial x^2} = -2\pi abc \int_0^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

$$\Delta V = -4\pi.$$

At infinity $\sigma = \infty$, and V and its derivatives accordingly vanish.

Hence the value of V found satisfies all the conditions.

115. Ivory's Theorem. If x, y, z is a point on the ellipsoid

$$(1) \qquad \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1,$$

the point $$x\frac{a_2}{a_1},\ y\frac{b_2}{b_1},\ z\frac{c_2}{c_1}$$

lies on the ellipsoid

$$(2) \qquad \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} + \frac{z^2}{c_2^2} = 1.$$

These will be called corresponding points. We shall now assume that these two ellipsoids are confocal, and (2) the smaller. Then

$$a_1^2 = a_2^2 + \lambda, \quad b_1^2 = b_2^2 + \lambda, \quad c_1^2 = c_2^2 + \lambda.$$

The action of (2) on the external point x, y, z is

$$X_2 = -2\pi a_2 b_2 c_2 x \int_\sigma^\infty \frac{du}{(a_2^2+u)\sqrt{(a_2^2+u)(b_2^2+u)(c_2^2+u)}},$$

where $$\frac{x^2}{a_2^2+\sigma} + \frac{y^2}{b_2^2+\sigma} + \frac{z^2}{c_2^2+\sigma} = 1,$$

and since $$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1,$$

we must have $\sigma = \lambda$.

If now we substitute

$$u = u' + a_1^2 - a_2^2 = u' + \sigma,$$

$$X_2 = -2\pi a_2 b_2 c_2 x \int_0^\infty \frac{du'}{(a_1^2+u')\sqrt{(a_1^2+u')(b_1^2+u')(c_1^2+u')}}.$$

Now the attraction of the ellipsoid (1) on the interior point $x\frac{a_2}{a_1}$, $y\frac{b_2}{b_1}$, $z\frac{c_2}{c_1}$ is

$$X_1 = -2\pi a_1 b_1 c_1\, x\frac{a_2}{a_1} \int_0^\infty \frac{du}{(a_1^2+u)\sqrt{(a_1^2+u)(b_1^2+u)(c_1^2+u)}}.$$

The definite integrals being the same in both cases, we have

$$\frac{X_2}{X_1} = \frac{b_2 c_2}{b_1 c_1},$$

$$\frac{Y_2}{Y_1} = \frac{c_2 a_2}{c_1 a_1},$$

$$\frac{Z_2}{Z_1} = \frac{a_2 b_2}{a_1 b_1}.$$

This is Ivory's theorem: Two confocal ellipsoids of equal density each act on corresponding points on the other with forces whose components are proportional to the areas of their principal sections normal to the components.*

116. Ellipsoids of Revolution. For an ellipsoid of revolution, the elliptic integrals reduce to inverse circular functions.

Put $b = c$, a being the axis of revolution,

(1) $$V = \pi a b^2 \int_\sigma^\infty \frac{du}{(b^2 + u)\sqrt{a^2 + u}} - \tfrac{1}{2}(Xx + Yy),$$

(2) $$X = 2\pi a b^2 x \int_\sigma^\infty \frac{du}{(b^2 + u)(a^2 + u)^{\frac{3}{2}}},$$

(3) $$Y = 2\pi a b^2 y \int_\sigma^\infty \frac{du}{(b^2 + u)^2\sqrt{(a^2 + u)}},$$

where $$\frac{x^2}{a^2 + \sigma} + \frac{y^2}{b^2 + \sigma} = 1.$$

Put $$b^2 + u = \frac{b^2 - a^2}{s^2},$$

$$a^2 + u = (b^2 - a^2)\left(\frac{1}{s^2} - 1\right) = (b^2 - a^2)\frac{(1 - s^2)}{s^2},$$

$$du = -\frac{2(b^2 - a^2)}{s^3} ds.$$

When $u = \infty$, $s = 0$; when $u = \sigma$, $s = \sqrt{\dfrac{b^2 - a^2}{b^2 + \sigma}}$, so that

$$V = \pi a b^2 \int_{u=\sigma}^{u=\infty} \frac{-2(b^2 - a^2)s^3\,ds}{s^3(b^2 - a^2)\sqrt{(b^2 - a^2)(1 - s^2)}} - \tfrac{1}{2}(Xx + Yy)$$

* Ivory, "On the attractions of homogeneous Ellipsoids." *Phil. Trans.*, 1809.

(4) $$= \frac{2\pi ab^2}{\sqrt{b^2-a^2}} \int_0^{\sqrt{\frac{b^2-a^2}{b^2+\sigma}}} \frac{ds}{\sqrt{1-s^2}} - \tfrac{1}{2}(Xx+Yy),$$

(5) $$V = \frac{2\pi ab^2}{\sqrt{b^2-a^2}} \left\{\sin^{-1}\sqrt{\frac{b^2-a^2}{b^2+\sigma}}\right\} - \tfrac{1}{2}\{Xx+Yy\},$$

(6) $$X = 2\pi ab^2 x \int - \frac{2(b^2-a^2)s^2}{s^3(b^2-a^2)(b^2-a^2)^{\frac{1}{2}}} \frac{s^3 ds}{(1-s^2)^{\frac{3}{2}}}$$

$$= \frac{4\pi ab^2 x}{(b^2-a^2)^{\frac{3}{2}}} \int_0^{\sqrt{\frac{b^2-a^2}{b^2+\sigma}}} \frac{s^2 ds}{(1-s^2)^{\frac{3}{2}}}.$$

Now $$\int \frac{s^2 ds}{(1-s^2)^{\frac{3}{2}}} = \frac{s}{\sqrt{1-s^2}} - \int \frac{ds}{\sqrt{1-s^2}},$$

so that

(7) $$X = \frac{4\pi ab^2 x}{(b^2-a^2)^{\frac{3}{2}}} \left\{\sqrt{\frac{b^2-a^2}{a^2+\sigma}} - \sin^{-1}\sqrt{\frac{b^2-a^2}{b^2+\sigma}}\right\},$$

(8) $$Y = 2\pi ab^2 y \int \frac{-2(b^2-a^2)}{s^3(b^2-a^2)^2} \frac{s^4 . s ds}{\sqrt{(b^2-a^2)(1-s^2)}}$$

$$= \frac{4\pi ab^2 y}{(b^2-a^2)^{\frac{3}{2}}} \int_0^{\sqrt{\frac{b^2-a^2}{b^2+\sigma}}} \frac{s^2 ds}{\sqrt{1-s^2}}.$$

Now $$\int \frac{s^2 ds}{\sqrt{1-s^2}} = \tfrac{1}{2}\{\sin^{-1} s - s\sqrt{1-s^2}\},$$

so that

(9) $$Y = \frac{2\pi ab^2 y}{(b^2-a^2)^{\frac{3}{2}}} \left\{\sin^{-1}\sqrt{\frac{b^2-a^2}{b^2+\sigma}} - \sqrt{\frac{(b^2-a^2)(a^2+\sigma)}{b^2+\sigma}}\right\}.$$

For $\sin^{-1}\sqrt{\frac{b^2-a^2}{b^2+\sigma}}$ we may write $\tan^{-1}\sqrt{\frac{b^2-a^2}{a^2+\sigma}}$,

for if $$\sin\theta = \sqrt{\frac{b^2-a^2}{b^2+\sigma}},$$

then $$\cos\theta = \sqrt{\frac{a^2+\sigma}{b^2+\sigma}},$$

$$\tan\theta = \sqrt{\frac{b^2-a^2}{a^2+\sigma}}.$$

These formulae all serve for an oblate spheroid, where $a < b$. For a prolate spheroid, $b > a$, they introduce imaginaries, from which they may be cleared as follows.

Call
$$\sin^{-1}(iu) = \theta,$$
then
$$iu = \sin\theta, \sqrt{1+u^2} = \cos\theta,$$
$$e^{-i\theta} = \cos\theta - i\sin\theta = \sqrt{1+u^2} + u,$$
therefore
$$-i\theta = \log\{\sqrt{1+u^2}+u\},$$
$$\sin^{-1}(iu) = \theta = i\log\{\sqrt{1+u^2}+u\}.$$

Put
$$u = \sqrt{\frac{a^2-b^2}{b^2+\sigma}},$$
$$\sin^{-1}\sqrt{\frac{b^2-a^2}{b^2+\sigma}} = i\log\left\{\frac{\sqrt{a^2+\sigma}+\sqrt{a^2-b^2}}{\sqrt{b^2+\sigma}}\right\}.$$

Hence

(10) $$V = \frac{2\pi ab^2}{\sqrt{a^2-b^2}}\log\left\{\frac{\sqrt{a^2-b^2}+\sqrt{a^2+\sigma}}{\sqrt{b^2+\sigma}}\right\} - \tfrac{1}{2}\{Xx+Yy\},$$

(11) $$X = \frac{4\pi ab^2x}{(a^2-b^2)^{\frac{3}{2}}}\left\{\log\frac{\sqrt{a^2-b^2}+\sqrt{a^2+\sigma}}{\sqrt{b^2+\sigma}} - \sqrt{\frac{a^2-b^2}{a^2+\sigma}}\right\},$$

(12) $$Y = \frac{2\pi ab^2y}{(a^2-b^2)^{\frac{3}{2}}}\left\{\frac{\sqrt{(a^2-b^2)(a^2+\sigma)}}{b^2+\sigma} - \log\frac{\sqrt{a^2-b^2}+\sqrt{a^2+\sigma}}{\sqrt{b^2+\sigma}}\right\}.$$

In all these formulae, σ is the larger root of the quadratic
$$\frac{x^2}{a^2+\sigma} + \frac{y^2}{b^2+\sigma} = 1,$$
for an outside point, and $\sigma = 0$ for an inside point. In the latter case, we have functions only of the ratio $\frac{a}{b}$*.

117. Energy of Distributions. Gauss's Theorem. If a particle of unit mass be at $p, (x, y, z)$ at a distance r from a particle of mass m_q, the work necessary to bring the unit particle from an infinite distance against the repulsion of the particle m_q will be

(1) $$W = \frac{m_q}{r} = V(x, y, z) = V_p.$$

If, instead of a particle of unit mass, we have one of mass m_p the work necessary will be m_p times as great,

(2) $$W_{pq} = \frac{m_q}{r}m_p = m_pV_p = m_qV_q,$$

where
$$V_q = \frac{m_p}{r}.$$

* Thomson and Tait. *Natural Philosophy*, Part II., § 527.

In other words, this is the amount of loss of the potential energy of the system on being allowed to disperse to an infinite distance from a distance apart r. Similarly, for any two systems of particles m_p, m_q,

$$(3)\qquad W_{pq} = \Sigma_q \Sigma_p \frac{m_p m_q}{r_{pq}} = \Sigma_p m_p V_p' = \Sigma_q m_q V_q,$$

V_p' being the potential *at* any point p due to *all* the particles q and V_q being the potential *at* any point q due to *all* the particles p. This sum is called the *mutual potential energy* of the systems p and q. If however we consider all the particles to belong to one system, we must write

$$(4)\qquad W = \tfrac{1}{2}\Sigma\Sigma \frac{m_p m_q}{r_{pq}} = \tfrac{1}{2}\Sigma m V,$$

where every particle appears both as p and q, the $\frac{1}{2}$ being put in because every *pair* would thus appear twice. This expression has been given in § 59, (33).

If the systems are continuously distributed over volumes τ, τ' we have

$$(5)\qquad W_{pq} = \iiiint\!\!\iint \frac{\rho_p \rho_q}{r_{pq}} d\tau_p d\tau_q = \iiint_\tau \rho_p V_p' d\tau_p = \iiint_{\tau'} \rho_q V_q d\tau_q.$$

The theorem expressed by the equality of the two integrals is known as Gauss's theorem on mutual energy, where V_p' represents the potential at p due to the whole mass M_q, V_q, that at q due to the whole mass M_p.*

The above equality may be also proved as follows. Since

$$(6)\qquad \rho_p = -\frac{1}{4\pi}\Delta V_p,$$

and

$$\rho_q = -\frac{1}{4\pi}\Delta V_q',$$

the triple integrals in (5) become respectively,

$$(7)\qquad -\frac{1}{4\pi}\iiint_\tau V_p' \Delta V_p d\tau_p,$$

and

$$-\frac{1}{4\pi}\iiint_{\tau'} V_q \Delta V_q' d\tau_q.$$

* Gauss. "Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte." *Werke*, Bd. v. p. 197.

Now since outside of τ, $\Delta V=0$ and outside of τ', $\Delta V'=0$ the integrals may be extended to all space. But by Green's theorem, both these integrals are equal to

$$\frac{1}{4\pi}\iiint_\infty \left\{\frac{\partial V}{\partial x}\frac{\partial V'}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial V'}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial V'}{\partial z}\right\}d\tau,$$

since the surface integrals

$$\iint V\frac{\partial V'}{\partial n}dS, \quad \iint V'\frac{\partial V}{\partial n}dS$$

vanish at infinity. Gauss's theorem accordingly follows from Green's theorem and Poisson's equation.

118. Energy in terms of Field. For the energy of any distribution consisting of both volume and surface distributions, the sum (4) becomes the integrals

$$W=\frac{1}{2}\iint_S V\sigma dS+\frac{1}{2}\iiint_\tau V\rho\, d\tau. \tag{8}$$

Now at a surface distribution Poisson's equation is

$$\sigma=-\frac{1}{4\pi}\left\{\frac{\partial V}{\partial n_1}+\frac{\partial V}{\partial n_2}\right\}.$$

If, as in § 85, we draw surfaces close to the surface distributions, and exclude the space between them, we may, as above, extend the integrals to all other space, so that

$$W=-\frac{1}{8\pi}\iint_S V\frac{\partial V}{\partial n}dS-\frac{1}{8\pi}\iiint_\infty V\Delta V d\tau, \tag{9}$$

the normals being from the surfaces S toward the space τ. But by Green's theorem, as before, this is equal to the integral

$$W=\frac{1}{8\pi}\iiint_\infty\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}d\tau. \tag{10}$$

Thus the energy is expressed in terms of the strength of the field

$$F^2=\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=P^2{}_V$$

at all points in space. This integral is of fundamental importance.

It is at once seen that this is always positive.

We may obtain the same expression as follows. Suppose that the matter at a point x, y, z is displaced to a point $x+\delta x$,

$y + \delta y$, $z + \delta z$. The amount of matter in the fixed infinitesimal parallelopiped $dxdydz$, $dm = \rho dxdydz$ is thereby changed, and the work necessary is the same as that required to bring the mass δdm from infinity to the point x, y, z, where the potential is V, namely, $\delta W = V\delta dm$. We have found in § 38,

$$\delta dm = -\left\{\frac{\partial(\rho\delta x)}{\partial x} + \frac{\partial(\rho\delta y)}{\partial y} + \frac{\partial(\rho\delta z)}{\partial z}\right\} dxdydz.$$

Consequently the whole increase of energy is

(11) $$\delta W = -\iiint V\left\{\frac{\partial(\rho\delta x)}{\partial x} + \frac{\partial(\rho\delta y)}{\partial y} + \frac{\partial(\rho\delta z)}{\partial z}\right\} d\tau.$$

Integrating by parts

$$-\iiint V\frac{\partial(\rho\delta x)}{\partial x}\,dxdydz = -\iint V\rho\,\delta x\,dydz + \iiint \rho\delta x\,\frac{\partial V}{\partial x}\,dxdydz,$$

(12) $$\delta W = \iiint \rho\left\{\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\right\} d\tau = \iiint \rho\,\delta V d\tau,$$

the integral being over all space, and the surface integrals vanishing at infinity.

But since $$\rho = -\frac{1}{4\pi}\Delta V,$$

this becomes

(13) $$\begin{aligned}\delta W &= -\frac{1}{4\pi}\iiint_\infty \Delta V\delta V d\tau\\ &= \frac{1}{4\pi}\iiint_\infty \left\{\frac{\partial V}{\partial x}\frac{\partial\delta V}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial\delta V}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial\delta V}{\partial z}\right\} d\tau\\ &= \frac{1}{4\pi}\iiint_\infty \frac{1}{2}\delta\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau\\ &= \delta\frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,\end{aligned}$$

so that

$$W = \frac{1}{8\pi}\iiint_\infty P^2{}_V d\tau.$$

For a third deduction, since in moving a mass dm a distance whose components are δx, δy, δz the energy lost is equal to the work done by the system

(14) $$\begin{aligned}-\delta W &= dm\{X\delta x + Y\delta y + Z\delta z\}\\ &= -dm\left\{\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\right\}\\ &= -dm\,\delta V = -\rho\delta V d\tau.\end{aligned}$$

The whole variation of the energy is

$$\delta W = \iiint_\infty \rho \delta V d\tau, \text{ as in (12).}$$

Applying Gauss's Theorem to the mutual energy of two distributions, one of which has density ρ, producing the potential V, the other the density $\delta\rho$, producing potential δV, we have

$$\iiint_\infty V \delta\rho d\tau = \iiint_\infty \rho \delta V d\tau = \delta W,$$

and

$$W = \frac{1}{2}\iiint_\infty \rho V d\tau$$

gives in agreement therewith

$$\delta W = \frac{1}{2}\iiint_\infty \delta(\rho V) d\tau$$

$$(15) \qquad = \frac{1}{2}\left\{\iiint_\infty \delta\rho \,.\, V d\tau + \iiint_\infty \rho\delta V d\tau\right\}$$

$$= \iiint_\infty \delta\rho \,.\, V d\tau = \iiint_\infty \rho \delta V d\tau.$$

The integrals may be now restricted to the space occupied by matter.

119. Maximum theorem for Energy. By making use of the two different expressions for the energy we can deduce an important theorem relating to the energy of a distribution. We may use the form, § 118, (8),

$$(1) \qquad W_d = \frac{1}{2}\iint \sigma V dS + \frac{1}{2}\iiint_\infty \rho V d\tau,$$

which is distinguished by the suffix d to denote that the densities occur explicitly. This form, by the definition of the potential, holds for any law of force, whether the Newtonian or not.* On the other hand we may use the form, § 118, (10),

$$(2) \qquad W_f = \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,$$

to which we give the suffix f in order to denote that it is expressed only in terms of the field at all points, and does not

* By this we mean any conservative law in which the action is proportional to the product of the masses, and to some function of their relative position.

explicitly contain the densities. This expression holds good only for a distribution acting according to the Newtonian Law. As these two expressions must be equal for all distributions, we may write

$$(3)\quad W = 2W_d - W_f$$

$$= \iint \sigma V dS + \iiint_\infty \left[\rho V - \frac{1}{8\pi}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\}\right] d\tau.$$

If in this latter expression for W we make an arbitrary variation in the form of the function V, we obtain for the varied value of W an integral containing the variations of V, σ, and ρ. If we suppose that the new distribution also acts according to the Newtonian law, in virtue of Poisson's equation there will be relations between δV, $\delta\sigma$, $\delta\rho$.

We shall however remove this restriction, and consider V, σ, ρ as *perfectly independent functions*, which can be varied independently.

We shall choose $\delta\sigma$ and $\delta\rho$ as zero, in other words we shall suppose V to be varied from the values that it actually has for the original Newtonian distribution, the variation being entirely arbitrary, while the densities are unchanged. Calling the variation under these circumstances $\delta_V W$,

$$(4)\quad W + \delta_V W = \iint \sigma (V + \delta V) dS + \iiint_\infty \rho (V + \delta V) d\tau$$

$$- \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial (V+\delta V)}{\partial x}\right)^2 + \left(\frac{\partial (V+\delta V)}{\partial y}\right)^2 + \left(\frac{\partial (V+\delta V)}{\partial z}\right)^2\right\} d\tau.$$

From this we obtain by subtraction of (3),

$$(5)\quad \delta_V W = \iint \sigma \delta V dS + \iiint_\infty \rho \delta V d\tau$$

$$- \frac{1}{4\pi}\iiint_\infty \left\{\frac{\partial V}{\partial x}\frac{\partial \delta V}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial \delta V}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial \delta V}{\partial z}\right\} d\tau$$

$$- \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial \delta V}{\partial x}\right)^2 + \left(\frac{\partial \delta V}{\partial y}\right)^2 + \left(\frac{\partial \delta V}{\partial z}\right)^2\right\} d\tau.$$

Integrating the third integral by Green's theorem, we have finally

$$(6)\qquad \delta_V W = \iint \left\{ \sigma + \frac{1}{4\pi}\left(\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2}\right)\right\} \delta V dS$$
$$+ \iiint_\infty \left\{ \rho + \frac{1}{4\pi}\Delta V \right\} \delta V d\tau$$
$$- \frac{1}{8\pi}\iiint_\infty \left\{ \left(\frac{\partial \delta V}{\partial x}\right)^2 + \left(\frac{\partial \delta V}{\partial y}\right)^2 + \left(\frac{\partial \delta V}{\partial z}\right)^2 \right\} d\tau.$$

Since the unvaried distribution is a Newtonian one, by Poisson's equation the factors multiplying δV in the first two integrands are zero. Consequently the variation of W is equal to minus the last integral, which, as the integrand is a sum of squares, is necessarily positive. Accordingly $\delta_V W < 0$ and we may state the theorem:

If the potential due to any given distribution of matter acting according to the Newtonian Law is known, the energy calculated by the formula (3) is a maximum for the actual distribution of potential as compared with arbitrary distributions differing by an infinitesimal amount from the actual.

We may state this theorem in physical language, avoiding specification of the form in which W is to be expressed, as follows. We may consider $V + \delta V$ as the potential due to a Newtonian distribution whose densities differ at each point of space by an infinitesimal amount from the densities of the given distribution, the differences being otherwise perfectly arbitrary. We will call the supposed distribution 2, the original distribution being 1. Then the terms

$$\iint \sigma (V + \delta V) dS + \iiint \rho (V + \delta V) d\tau,$$

are the mutual energy W_{12} of the distributions 1 and 2, by § 117, (5).

The integral

$$\frac{1}{8\pi}\iiint \left\{ \left(\frac{\partial (V+\delta V)}{\partial x}\right)^2 + \left(\frac{\partial (V+\delta V)}{\partial y}\right)^2 + \left(\frac{\partial (V+\delta V)}{\partial z}\right)^2 \right\} d\tau$$

is the energy W_2 of the distribution 2, by § 118, (10).

Accordingly equation (4) is

$$W_1 + \delta_V W = W_{12} - W_2,$$

so that

$$\delta_V W = W_{12} - (W_1 + W_2) < 0, \quad W_{12} < W_1 + W_2.$$

That is: The mutual energy of any two Newtonian distributions differing infinitesimally from each other, and supposed co-existing, is less than the sum of their individual energies. This theorem is probably true for all repulsive forces. We shall make use of it in § 180 to deduce the laws of dielectric and magnetic actions.

120. Potential of Polarized Bodies. Double Distributions. If a particle of mass dm be placed at a point where the field-strength is F, it experiences a force of amount Fdm in the direction of the field. If any distribution m be placed in a field which is uniform, that is, for which at all points F is constant in value and direction, the force P experienced by the whole mass is

$$P = \iiint F dm = F \iiint dm = Fm.$$

Certain bodies exist in nature which, on being placed in a uniform field, experience no tendency to move in any direction, so that P is zero. Accordingly for such bodies m must be zero, or their density must be in some points positive and in others negative. Such bodies experience a couple when placed in a uniform field, although the resultant force vanishes. Not only does the above property hold for the whole body, but if it be broken into any number of parts the resultant force on each part is zero. A magnet is the most familiar example of such a body, for placed in a uniform field of magnetic force it experiences no resultant force, no matter into how many pieces it may be broken. In such a body then the mass of any part, however small, must be zero. Let us consider how such a condition is possible.

Let us suppose that any portion of space τ is occupied by a body A of constant density ρ and that occupying identically the same space is a second body B of constant density $-\rho$. The two bodies will then completely neutralize each other's action in every way, and when placed in a uniform field would experience neither force nor couple. Now suppose that the first body A is displaced by an infinitesimal amount, so that every point in it moves a distance h in the same direction. The effect will be that while in the space occupied in common by the two bodies the densities neutralize each other, there is a space on one side filled with positive matter, and on

FIG. 51.

the other a space filled with negative matter, (Fig. 51). The volumes of these two spaces must be equal, since the bodies A and B originally coincided. The effect of the system is now that of a body covered on part of its surface with a positive, and on the remainder with a negative surface distribution. If n, Fig. 52, is the normal drawn inwards, and h represent the direction as well as magnitude of the displacement, the amount of matter contained in a right prism standing on the element of surface dS will be

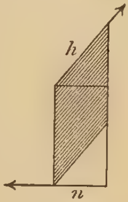

FIG. 52.

$$-\rho h \cos (hn)\, dS,.$$

but this is equal to σdS where σ is the surface density. Accordingly the surface density of the equivalent distribution is

$$(1) \qquad \sigma = -\rho h \cos (hn).$$

If we now decrease h and increase ρ without limit, keeping their product finite and equal to I, we obtain a body charged with surface density

$$(2) \qquad \sigma = -I \cos (hn),$$

possessing the property of experiencing a couple, but no resultant force when placed in a uniform field.

To find the magnitude of the couple let us divide the body up into prisms with their generators parallel to h and standing on the elements dS. Such a cylinder of length l carries upon one end the charge σdS which experiences the force $F\sigma dS$, and upon the other the charge $-\sigma dS$ which experiences the force $F\sigma dS$ in the opposite direction. The moment of the couple thus produced is

$$l \sin (hF)\, F\sigma dS.$$

For the whole moment we must take the integral of this over the positively charged surface,

$$(3) \qquad \iint l \sin (hF)\, F\sigma dS = -F \sin (hF) \iint lI \cos (hn)\, dS.$$

Now $-\cos (hn)\, dS$ is the area of a right section of the prism on dS, so that $-l \cos (hn)\, dS$ is its volume $\int_0^l d\tau$, and the total moment becomes

$$(4) \qquad F \sin (hF) \iiint I d\tau = FI \sin (hF)\,.\,\tau.$$

Such a distribution may be called a *double* or *sliding distribution*, and a body possessing such a distribution is said to be *polarized* in the direction h. The conception of the sliding distribution is due to Poisson. The moment of the couple experienced by the body when placed in a uniform field of strength unity whose direction is perpendicular to the direction of polarization, is called its *moment of polarization*, and I, the moment of unit volume, the *intensity* of polarization. I is a vector quantity, for the whole couple may evidently be supposed to arise from three bodies occupying the same space, and polarized in three mutually perpendicular directions, the intensities of polarization being respectively

$$A = I\cos(Ix), \quad B = I\cos(Iy), \quad C = I\cos(Iz).$$

For if the components of the field are

$$X = F\cos(Fx), \quad Y = F\cos(Fy), \quad Z = F\cos(Fz),$$

the polarization B produces the couple BZ about the X-axis, and the polarization C the couple $-CY$ about the same. In like manner the couples about the other axes are obtained,

(5) $$L = BZ - CY, \quad M = CX - AZ, \quad N = AY - BX.$$

We accordingly find that the resultant couple is the vector product of the intensity of polarization and the field-strength, whose magnitude is

$$FI\sin(FI), \text{ agreeing with (4).}$$

Suppose now that we have a body of such a nature that every element of its volume has a double distribution, although the direction and magnitude of the polarization I may vary from element to element. Such a body is polarized in the most general manner, and the volume-density will not vanish throughout. Let us seek its value in terms of the polarization at each point. Consider a rectangular element of volume $d\tau$, whose sides are dx, dy, dz, and in which the values of the component polarizations are A, B, C. Then the face $dydz$ on the side next to the origin has the charge $-A\,dydz$, while the opposite face has the equal and opposite charge $A\,dydz$. In the next element of volume on the right, whose center is at a distance dx from the center of the first, the X-component of polarization will be

$$A + \frac{\partial A}{\partial x}dx,$$

so that the face $dydz$ next the origin will have the charge

$$-\left(A+\frac{\partial A}{\partial x}dx\right)dydz.$$

This is superposed on the charge $+Adydz$, so that the whole charge on that face is

$$-\frac{\partial A}{\partial x}dxdydz.$$

Similarly the faces $dzdx$ and $dxdy$ on the sides farthest from the origin, have the charges

$$-\frac{\partial B}{\partial y}dxdydz.$$

and
$$-\frac{\partial C}{\partial z}dxdydz.$$

If we consider these faces as belonging to the element $d\tau$, while the opposite faces belong to adjacent elements, the whole charge belonging to the element $d\tau$ is

$$-\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right\}dxdydz.$$

Thus the charge per unit volume is

(6) $$\rho=-\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right\}.$$

Integrating this throughout the volume occupied by the body

$$\iiint\rho\,d\tau=-\iiint\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right\}\partial\tau$$

(7) $$=\iint\{A\cos(nx)+B\cos(ny)+C\cos(nz)\}\,dS$$

$$=\iint I\cos(In)\,dS=-\iint\sigma\,dS,$$

so that the total mass of the volume and surface charges is zero.

121. Induction. Comparing the expression for the density with the ordinary expression for the density as $1/4\pi$ times the divergence of the force, we have

(8) $$\rho=-\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right\}=\frac{1}{4\pi}\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right\},$$

so that the vector $\mathfrak{F}$ whose components are

(9) $$\mathfrak{X} = X + 4\pi A, \quad \mathfrak{Y} = Y + 4\pi B, \quad \mathfrak{Z} = Z + 4\pi C,$$

has no divergence anywhere. Its normal component is continuous at the surface of the body, for since by § 82,

$$\sigma = \frac{1}{4\pi}\{F_i \cos(F_i n_i) + F_e \cos(F_e n_e)\} = -I_i \cos(I_i n_i),$$

(where the suffixes i and e denote values inside and outside, and the opposite directions of the corresponding normals), we have

(10) $$\mathfrak{F}_i \cos(\mathfrak{F}_i n_i) = F_i \cos(F_i n_i) + 4\pi I_i \cos(I_i n_i)$$
$$= -F_e \cos(F_e n_e) = -\mathfrak{F}_e \cos(\mathfrak{F}_e n_e) = \mathfrak{F}_e \cos(\mathfrak{F}_e n_i).$$

The vector $\mathfrak{F}$ being everywhere solenoidal, its surface integral over any closed surface vanishes, so that as many unit tubes enter as leave the surface. Tubes leave the polarized body where σ is positive, and enter it where σ is negative. They form closed tubes, every one of which passes through the body. The vector $\mathfrak{F}$ is called by Maxwell the *induction*, and is characterized by the solenoidal property. The line separating the region of positive σ from those of negative is linked with all the tubes of induction belonging to the body. The induction is not in the same direction as the force F unless the polarization I is.

We obtain another physical conception of the induction by considering the force in a cavity in the conductor. By hollowing out a space in the body we remove a portion of the volume distribution, but give rise to a new surface distribution. We shall suppose the cavity so small that the volume-density of the part removed may be considered constant. Now if we consider the forces at corresponding points of geometrically similar distributions of constant densities, we have for the action of the volume-density,

$$V = \rho \iiint \frac{d\tau}{r},$$

and if we increase the dimensions in the ratio n, the element of volume and the potential at a corresponding point are $d\tau = n^3 d\tau$,

$$V' = \rho \iiint \frac{d\tau'}{r'} = \rho \iiint \frac{n^3 d\tau}{nr} = n^2 V,$$

and the force

$$-\frac{\partial V'}{\partial s'} = -\frac{\partial(n^2 V)}{\partial(ns)} = -n\frac{\partial V}{\partial s},$$

while for surface distributions

$$V' = \sigma \iint \frac{dS'}{r'} = \sigma \iint \frac{n^2 dS}{nr} = nV, \quad -\frac{\partial V'}{\partial s'} = -\frac{\partial V}{\partial s}.$$

Accordingly as we decrease the linear dimensions indefinitely, the force from the volume distribution decreases indefinitely, while the force from the surface distribution remains finite. Consequently in an infinitely small cavity the force does not depend on the volume density of the part removed, but only on the surface densities formed on the surface of the cavity. This will be at all points

$$\sigma = I \cos (In),$$

the normal being directed *into* the cavity. Suppose the cavity is in the form of a cylinder with generators in the direction of the polarization. Then the density on the sides is zero, and on the ends I and $-I$. If a is the radius of the cylinder, $2b$ its length, the action of the ends on a point at the center of the cylinder is the same as the action of two circular discs, of surface density I and $-I$, which, by § 81, is

$$4\pi I \left\{ \frac{b}{\sqrt{b^2 + a^2}} - 1 \right\}.$$

This is a function only of a/b, as we have just shown that the action is independent of the linear dimensions. If the radius is infinitely small in comparison with the length the action vanishes. Accordingly in such a cavity the force is that due to the action of the rest of the body, or

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}.$$

If on the other hand the length of the cylinder is infinitesimal in comparison with the radius, the force is $4\pi I$, so that the total force in the cavity is

$$X + 4\pi A = \mathfrak{X}, \quad Y + 4\pi B = \mathfrak{Y}, \quad Z + 4\pi C = \mathfrak{Z},$$

or the induction is equal to the force in a thin crack *perpendicular* to the lines of polarization.

122. Potential due to Polarized Distribution. If we introduce the expressions of the volume and surface densities in terms of the polarization, we obtain for the potential due to a polarized distribution

$$(1)\qquad V=\iint\frac{\sigma dS}{r}+\iiint\frac{\rho}{r}\,d\tau$$

$$=-\iint\frac{\{A\cos(nx)+B\cos(ny)+C\cos(nz)\}}{r}\,dS$$

$$-\iiint\frac{1}{r}\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right\}d\tau.$$

Integrating the volume integral by parts by Green's method, the surface integral cancels the surface integral in V, leaving V as the volume integral

$$(2)\qquad V=\iiint\left\{A\frac{\partial\left(\frac{1}{r}\right)}{\partial x}+B\frac{\partial\left(\frac{1}{r}\right)}{\partial y}+C\frac{\partial\left(\frac{1}{r}\right)}{\partial z}\right\}d\tau.$$

If as usual we use x, y, z to denote the coordinates of the attracted point, and a, b, c for the coordinates of the point of integration, we must write

$$(3)\qquad V=\iiint\left\{A\frac{\partial\left(\frac{1}{r}\right)}{\partial a}+B\frac{\partial\left(\frac{1}{r}\right)}{\partial b}+C\frac{\partial\left(\frac{1}{r}\right)}{\partial c}\right\}d\tau,\qquad d\tau=da\,db\,dc.$$

Now since

$$r^2=(x-a)^2+(y-b)^2+(z-c)^2,$$

$$(4)\qquad \frac{\partial\left(\frac{1}{r}\right)}{\partial a}=\frac{1}{r^2}\frac{x-a}{r}=\frac{1}{r^2}\cos(rx),$$

$$\frac{\partial\left(\frac{1}{r}\right)}{rb}=\frac{1}{r^2}\frac{y-b}{r}=\frac{1}{r^2}\cos(ry),$$

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial c}=\frac{1}{r^2}\frac{z-c}{r}=\frac{1}{r^2}\cos(rz),$$

the integrand is the geometrical product of the intensity of polarization and r the vector distance from the polarized element to the attracted point, divided by the cube of the distance. We might have obtained this result from the consideration of a doublet, or pair of points of equal masses of opposite signs, placed at a distance apart h, so that the moment of the doublet is $M=mh$. Then if r_1 and r_2 are the distances of the attracted point from the positive and negative ends of the doublet, we have

$$V=\frac{m}{r_1}-\frac{m}{r_2}=\frac{m(r_2-r_1)}{r_1r_2}.$$

But if h is infinitesimal, we have, neglecting infinitesimals of the second order, $r_2 - r_1 = h\cos(hr)$ and $r_1 r_2 = r^2$,

$$(5) \qquad V = \frac{mh\cos(hr)}{r^2} = \frac{M}{r^2}\cos(hr).$$

Now in a polarized body in general, the element of volume $da\,db\,dc$ contributes the potential

$$\frac{I\,da\,db\,dc\cos(Ir)}{r^2} = \frac{\widehat{Ir}\,.\,d\tau}{r^3},$$

so that we obtain the form already given.

If the direction of the axis of the doublet is that of the Z-axis, we have

$$V = \frac{Mz}{r^3},$$

to which, by § 103 (10), we find the conjugate function

$$\Psi = -\frac{M\rho^2}{r^3} = -\frac{M\sin^2(zr)}{r}.$$

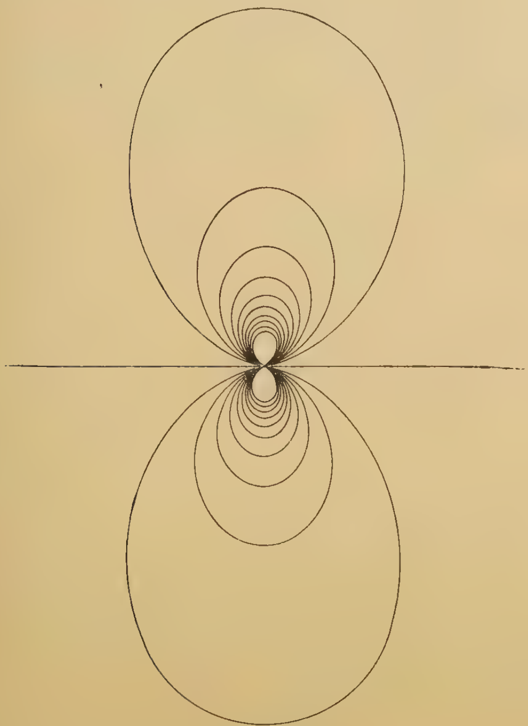

FIG. 53.

From this, the lines of force due to the doublet, or the lines $\Psi = \text{const.}$, are drawn by a simple geometrical construction, Fig. 53. (*Nature*, Vol. XXI., p. 371.) The combination of this field with a straight field in the direction of the axis of the doublet, drawn by the method of § 106, is shown in Figs. 74 and 75, according as the axis of the doublet is in the direction of, or opposite to that of the field. The lines are drawn for equal increments of Ψ.

123. Potential due to uniform Polarization. We may easily find a convenient expression for the potential at any point due to uniform polarization, $I = \text{constant}$. Let Ω represent the potential at a point P whose coordinates are x, y, z of a body occupying the space τ and filled with a single or ordinary distribution of the uniform density unity. Then after the body A of the double distribution has been displaced the distance h in the direction I, the potential at P is the same as if the body had remained fixed while P had been displaced the distance h in the opposite direction, that is,

$$\rho\left\{\Omega - \frac{\partial\Omega}{\partial h}h\right\}.$$

The potential at P of the negative body B is

$$-\rho\Omega,$$

and the potential of the double distribution is the sum of these two, or

$$-\rho h\frac{\partial\Omega}{\partial h},$$

and inserting the value of the polarization,

$$(6)\qquad V = -I\frac{\partial\Omega}{\partial h} = -I\left\{\frac{\partial\Omega}{\partial x}\cos(Ix) + \frac{\partial\Omega}{\partial y}\cos(Iy) + \frac{\partial\Omega}{\partial z}\cos(Iz)\right\}.$$

Consequently if we know the value of the potential due to a single distribution of constant density, we may obtain by differentiation the potential for a body of the same form uniformly polarized. The expression holds both for inside and outside points. The potential due to a doublet illustrates this, for the potential due to a point m is

$$\Omega = \frac{m}{r},$$

while the potential due to the doublet is

$$(7)\qquad -h\frac{\partial\Omega}{\partial h}=-mh\frac{\partial\left(\frac{1}{r}\right)}{\partial h}=\frac{M\cos(hr)}{r^2},$$

a spherical harmonic of degree -2.

124. Solenoidal and Lamellar Polarizations. The volume-density of polarized matter has been found, § 120 (6), to be equal to the convergence of the polarization. If the polarization is solenoidal, the volume-density vanishes, and the polarization is equal to a surface distribution, as in the original assumption of § 120. We may then divide the body into tubes of polarization, or *polarized solenoids*. Such a solenoid possesses the property that if it be cut anywhere the two cut ends will bear equal and opposite charges, their amounts being the same wherever the cut be made. The potential due to a solenoid of infinitesimal section depends only on the position of its ends, and a solenoid may be considered as equivalent to a doublet of points at a finite distance apart. Again the polarization may be lamellar, that is it may be the vector differential parameter of a function ϕ which will be called the potential of polarization. We then have

$$(8)\qquad A=\frac{\partial\phi}{\partial a},\quad B=\frac{\partial\phi}{\partial b},\quad C=\frac{\partial\phi}{\partial c}.$$

Outside the polarized body, since $I=0$, ϕ must be constant, and accordingly discontinuous at the surface.

Inserting this in the value of V the potential becomes

$$(9)\qquad V=\iiint\left\{\frac{\partial\phi}{\partial a}\frac{\partial\left(\frac{1}{r}\right)}{\partial a}+\frac{\partial\phi}{\partial b}\frac{\partial\left(\frac{1}{r}\right)}{\partial b}+\frac{\partial\phi}{\partial c}\frac{\partial\left(\frac{1}{r}\right)}{\partial c}\right\}d\tau.$$

Applying Green's theorem we obtain

$$(10)\qquad V=-\iint\phi\frac{\partial\left(\frac{1}{r}\right)}{\partial n}dS-\iiint\phi\Delta\left(\frac{1}{r}\right)d\tau.$$

But since $1/r$ is harmonic except for $r=0$, if the attracted point is outside of the polarized body, V is given by the surface integral,

$$(11)\qquad V=-\iint\phi\frac{\partial\left(\frac{1}{r}\right)}{\partial n}dS.$$

Accordingly the potential at points outside of a lamellarly polarized body depends only on its form and position, and on the values of the potential of polarization at the surface.

If the attracted point is within the substance of the polarized body, we may integrate (9) in the other manner, interchanging ϕ and $1/r$, obtaining

$$(12)\qquad V = -\iint \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS - \iiint \frac{1}{r}\Delta\phi\,d\tau,$$

which, by the theorem of § 83 (5) or § 84 (12) applied to ϕ, becomes

$$(13)\qquad V = 4\pi\phi - \iint \phi \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\,dS.$$

In the case of lamellar polarization the induction becomes

$$(14)\quad \mathfrak{X} = -\frac{\partial V}{\partial x} + 4\pi\frac{\partial\phi}{\partial x},\ \mathfrak{Y} = -\frac{\partial V}{\partial y} + 4\pi\frac{\partial\phi}{\partial y},\ \mathfrak{Z} = -\frac{\partial V}{\partial z} + 4\pi\frac{\partial\phi}{\partial z},$$

so that the induction, being the parameter of the function $-V + 4\pi\phi$, is also lamellar. For both inside and outside points, this function is equal, except for a constant, to the surface integral

$$(15)\qquad \iint \phi \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\,dS,$$

as we see from (13) and (11), together with the fact that outside ϕ is constant.

125. Polarized Shells. The characteristic of a lamellar polarization is that if we construct two infinitely near equipotential surfaces of polarization $\phi = \phi_1$ and $\phi = \phi_2$, the polarization is normal to them at all points, inversely proportional to the distance between them, and in the direction from the smaller to the larger value of ϕ. The portion of matter included between the two surfaces, which need not be closed, is called a simple polarized shell. If we consider the infinitesimal portion of the potential due to such an unclosed shell, the surface integral (11) is taken over both sides of the shell, the portion over the edge vanishing, since the width of the edge is infinitesimal. Consequently, replacing n, the internal normal, by n_1 and n_2, away from

the shell,

$$(16)\quad V = \phi_1 \iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n_1} dS + \phi_2 \iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n_2} dS = (\phi_1 - \phi_2) \iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n_1} dS.$$

The geometrical integral,

$$(17)\qquad \iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n_1} dS = -\iint \frac{1}{r^2}\frac{\partial r}{\partial n_1} dS = -\iint \frac{\cos(rn_1)\, dS}{r^2},$$

has been found in § 39 to be equal to the solid angle ω subtended at P by the surface S, if n_1 points toward the side of S on which P lies. Consequently the potential at any point P due to the shell is equal to the product of the difference of potential of polarization on the two sides of the shell by the solid angle subtended by the shell at P, the potential being positive if P is on the positive side of the shell, that is, the side toward which the polarization is directed.

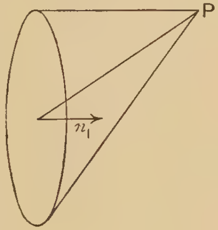

FIG. 54.

Now we have seen in § 39 (5) that the solid angle integral is equal to -4π for a point inside a closed surface, and to zero for an outside point, that is, it experiences a discontinuity of 4π as P crosses the surface. When the surface is not closed the same thing takes place. For the integral

$$\omega = -\iint \frac{\cos(nr)}{r^2} dS$$

is a continuous function of P so long as r is not zero, that is, so long as P does not lie on the surface. If P lies on the surface, the integral has an infinite element. We remove this by cutting out a small area around P. If now ω' be that part of the integral due to the remainder of the surface, ω' is finite and continuous even when P passes through the surface. As P approaches the surface the solid angle subtended by the small area cut out, which may be treated as plane, approaches 2π, so that at the surface on the side 1, $\omega_1 = \omega' + 2\pi$. At an infinitely near point on the side 2, however, the cosine in the numerator has changed sign, for the small area, so that the solid angle subtended by the latter is to have the negative sign. Accordingly on the side 2, $\omega_2 = \omega' - 2\pi$, and accordingly,

$$(18)\qquad \omega_1 - \omega_2 = 4\pi,$$

and the potential V experiences a discontinuity of $-4\pi(\phi_1 - \phi_2)$ in passing through the shell from the positive to the negative side.

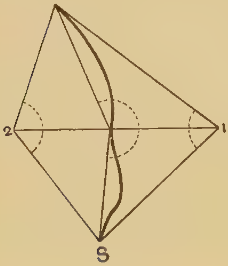

FIG. 55.

The discontinuity may be also explained by considering the solid angles subtended at points 1 and 2 approaching a point on the surface from opposite sides. If the solid angles have different signs on opposite sides, as the points come together the sum of the absolute values of the two angles approaches 4π, so that at the surface

$$\omega_1 + (-\omega_2) = 4\pi.$$

If the thickness of the shell is ϵ, the polarization is $(\phi_1 - \phi_2)/\epsilon$, and the moment of the equal and opposite charges on the element of surface dS on the opposite sides of the shell is, since the volume of the element is ϵdS, equal to

$$(\phi_1 - \phi_2)\, dS.$$

Thus the surface density times the thickness, or the moment of polarization per unit of surface of a simple polarized shell, is constant. The value of the constant $\Phi = \phi_1 - \phi_2$ is called the *strength* of the shell, and it is this strength that is multiplied by the solid angle in the expression for the potential.* Suppose now that the intensity of polarization increases without limit, so that the strength of the shell $\phi_1 - \phi_2$ is finite, instead of infinitesimal. Then the difference of potential on the two sides of the shell is finite, or the potential is discontinuous in crossing the shell, by the amount

$$V_1 - V_2 = 4\pi\Phi.$$

The derivative, $\partial V/\partial n$, is however continuous. We may prove the converse of this proposition. If a function satisfies Laplace's equation, vanishes at infinity, and is continuous everywhere except at a certain surface, its first derivatives being everywhere continuous, the function represents the potential of a double distribution on the surface of discontinuity. If the function were uniform and continuous, it must, by Dirichlet's principle, vanish everywhere. The demonstration will be given in § 210.

* Gauss. "Allgemeine Theorie des Erdmagnetismus," § 38, 1839. *Werke*, Bd. v., p. 119.

126. Energy of Polarized Distributions. If a polarized distribution is placed in a field of which the potential is V, their mutual energy is, by § 117,

$$W = \iint V\sigma dS + \iiint V\rho d\tau,$$

which, by § 120, is equal to

$$(1) \quad W = -\iint V\{A\cos(nx) + B\cos(ny) + C\cos(nz)\}\, dS \\ - \iiint V\left\{\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right\} d\tau.$$

Integrating by Green's theorem, this becomes

$$(2) \qquad W = \iiint \left\{A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} + C\frac{\partial V}{\partial z}\right\} d\tau.$$

The integrand is the negative of the geometric product of the polarization and the force of the field. This result may be obtained directly for a doublet as we obtained the potential in § 122.

If the polarization is lamellar, the energy of the distribution is

$$(3) \quad W = \iiint \left\{\frac{\partial \phi}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial V}{\partial z}\right\} d\tau \\ = -\iint \phi \frac{\partial V}{\partial n}\, dS - \iiint \phi \Delta V d\tau.$$

For a polarized shell the volume integral disappears, and the surface integral becomes

$$(4) \qquad W = \iint \phi_1 \frac{\partial V}{\partial n_1}\, dS + \iint \phi_2 \frac{\partial V}{\partial n_2}\, dS_2 = \Phi \iint \frac{\partial V}{\partial n_1}\, dS.$$

Accordingly the energy of a polarized shell is equal to the product of its strength by the flux of force through it in the direction *opposite* to the polarization.

If we wish to find the energy of the polarized distribution itself, we must put for V in the above formulae the potential due to the distribution itself, and multiply by the factor one-half, as in § 117. It is important to notice that the energy of polarized distributions is defined as the work that they are capable of doing if every particle is allowed to retire to infinity carrying

its own charge. But if the distribution should be cut up into small parts, new surface densities would appear on each part. To prevent this the distribution must be supposed split up into infinitely thin shreds along the lines of polarization—on separating these from each other no new surface densities would be formed, so that the energy as calculated would be the work obtained by letting these shreds be bodily removed to infinite distances from each other. Similarly polarized shreds side by side of course repel each other, so that this energy is positive. If we should further break up each shred into infinitely short lengths, and separate these from each other, we should have to do positive work to pull them apart, and if we should remove all the parts to infinite distances from each other, it has been shown by Lord Kelvin* that we should have to do exactly as much work as was before obtained by separating the shreds. Consequently the energy must be defined by the first operation alone.

127. Development of Potential of Polarized Body in Spherical Harmonics. We have seen in § 123 (7) that the potential due to a doublet placed at the origin is a spherical harmonic. We may develop the potential due to any polarized distribution in a series of spherical harmonics. If we call r and r' the distances from the origin of the attracted point x, y, z, and the point of integration a, b, c, so that

$$r^2 = x^2 + y^2 + z^2, \quad r'^2 = a^2 + b^2 + c^2,$$

we have for the distance between the two points, by § 100 (22), if $r' < r$,

$$\frac{1}{d} = \frac{1}{r}\left\{P_0(\mu) + \frac{r'}{r}P_1(\mu) + \frac{r'^2}{r^2}P_2(\mu) + \dots\right\},$$

where μ, the cosine of the angle between r, r', is

$$(ax + by + cz)/rr'.$$

Inserting this value and those of P_0, P_1, P_2, § 100, we have

$$\frac{1}{d} = \frac{1}{r} + \frac{ax + by + cz}{r^3} + \frac{1}{2}\frac{3(ax + by + cz)^2 - r^2r'^2}{r^5} + \dots.$$

* Thomson. "On the Mechanical Values of Distributions of Matter, and of Magnets." *Papers on Electrostatics and Magnetism*, p. 437.

Now inserting this value of $1/d$ in the expression for the potential in § 122 (3), in which $1/r$ is to be replaced by $1/d$,

$$V = \iiint \left\{ A \frac{\partial \left(\frac{1}{d}\right)}{\partial a} + B \frac{\partial \left(\frac{1}{d}\right)}{\partial b} + C \frac{\partial \left(\frac{1}{d}\right)}{\partial c} \right\} d\tau,$$

or performing the differentiations, x, y, z, r being constant,

$$V = \iiint A \left(\frac{x}{r^3} + \frac{3(ax + by + cz)x - r^2 a}{r^5} + \dots \right) d\tau$$

$$+ \iiint B \left(\frac{y}{r^3} + \frac{3(ax + by + cz)y - r^2 b}{r^5} + \dots \right) d\tau$$

$$+ \iiint C \left(\frac{z}{r^3} + \frac{3(ax + by + cz)z - r^2 c}{r^5} + \dots \right) d\tau,$$

and collecting in powers of x, y, z, which can be taken out from under the integral signs, we get the development in spherical harmonics

$$V = V_{-2} + V_{-3} + V_{-4} + \dots,$$

$$V_{-2} = \frac{\bar{A}x + \bar{B}y + \bar{C}z}{r^3},$$

$$V_{-3} = \frac{(2L - M - N)x^2 + (2M - N - L)y^2 + (2N - L - M)z^2 + 3(Pyz + Qzx + Rxy)}{r^5},$$

where the coefficients are the definite integrals

$$\bar{A} = \iiint A d\tau, \quad \bar{B} = \iiint B d\tau, \quad \bar{C} = \iiint C d\tau,$$

$$L = \iiint Aa d\tau, \quad M = \iiint Bb d\tau, \quad N = \iiint Cc d\tau,$$

$$P = \iiint (Bc + Cb)\, d\tau, \quad Q = \iiint (Ca + Ac)\, d\tau, \quad R = \iiint (Ab + Ba)\, d\tau.$$

In like manner the coefficients in the harmonics of higher orders are definite integrals throughout the polarized body of the components of polarization multiplied by powers of the coordinates of the point of integration. By a change of the origin the integrals L, M, N may be made to vanish. For putting

$$a = a_0 + a', \quad b = b_0 + b', \quad c = c_0 + c',$$

we have

$$L = a_0 \iiint A\,d\tau + \iiint A a'\,d\tau = a_0\overline{A} + L',$$

$$M = b_0 \iiint B\,d\tau + \iiint B b'\,d\tau = b_0\overline{B} + M',$$

$$N = c_0 \iiint C\,d\tau + \iiint C c'\,d\tau = c_0\overline{C} + N'.$$

and if we choose

$$a_0 = \frac{L}{\overline{A}}, \quad b_0 = \frac{M}{\overline{B}}, \quad c_0 = \frac{N}{\overline{C}},$$

the integrals L', M', N' vanish, and V_{-3} reduces to three terms. The values of the integrals $\overline{A}$, $\overline{B}$, $\overline{C}$, are not changed by this change of origin, but those of all the others are.

The new origin is called the *center* of the polarized distribution. If the polarization is uniform, it is the center of gravity of the body. If we find a vector M whose components are $\overline{A}$, $\overline{B}$, $\overline{C}$, we have

$$V_{-2} = \frac{M\{\cos(Mx)\cos(rx) + \cos(My)\cos(ry) + \cos(Mz)\cos(rz)\}}{r^2}$$

$$= \frac{M\cos(Mr)}{r^2}.$$

But this is equal to the potential due to a doublet of moment M situated at the center. M is called the *moment* of the polarized body, and since at great distances the first terms are relatively the most important, we see that at great distances the body acts as if concentrated at its center. The line through the center having the direction of M is called the *axis of* the distribution.

PART II.

ELECTROSTATICS, ELECTROKINETICS AND MAGNETISM.

CHAPTER VI.

ELECTRICAL PHENÔMENA. SYSTEMS OF CONDUCTORS.

128. Fundamental Experiments. We shall begin the treatment of Electricity by the description of a number of simple experiments, for the most part due to Faraday and described by Maxwell, the explanation of which will devolve upon the theory, when mathematically established.

EXPERIMENT I. Let a piece of glass and a piece of resin, neither of which exhibits properties different from those of ordinary bodies, be hung up near each other by silk threads. They do not affect each other, and the threads hang vertically. Let the glass and the resin be rubbed together, and left in contact. They still exhibit no peculiar properties. Let them now be separated. They attract each other, and the strings take an inclined position. The system composed of the glass and resin has now acquired energy, which has enabled it to do work against the force of gravity in lifting the two bodies through a certain distance.

Let a second piece of glass be rubbed with a second piece of resin, and be similarly suspended. Then it may be observed that the two pieces of glass repel each other, and have therefore acquired energy, which is evinced by their overcoming gravity in lifting themselves.

The two pieces of resin in like manner repel each other. Each piece of glass attracts each piece of resin. All of these phenomena, each of which indicates the acquisition of a positive amount of potential energy, are known as Electrical phenomena, and the bodies exhibiting them are said to be *electrified*, or charged with Electricity.

The properties of the two pieces of glass are similar, but opposite to those of the resin. What the glass attracts the resin repels, and vice versa. Bodies repelled by the glass and attracted by the resin are said to be *vitreously*, those attracted by the glass and repelled by the resin, *resinously* electrified. By general convention we say *positive*, instead of vitreous, *negative* for resinous.

EXPERIMENT II. Let a hollow metal vessel be hung up by silk threads, and let a lid completely closing it be also so hung, so that it may be removed and replaced without touching it. Then if the electrified glass be hung inside the vessel without touching it, and the lid placed on, the outside of the vessel will be found vitreously electrified, and the manner of the electrification will be exactly the same in whatever part of the interior the glass may be. That is to say, if we place successively at different points of the external space the same small electrified body, it will be acted upon at each point by a certain force. The direction and magnitude of this force determine a vector called the strength of the electrical field of force. The field may be geometrically represented by lines of force in the usual manner. The electric field is the tangible evidence of the electrification, and the measurement of a force is the means of its measurement. We may therefore describe the above experiment by saying that the field external to the closed metal vessel is independent of the position of the charged body within. If the glass be removed without touching the vessel, the electrification of the glass will be unchanged, and that of the vessel will have disappeared. If resin be substituted for glass the outside of the vessel will be negatively electrified. Such electrification, which depends on the proximity of electrified bodies, is called electrification by *influence*, or *induction*. In this manner a body may acquire energy without contact with other bodies, and it is natural to suppose that the energy has passed through the intervening medium from the electrified body. Such a medium,

which allows electrical influences to pass through it, is called a *dielectric*, as was proposed by Faraday*.

Experiment III. Let the vessel be positively electrified by induction as before, let a second vessel be suspended by silk threads, and let a metallic wire, similarly suspended, be made to touch both simultaneously. The second will be found to be positively electrified, and the positive electrification of the first is lessened.

Experiment IV. If instead of a metal wire we had used a rod of glass, sealing wax, or hard rubber, no such effect would have been produced. Bodies may accordingly be divided into two classes, 1°, those which, like metals, allow a transference of electrification from place to place. These are called *conductors*. The second body above is said to be electrified by *conduction*: 2°, those which do not allow such transfer. These are called non-conductors or *insulators*. The dividing line cannot be drawn with perfect sharpness, since no bodies have been found to be absolutely non-conducting. All insulators are dielectrics, but not all dielectrics are necessarily insulators.

Experiment V. In Experiment II it was shown that the external electrification of the vessel due to the introduction of the electrified glass was independent of the position of the latter in the vessel. If we now introduce the piece of glass together with the piece of resin with which it was rubbed, without touching the vessel, the electrification of the latter disappears. We therefore conclude that the electrification of the glass and resin, which are able to counteract each other's effects, are equal in amount. By putting in a number of bodies, and examining the external field, we may show that the induced electrification is proportional to their algebraic sum. We thus have an experimental method of adding the effects of several electrifications without altering the electrifications.

Experiment VI. Let there be two insulated metallic vessels, A and B, and let the glass be introduced into A and the resin into B, and let them be connected by a wire. All electrification disappears, as was to be expected. Now let the wire be removed, and then let the glass and resin be taken out. It will be found

* Exp. Res., § 1168.

that A is now negatively electrified, and B positively. By introducing A and the glass together into a larger metal vessel C, its outside will be found to have no charge, consequently the induced charge on A is equal and opposite to that of the glass. In like manner the charge of B may be shown to be equal and opposite to that of the resin. The charge of A, which is not apparent as long as the glass is within, is said to be *bound* by the inducing charge of the glass, and resides on the inside of A. By the withdrawal of the glass it becomes *free*, and appears on the outside of A. We have thus a method of charging a vessel with an electrification equal in amount and opposite in kind to that of a given electrified body without changing its electrification.

EXPERIMENT VII. Let the vessel B, charged with a quantity of positive electricity, which we shall take for a provisional unit, be introduced into the vessel C without touching it. C will be found charged on the outside with a unit of positive electricity. Now let B touch the inside of C. The external electrification is unchanged. If B be now removed from C without touching it, and taken to a distance, the field external to C is still unchanged, that is, C is charged with a unit of electricity, but B is *completely discharged.* If B be now recharged with a unit of positive electricity, and again introduced and made to touch C, on removal it will again be found to be completely discharged, and the charge of C will be increased by one unit. This may be repeated indefinitely, and no matter how highly C may become charged, it will be found that B is always completely discharged. This is a cardinal point in the theory of electricity. Since when in contact B virtually forms a part of the conductor C, we may state that there is no electrification on the inside of a charged conductor left to itself. We now have a means of charging a body with any number of units of electricity. A machine for the purpose of generating electricity on this principle is Kelvin's Replenisher, whose theory will be considered later.

The last experiment may be modified by examining the field of force within a hollow charged conductor. This cannot be done by introducing anything through a hole, but was accomplished by Faraday by building a closed conductor large enough for a person to remain inside. Even when the outside was so highly electrified that large sparks were flying off from it, the strength of the field at points within was absolutely zero.

Experiment VIII. Suppose that while the pieces of electrified glass are suspended as in Experiment I, we surround them with a dielectric fluid insulator, such as turpentine, kerosene, or melted paraffin. It will be found that, if the buoyancy of the liquid be just counterbalanced by weights, the threads will now hang more nearly vertically, showing that the repulsion is less. The energy of the system is consequently less. We see then that the energy of a system of electrified bodies depends not only on their charges and positions, but on the nature of the dielectric medium in which they are placed. The consideration of the part played by the medium is now one of the principal parts of electrical theory.

If any of the other experiments be repeated with the closed vessel filled with any dielectric fluid, the results will be unchanged, showing that the values of *charges induced on a closed conductor by charges within are unaffected by the dielectric.*

We will now briefly recapitulate the results of our experiments.

We may examine the nature and the magnitude of the charge of any electrified body without altering it, by placing it within an insulated hollow conductor without touching it, and examining the charge induced on the outside of the latter.

The amount of electricity on a body remains unchanged, unless it be put in conducting communication with another body.

When a body electrifies another by conduction, the quantity of electricity on the two remains unchanged.

When electricity is produced by friction (or otherwise, as we shall find) equal quantities of positive and negative electricity are produced.

When electrification is caused by induction from a body surrounded by a conductor, the amount of electricity on the inside of the conductor is equal in quantity and opposite in sign to the charge of the inducing body.

There is no electricity on the inside surface of a closed hollow conductor, charged but under the action of no internal bodies.

The forces between charged bodies, and their electrical energy, depend on the dielectric medium in which they are placed. The charges induced on closed conductors by charges within do not.

129. Mathematical Conclusions. Law of Force. We have used the words electrification, electricity, and charge to denote a measurable quantity, which possesses the property of conservation, that is of remaining unchanged in amount. For if electricity disappears, it is by the disappearance of two equal quantities of opposite sign, whose algebraic sum was zero. We need define these terms no further than by their properties, and for the present, the single property of exerting force is sufficient. We may speak of electrification occupying definite portions of space, for the field of force is such that lines of force issue from positive electrifications and run into negative electrifications. Electrifications being always examined by examining their fields of force, we may consider the field of force as specifying the electrification. Certain writers have gone farther, and insisted that electricity does not exist, but that lines of force and electrical energy are the only real entities. Such a question is purely metaphysical, and of no importance to the physicist. It is obviously of no importance whether we define electricity as that which exists where lines of force converge, or say that the electricity exerts force upon other electrifications. If we wish to use the term "electrical fluid" or "matter" we may do so, provided we use "fluid" or "matter" simply as convenient terms, without attributing to electricity any of the properties of ordinary fluids or of matter. It has, so far as we know, no inertia, the fundamental property of matter, nor is it incompressible. We may then define a charge of electricity as a "something," "fluid," or "matter," which possesses the unique property of repelling or attracting other charges of electricity, according to the signs of the two charges. By definition the force is proportional to the charge, and it is natural to suppose that the force between two electrified elements will be in the line joining them, and proportional to some function of the distance. Experiment VIII shows that the force depends on something beside the distance, but if we suppose all space to be filled with the same dielectric medium, such as air, the assumption is justified by experiment. This supposition will accordingly be made for the present. We shall also suppose all conductors to be made of a single material.

We shall now deduce the law of the force from the result of Experiment VII,—that there is no force within a hollow conductor. Let the conductor be in the form of a sphere. On account of

symmetry the charge is so distributed that equal areas possess equal charges. Let the charge per unit area be σ, and let us find the form of $f(r)$, so that the resultant of all the forces $\sigma dS f(r)$ due to all the elements dS at distances r from a given point Q within the sphere, shall be zero when resolved in any direction.

On account of symmetry, the force acting on Q must be in the direction of the radius OQ. We shall accordingly consider the radial component R. Let $OQ = b$, and let the radius of the sphere be a. Let the distance PQ, where P is any point on the surface of the sphere, be r, and let the polar coordinates of P be θ, ϕ, the co-latitude θ being measured from the radius OQ. Let the angle PQO be δ. Then the whole force at Q resolved along the radius OQ is proportional to

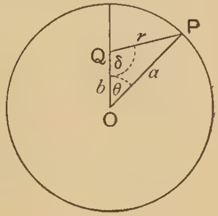

FIG. 56.

$$(1)\qquad R = \iint \sigma dS \,.\, f(r) \cos \delta = \sigma \int_0^{\pi} \int_0^{2\pi} f(r) \cos \delta \,.\, a^2 \sin \theta d\theta d\phi.$$

We may at once integrate with respect to ϕ,

$$(2)\qquad R = 2\pi a^2 \sigma \int_0^{\pi} f(r) \cos \delta \sin \theta d\theta.$$

Now OQ is the sum of the projections of OP and PQ on the radius

$$r \cos \delta + a \cos \theta = b,$$

$$(3)\qquad \cos \delta = \frac{b - a \cos \theta}{r}.$$

From the relation between the sides of the triangle POQ,

$$(4)\qquad r^2 = a^2 + b^2 - 2ab \cos \theta,$$

we get on partial differentiation with respect to b,

$$(5)\qquad r \frac{\partial r}{\partial b} = b - a \cos \theta,$$

$$\frac{\partial r}{\partial b} = \frac{b - a \cos \theta}{r} = \cos \delta.$$

Substituting this value of $\cos \delta$ in the integral,

$$R = 2\pi a^2 \sigma \int_0^{\pi} f(r) \frac{\partial r}{\partial b} \sin \theta d\theta,$$

and if we call $\qquad f(r) = \Phi'(r),$

so that

$$f(r)\frac{\partial r}{\partial b}=\frac{\partial\Phi(r)}{\partial b},$$

we have

(6) $$R=2\pi a^2\sigma\frac{\partial}{\partial b}\int_0^\pi\Phi(r)\sin\theta\,d\theta.$$

We may now change the variable from θ to r by differentiating the relation

$$r^2=a^2+b^2-2ab\cos\theta,$$

$$r\,dr=ab\sin\theta\,d\theta,$$

(7) $$\sin\theta\,d\theta=\frac{r\,dr}{ab}.$$

For $\theta=0$, $r=a-b$, and for $\theta=\pi$, $r=a+b$, so that

$$R=2\pi a^2\sigma\frac{\partial}{\partial b}\int_{a-b}^{a+b}\frac{\Phi(r).\,r\,dr}{ab}.$$

Calling $$r\Phi(r)=\Psi'(r),$$

we have

(8) $$\int_{a-b}^{a+b}\Psi'(r)\,dr=\Psi(a+b)-\Psi(a-b),$$

and

(9) $$R=2\pi a\sigma\frac{\partial}{\partial b}\left\{\frac{1}{b}(\Psi(a+b)-\Psi(a-b))\right\}.$$

By the conditions of the problem this must vanish, so that we have the differential equation for Ψ,

(10) $$\frac{d}{db}\left\{\frac{1}{b}(\Psi(a+b)-\Psi(a-b))\right\}=0,$$

which being integrated gives

(11) $$\frac{1}{b}\{\Psi(a+b)-\Psi(a-b)\}=C,$$

$$\Psi(a+b)-\Psi(a-b)=Cb,$$

a functional equation to determine Ψ. Differentiating twice with respect to b,

$$\Psi''(a+b)-\Psi''(a-b)=0,$$

(12) $$\Psi''(a+b)=\Psi''(a-b).$$

This equation holding for *all* values of a and b, since $a+b$ and $a-b$ are entirely independent variables, Ψ'' must have the same value for all arguments. Accordingly, putting r for the argument,

$$(13)\qquad \Psi''(r) = A,$$

$$(14)\qquad \Psi'(r) = Ar + B = r\Phi(r),$$

$$(15)\qquad \Phi(r) = A + \frac{B}{r},$$

$$(16)\qquad \Phi'(r) = f(r) = -\frac{B}{r^2}.$$

Consequently the force $f(r)$ is inversely proportional to the square of the distance. This proof is due to Laplace*. The law of force was also deduced by Cavendish as a consequence of the fact that a conductor is completely discharged by contact with the interior of a closed conductor. The experiment was repeated very carefully by Maxwell†. The law of the force may also be deduced from the result of Experiment II.

The law of the inverse square was obtained by Coulomb by direct experiment with the torsion balance, but such experiments could not be exact enough to demonstrate the law with the same accuracy as by reasoning from the results of the experiments of Cavendish and Faraday.

130. Dimensions of Electrical Quantities. Since charges of electricity in a uniform dielectric medium act on each other according to the Newtonian Law, the whole mathematical investigation of Newtonian forces and potentials at once becomes applicable. The volume density of electrification, or the charge per unit of the volume, will be denoted by ρ, and the surface density, or the charge of unit area of a superficial distribution, by σ. The charge of a body e is

$$(1)\qquad e = \iiint \rho\, d\tau + \iint \sigma\, dS,$$

and the potential at a point,

$$(2)\qquad V = \iiint \frac{\rho}{r}\, d\tau + \iint \frac{\sigma}{r}\, dS$$

* *Mécanique Céleste*, I. 2.

† *Electricity and Magnetism*, I. p. 79.

is the work that must be done against the electrical forces in bringing a unit of positive electricity from an infinite distance to the given point. Positive electricity tends to move from places of greater to places of less potential, and negative electricity the contrary.

The unit of electricity must be defined as the amount of electricity, which if concentrated on a very small body, will repel a body similarly charged and placed at unit distance from it in vacuo, with unit force. This unit is the basis of the *electrostatic system* of electrical units. In the C.G.S. system, the unit of electricity repels a similar unit at a distance of one centimeter with a force of one dyne.

It is necessary, as shown by Experiment VIII, to specify the medium in defining the unit. If air were adopted instead of vacuum, that is, ether, the difference would be so slight* as to escape detection by all but the most refined electrostatic measurements at present in use, hence we may with very slight error consider all experiments to be made in air.

The dimensions of the unit of electricity are found from the equation of force

$$\frac{ee'}{r^2} = \text{Force},$$

$$\frac{[e^2]}{[L^2]} = \left[\frac{ML}{T^2}\right], \quad [e] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}],$$

and the C.G.S. unit of electricity is

$$1 \text{ gm.}^{\frac{1}{2}} \text{ cm.}^{\frac{3}{2}} \text{ sec.}^{-1}.$$

The dimensions of ρ, σ, V are found from

$$[\rho] = \text{Volume density} = \frac{\text{Electrification}}{\text{Volume}} = \frac{[e]}{[L^3]} = [M^{\frac{1}{2}} L^{-\frac{3}{2}} T^{-1}],$$

$$[\sigma] = \text{Surface density} = \frac{\text{Electrification}}{\text{Surface}} = \frac{[e]}{[L^2]} = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}],$$

$$[V] = \text{Potential} = \frac{\text{Electrification}}{\text{Distance}} = \frac{[e]}{[L]} = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}].$$

* Less than one part in a thousand.

The strength of electric field, or the intensity of electric force at a point, F, is defined as the force acting on unit of electricity placed at the point. Its dimensions are obtained from

$$[Fe] = \text{Force} = \left[\frac{ML}{T^2}\right],$$

$$[F] = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}].$$

This agrees with the definition $F = -\dfrac{\partial V}{\partial n}$ which is of dimensions

$$\frac{\text{Potential}}{\text{Length}} = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}].$$

The energy of the system may be written in either of the forms, § 118 (8),

$$W_d = \frac{1}{2}\iint \sigma V dS + \frac{1}{2}\iint \rho V d\tau,$$

the integrals having dimensions

$$[\text{Surface-density} \times \text{Potential} \times \text{Surface}] = [ML^2T^{-2}],$$

and $$[\text{Volume-density} \times \text{Potential} \times \text{Volume}],$$

or, § 118 (10),

$$W_f = \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,$$

the integral having the dimensions

$$[\text{Field-strength}^2] \times [\text{Volume}] = [ML^2T^{-2}],$$

giving in either case the proper dimensions for energy.

131. Electrical Equilibrium. Suppose we have an electric field due to the presence of a number of charged insulating bodies D, together with a number of conductors K, insulated and originally either charged or not. The charges of the bodies D cannot move in the bodies, since they are insulators. We shall assume that the dielectric properties of the bodies D are the same as those of air. The electrification of the conductors, however, may move in any manner in the conductors, subject to the condition that the total charge e_s of any conductor K_s is constant. By the principle of virtual work, we can find the condition for equilibrium.

Suppose that in any assumed distribution V is the potential due to the total electrification of the conductors K, in which the volume and surface densities are ρ and σ. Let V' be the potential

due to the fixed electrification of D in which the volume density is ρ'. The total energy of the system will be

$$(1)\qquad W = \frac{1}{2}\iiint_K (V+V')\,\rho\, d\tau + \frac{1}{2}\iint_K (V+V')\,\sigma dS + \frac{1}{2}\iiint_D (V+V')\,\rho' d\tau,$$

or

$$(2)\qquad W = \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial (V+V')}{\partial x}\right)^2 + \left(\frac{\partial (V+V')}{\partial y}\right)^2 + \left(\frac{\partial (V+V')}{\partial z}\right)^2\right\} d\tau.$$

Suppose now that if we change ρ in the conductors to $\rho+\delta\rho$ and σ to $\sigma+\delta\sigma$, the integral W becomes $W+\delta W$, while ρ' and V' are unchanged at all points, since the electrification of D is unchanged. Also since the charges of the conductors are to be unchanged, we must have for any conductor K_s

$$(3)\qquad \delta e_s = \delta\left[\iiint_{K_s} \rho\, d\tau + \iint_{K_s} \sigma dS\right] = 0.$$

The condition for stable equilibrium is that for all possible values of the functions $\delta\rho$ and $\delta\sigma$ subject to the conditions $\delta e_s = 0$, we must have $\delta W > 0$, (§ 58).

Making the above changes in the integral (1), we have

$$(4)\qquad W+\delta W = \frac{1}{2}\iiint_K (V+\delta V+V')(\rho+\delta\rho)\, d\tau + \frac{1}{2}\iint_K (V+\delta V+V')(\sigma+\delta\sigma)\, dS + \frac{1}{2}\iiint_D (V+\delta V+V')\,\rho' d\tau,$$

and subtracting W, we get

$$(5)\qquad \delta W = \frac{1}{2}\iiint_K \{(V+\delta V+V')\,\delta\rho + \rho\,\delta V\}\, d\tau + \frac{1}{2}\iint_K \{(V+\delta V+V')\,\delta\sigma + \sigma\delta V\}\, dS + \frac{1}{2}\iiint_D \rho'\delta V d\tau.$$

Now V, V', δV are potentials due respectively to distributions of densities ρ, σ in the space K for V, ρ' in the space D for V', and $\delta\rho$, $\delta\sigma$ in the space K for δV, and accordingly by Gauss's theorem of mutual potential energy, § 117 (5),

$$\iiint_K \rho\delta V d\tau = \iiint_K V\delta\rho d\tau,$$

(6)
$$\iint_K \sigma\delta V dS = \iint_K V\delta\sigma dS,$$

$$\iiint_D \rho'\delta V d\tau = \iiint_K V'\delta\rho d\tau + \iint_K V'\delta\sigma dS.$$

In virtue of these equalities, the integral reduces to

(7)
$$\delta W = \iiint_K (V+V')\,\delta\rho d\tau + \iint_K (V+V')\,\delta\sigma dS$$
$$+\frac{1}{2}\iiint_K \delta V\delta\rho d\tau + \frac{1}{2}\iint_K \delta V\,\delta\sigma dS,$$

all the integrals being taken throughout all the conductors only. In order to take account of the conditions $\delta e_s = 0$ we must multiply each such equation by an arbitrary constant, $-c_s$, and add the sum to the above value of δW,

(8)
$$\delta W - \Sigma_s c_s \delta e_s \geqq 0,$$

that is, $=0$ to the first order of small quantities, while the terms of second order must be positive for a minimum.

Introducing the values of δe, (3)

(9)
$$\delta e_s = \delta\iiint_{K_s} \rho d\tau + \delta\iint_{K_s} \sigma dS = \iiint_{K_s} \delta\rho d\tau + \iint_{K_s} \delta\sigma dS,$$

we get

(10)
$$\Sigma_s \left\{\iiint_{K_s} (V+V'-c_s)\,\delta\rho d\tau + \iint_{K_s} (V+V'-c_s)\,\delta\sigma dS \right.$$
$$\left. +\frac{1}{2}\iiint_{K_s} \delta V\delta\rho d\tau + \iint_{K_s} \delta V\delta\sigma dS\right\} \geqq 0.$$

The equations of condition having been introduced, we may treat $\delta\rho$ and $\delta\sigma$ as arbitrary, and if we put in each conductor

(11)
$$V + V' - c_s = 0,$$

the above reduces to the terms of second order

$$\frac{1}{2}\iiint_K \delta V\delta\rho d\tau + \frac{1}{2}\iint_K \delta V\delta\sigma dS,$$

which being the energy of the distribution $\delta\rho$, $\delta\sigma$ may be written

$$\frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial \delta V}{\partial x}\right)^2 + \left(\frac{\partial \delta V}{\partial y}\right)^2 + \left(\frac{\partial \delta V}{\partial z}\right)^2\right\} d\tau,$$

which is necessarily positive. Consequently the condition for stable equilibrium is that in each conductor the total potential $V + V'$ is constant*.

The integral which takes a minimum in the above investigation is the same one that appears in the demonstration of Kelvin and Dirichlet's principle, § 86. We saw that in general there was a doubt as to the existence of a function making the integral a minimum. In the electrical case, experiment shows that there is always an equilibrium distribution, so that the only doubt which may affect the mathematician does not trouble the physicist. Reasoning depending upon such physical facts was frequently made use of by Green, and while not legitimate for purposes of mathematical demonstration is frequently of service to the physicist.

Since in any conductor $V + V' = c$,

(12) $$\frac{\partial (V+V')}{\partial x} = \frac{\partial (V+V')}{\partial y} = \frac{\partial (V+V')}{\partial z} = 0,$$

or there is no force in the substance of a conductor; further

$$\Delta (V + V') = 0.$$

But since the distribution causing V' lies outside of the conductor, $\Delta V' = 0$ in the conductor, and

(13) $$\Delta V = 0 = -4\pi\rho.$$

Consequently, in every conductor $\rho = 0$, or the distribution is superficial. Now at the surface distribution σ we have a discontinuity in the derivative $\frac{\partial V}{\partial n}$, and

(14) $$\frac{\partial V}{\partial n_i} + \frac{\partial V}{\partial n_e} = -4\pi\sigma.$$

But since within the conductor $V + V' = c_s$,

(15) $$\frac{\partial (V+V')}{\partial n_i} = 0, \quad \frac{\partial V}{\partial n_i} = -\frac{\partial V'}{\partial n_i} = \frac{\partial V'}{\partial n_e},$$

* The above demonstration is given by Betti, *Teorica delle Forze Newtoniane*, p. 164.

for the derivative of V' is continuous on crossing the surface, as none of the distribution causing V' lies on the surface. Accordingly the surface equation becomes

$$(16)\qquad \frac{\partial (V+V')}{\partial n_e} = -4\pi\sigma.$$

The surface density at any point of a conductor is the derivative of the total potential in the direction of the *external* normal to the conductor multiplied by $-\frac{1}{4\pi}$, that is to the field-strength directed away from the conductor divided by 4π. This theorem is due to Coulomb, at least by implication*. The total charge of a conductor K_s is

$$(17)\qquad e_s = \iint_{K_s} \sigma dS = -\frac{1}{4\pi}\iint_{K_s} \frac{\partial(V+V')}{\partial n_e} dS.$$

The following form of the investigation is shorter, and depends on the variation of the second form of the integral

$$(2)\qquad W = \frac{1}{8\pi}\iiint_\infty \left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,$$

where we put, as we shall hereafter do, V for the *total* potential, heretofore denoted by $V+V'$,

$$(18)\qquad \delta W = \frac{1}{4\pi}\iiint_\infty \left\{\frac{\partial V}{\partial x}\frac{\partial \delta V}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial \delta V}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial \delta V}{\partial z}\right\} d\tau$$

$$= -\frac{1}{4\pi}\iint_K V\left\{\frac{\partial \delta V}{\partial n_i} + \frac{\partial \delta V}{\partial n_e}\right\} dS - \frac{1}{4\pi}\iiint_\infty V\Delta\delta V d\tau$$

$$= -\frac{1}{4\pi}\iint_K V\delta\left\{\frac{\partial V}{\partial n_i} + \frac{\partial V}{\partial n_e}\right\} dS - \frac{1}{4\pi}\iiint_\infty V\delta\Delta V d\tau$$

$$= \iint_K V\delta\sigma dS + \iiint_\infty V\delta\rho d\tau.$$

Now in D, $\delta\rho = 0$, and in external space $\rho = 0$, so that the volume integral can be taken through K only, and introducing the equations of condition $\delta e_s = 0$,

$$(19)\qquad \Sigma_s\left[\iint_{K_s}(V-c_s)\,\delta\sigma dS + \iiint_{K_s}(V-c_s)\,\delta\rho\right] = 0,$$

* Coulomb. "Suite des recherches sur la distribution du fluide électrique entre plusieurs corps conducteurs." 1788. *Collection de mém. rel. à la physique, pub. par la Soc. franç. de Phys.* Tom. I. p. 230.

necessitating

$$V = c_s, \quad \Delta V = 0, \quad \rho = 0,$$

$$\frac{\partial V}{\partial n_i} = 0, \quad \frac{\partial V}{\partial n_e} = -4\pi\sigma,$$

$$e_s = -\frac{1}{4\pi}\iint_{K_s} \frac{\partial V}{\partial n_e} dS,$$

the same results as above, writing V for $V + V'$ above.

We may easily show that there is but one equilibrium distribution. For if there be another, $\overline{V}$, $\overline{\sigma}$, for which the constant values of $\overline{V}$ are $\overline{c}_s$, let us apply Green's theorem to the difference

$$u = V - \overline{V},$$

$$(20) \qquad J(u) = \iiint_{\infty} \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\} d\tau$$

$$= -\iint_K u \frac{\partial u}{\partial n_e} dS - \iiint u \Delta u d\tau,$$

the volume integral on the right being extended to all space outside of the conductors. But in that space $\Delta V = \Delta \overline{V} = 0$ or $-4\pi\rho'$, and accordingly $\Delta u = 0$. Also at the surface of any conductor K_s,

$$u = c_s - \overline{c}_s.$$

The integral $J(u)$ therefore becomes

$$\Sigma_s \left[(\overline{c}_s - c_s)\iint \frac{\partial u}{\partial n_e} dS\right].$$

Now the surface integral is equal to $-1/4\pi$ times the difference of the charges of K_s in the two distributions. But the charge being originally given this is 0. Accordingly the integral $J(u)$ vanishes, and everywhere

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \quad u = V - \overline{V} = \text{const.}$$

Since V and $\overline{V}$ vanish at infinity, the constant is 0, and the distribution σ is the same in either case. Consequently we see that the constant values of the potential on the surface and throughout the substance of the conductors, or as we shall say, the potentials *of* the conductors, are determined by the electrifications of D and the total charges of the conductors.

132. Zero Potential. If we have a single charged conductor in the form of a sphere, uninfluenced by other bodies, since the surface density will be constant, the potential at the center will be $\frac{e}{r}$ where r is the radius of the sphere, and since the potential throughout the conductor is constant, its value will be $\frac{e}{r}$. As r increases the potential decreases in absolute value. Now the earth may be considered as a conductor of a radius which is infinite in comparison with the dimensions of our apparatus. Its potential may be therefore regarded as zero, and any conductor may be kept at zero potential by being connected with the earth.

If within a conductor there is a hollow space, not containing any electrified body, the function V is harmonic in the cavity, and being equal to a constant at the inner surface of the conductor, must by Kelvin and Dirichlet's principle be constant throughout the cavity, consequently there is no force at any point in the cavity. Or a closed conductor screens the interior from the effect of an external field of force. This explains Faraday's Experiment VII.

If the system is composed of a single hollow conductor in communication with the earth, containing within several rigidly electrified bodies D, then the total potential V being zero at the outer surface of the conductor and at infinity is by Dirichlet's principle zero everywhere in the external space, and there is no force there. That is, a closed conductor connected to earth shields external space from the action of an electric field of force within. This principle of electric screens is of great importance in practice in connection with electrostatic instruments.

The surface density at the outer surface of the conductor

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n_e}$$

then vanishes, and the outside is without charge. On the inside surface, the charge is

$$e = -\frac{1}{4\pi}\iint \frac{\partial V}{\partial n_i} dS,$$

the normal being, as usual, drawn *away* from the conductor. But

$$-\frac{1}{4\pi}\iint_K \frac{\partial V}{\partial n_i} dS = \frac{1}{4\pi}\iiint \Delta V d\tau = -\iiint_D \rho d\tau,$$

ΔV vanishing except in the bodies D, in which it is equal to $-\frac{1}{4\pi}\rho$. Consequently the charge e on the inside of the conductor is equal and opposite to the whole charge of the bodies D. This explains Experiment VI.

If the conductor is not connected to earth, the same result as far as the inner charge follows. There is, however, an outer charge, but as there is no force in the substance of the conductor, the distribution of this charge is unaffected by what is within. This explains Experiment II.

133. Tubes of Force. If we apply the above reasoning

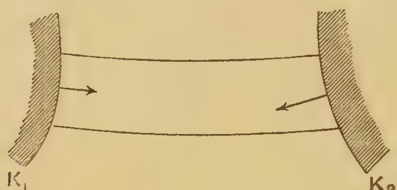

FIG. 56*a*.

to the space inclosed by any tube of force, which must end either at infinity or at conducting surfaces, we have, since $\Delta V = 0$,

$$-\iiint \Delta V d\tau = \iint_{K_1} \frac{\partial V}{\partial n_e} dS + \iint_{K_2} \frac{\partial V}{\partial n_e} dS = 0,$$

$$\iint_{K_1} \sigma dS = -\iint_{K_2} \sigma dS,$$

or the ends of any tube of force cover equal and opposite charges, the flux of force through the tube or the number of *unit tubes* contained in it being 4π times the absolute value of the charge.

134. Theorems on Sign of Electrification. By means of the properties of tubes of force and of the Potential Function, we may deduce a number of theorems on the sign of the electrification on the surfaces of conductors. These theorems are taken from the excellent *Leçons sur l'Electricité et le Magnétisme*, by P. Duhem.

We shall call a distribution in which the sign of the surface-density is everywhere the same, *monogenic*. If the density varies in sign we shall call the distribution *amphigenic*. We suppose that all the conductors are external to each other, and that in each case there are no conductors present except those mentioned.

THEOREM I. If the system consists of a single electrified conductor, the distribution is monogenic. For on the conductor the potential is constant, while at infinity it is zero. In free space, being harmonic, it has neither maximum nor minimum (§ 34), hence all the equipotential surfaces are closed surfaces surrounding the conductor, and the tubes of force proceed from the conductor to infinity. Thence $\frac{\partial V}{\partial n_e}$ is of the same sign all over the surface of the conductor, and the theorem is proved.

THEOREM II. If the system is composed of two conductors, the distribution of at least one of them is monogenic. For the greatest and least values of the potential are two of the three values of the potential on the two conductors, and at infinity. The potential on one of the conductors is accordingly an extreme value, so that the derivative has the same sign over its surface.

THEOREM III. If an insulated conductor with zero charge is placed in presence of a charged conductor, the charge of the former is amphigenic, of the latter monogenic. For since the charge of the first is zero, the surface-density and hence the derivative $\frac{\partial V}{\partial n_e}$ must be positive in some regions, negative in others, consequently its potential lies between the extreme values, which are accordingly on the second conductor and at infinity.

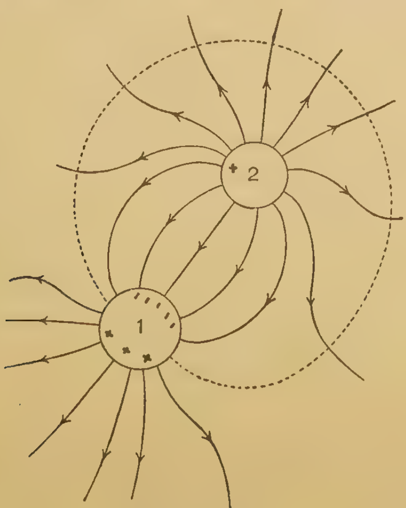

FIG. 57.

On the line on the first conductor separating positive and negative values of σ, the potential does not vary as we leave the surface in the direction of the normal, or in other words the equipotential surface of which the conductor forms a part has a sheet cutting the conductor normally. This sheet is closed, containing the other conductor with the monogenic charge. This sheet is dotted in Fig. 57. The direction of the lines of force is shown by the arrows.

THEOREM IV. If the system consists of two bodies with equal and opposite charges, the distribution on each is monogenic. For if not, it is evident on inspection of Fig. 57, that if we draw a sufficiently large surface including both conductors, the tubes of force will cross it everywhere in the same direction (outward or inward). But the total outward flux of force is equal to 4π times the total charge within the surface, which is zero, accordingly all the tubes of force must issue from one conductor and end on the other.

THEOREM V. A charge concentrated at a point produces a monogenic charge on a conductor whose charge is of equal amount and opposite sign. For this is a particular case of the preceding theorem.

THEOREM VI. A negative charge concentrated at a point produces a monogenic distribution on a conductor with a positive charge of greater absolute value. For the charged point may be considered as the limit of a conductor with potential $-\infty$. This is then the lowest value of the potential occurring. The value at infinity, namely zero, is not the greatest value, for then all the values occurring would be negative, but as we approach infinity the value approached by the potential is M/r, § 74 (7), where M is the total charge of all the distributions, which is here positive. Since positive values occur, the highest value attained must be on the conductor, whose distribution is therefore monogenic.

The remaining theorems are expressed in terms of known potentials, instead of charges, of the conductors.

THEOREM VII. If two conductors have potentials of the same sign, the distribution is monogenic on the one whose potential has the greatest absolute value, and the density has the same sign as the potential.

For the potential of the conductor having the greatest absolute value is, if positive, the highest, and if negative, the lowest value occurring, so that in the former case σ is positive, in the latter, negative.

THEOREM VIII. On each of two conductors whose potentials are of opposite signs the distribution is monogenic. For the potentials of the conductors are the highest and lowest occurring.

THEOREM IX. If one of two conductors has the potential zero, the other a potential not zero, the distribution is monogenic on both, and on the second the density has the sign of the potential, on the first the contrary sign. For this is a limiting case of the preceding, as the potential of one of the conductors approaches zero.

THEOREM X. On a conductor connected to earth, a charge concentrated at a point causes a monogenic charge of sign opposite to its own. For this is a particular case of the preceding theorem.

Theorem I may be generalized as

THEOREM XI. In a system formed of any number of conductors, the distribution on at least one is monogenic. For the highest or lowest value of the potential must be on one of the conductors.

135. General Problem of Electrostatics. If we have a number of conductors in a state of equilibrium, of which some are insulated and charged with quantities e_s, others connected to earth, or kept, by means to be hereafter described, at given constant potentials V_s, and influenced by certain bodies D rigidly electrified with density ρ, the problem to be solved consists in finding a potential function V which, 1°, is constant in each conductor, taking the values V_s in those conductors for which the constant is given, 2°, in the bodies D satisfies the equation

$$\Delta V = -4\pi\rho,$$

and 3°, in the rest of space is harmonic,

$$\Delta V = 0.$$

We can satisfy these conditions if we can solve $n+1$ independent problems, n being the number of conductors.

I. To find for each value of s from 1 to n, a function v_s, which at the surface K_s takes the constant value 1, at all the surfaces K_r, where r is different from s, takes the constant value 0, and in all space external to K is harmonic. Each of these n problems is a different problem of Dirichlet.

II. To determine a function w, which in all the conductors is zero, in the bodies D satisfies the equation

$$\Delta w = -4\pi\rho,$$

and in the rest of space is harmonic.

These $n+1$ functions being found, the required function V is given by the linear function

$$V = V_1 v_1 + V_2 v_2 \ldots + V_n v_n + w, \tag{1}$$

where $V_1, V_2 \ldots V_n$ are the given constant values. For each of the functions v_s and w is harmonic in all space except D, where the v_s's are harmonic, and w satisfies $\Delta w = -4\pi\rho$; therefore the sum V is harmonic everywhere except in D, where it satisfies

$$\Delta V = -4\pi\rho.$$

On any conductor K_s, w and all the v's vanish except v_s, which is 1, hence

$$V = V_s.$$

From any of the functions v_s and w let us calculate for any surface K_r the integrals

$$q_{sr} = -\frac{1}{4\pi}\iint_{K_r} \frac{\partial v_s}{\partial n_e}\,dS, \quad Q_r = -\frac{1}{4\pi}\iint_{K_r} \frac{\partial w}{\partial n_e}\,dS.$$

Since the finding of the function v_s is a purely geometrical problem, depending on the form and position of the surfaces K_s, all the n^2 quantities q_{rs} are geometrical constants for the given system of conductors. We have now for the charge of any conductor K_s

$$e_s = -\frac{1}{4\pi}\iint_{K_s} \frac{\partial V}{\partial n_e}\,dS = -\frac{1}{4\pi}\left[V_1\iint_{K_s} \frac{\partial v_1}{\partial n_e}\,dS \right. \tag{2}$$

$$\left. + V_2\iint_{K_s} \frac{\partial v_2}{\partial n_e}\,dS \ldots + V_n\iint_{K_s} \frac{\partial v_n}{\partial n_e}\,dS + \iint_{K_s} \frac{\partial w}{\partial n_e}\,dS\right],$$

or inserting the above notation for the integrals,

$$e_s = q_{1s}V_1 + q_{2s}V_2 \ldots + q_{ns}V_n + Q_s. \tag{3}$$

There is one such equation for each conductor. These n equations determine the charges in terms of the potentials, and if the potentials of some of the conductors are given, and the charges of the rest, all the remaining charges and potentials are determined. Q_s is the charge of the conductor K_s by induction from D when all the conductors (including K_s) are connected to earth, and consequently

$$V_1 = V_2 = \dots = V_n = 0.$$

136. Coefficients of Induction. Reciprocal Relation. We shall now suppose the system of conductors to be under the action only of their own field, so that $Q = 0$. Then we have

$$(4)\quad \begin{aligned} e_1 &= q_{11}V_1 + q_{21}V_2 \dots + q_{n1}V_n, \\ e_2 &= q_{12}V_1 + q_{22}V_2 \dots + q_{n2}V_n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ e_n &= q_{1n}V_1 + q_{2n}V_2 \dots + q_{nn}V_n. \end{aligned}$$

The constants q_{rs} are called *coefficients of induction*, and any q_{rs} is defined as the charge induced on the conductor K_s when it and all the others are earthed, except K_r which is brought to potential 1. Any coefficient with double suffix q_{ss} is the charge of K_s when it is at potential unity, and all the other conductors are earthed. It is called the *capacity* of the conductor K_s. The dimensions of the q's are $\left[\frac{e}{V}\right] = [L]$. We shall now show that the order of the suffixes in q_{rs} is immaterial. Applying Green's theorem in the second form to the functions v_s and v_r, we have

$$(5)\quad \iint_K \left(v_s \frac{\partial v_r}{\partial n_e} - v_r \frac{\partial v_s}{\partial n_e}\right) dS = \iiint (v_r \Delta v_s - v_s \Delta v_r)\, d\tau.$$

The volume integral being taken throughout the space external to the conductors where v_r and v_s are both harmonic, vanishes, and since v_r vanishes on all conductors except K_r where it is constant and equal to unity, and v_s vanishes on all conductors except K_s where it is equal to unity, (5) becomes

$$\iint_{K_s} \frac{\partial v_r}{\partial n_e} dS - \iint_{K_r} \frac{\partial v_s}{\partial n_e} dS = 0,$$

$$(6)\quad -\frac{1}{4\pi}\iint_{K_s} \frac{\partial v_r}{\partial n_e} dS = -\frac{1}{4\pi}\iint_{K_r} \frac{\partial v_s}{\partial n_e} dS,$$

$$q_{rs} = q_{sr}.$$

We may accordingly state the reciprocal theorem:

The quantity of electricity induced upon a conductor A of a system when another conductor B is brought to a given potential V and all the others including A are earthed, is the same as the quantity induced on B when it and the others are earthed, except A, which is brought to the same potential V.

137. Energy of System. The energy of the system is

$$W = \frac{1}{2}\iint_K \sigma V dS = \tfrac{1}{2}\Sigma_s V_s \iint_{K_s} \sigma dS \tag{7}$$
$$= \tfrac{1}{2}(V_1 e_1 + V_2 e_2 \ldots\ldots V_n e_n),$$

or introducing the values of e_s and bearing in mind the relation $q_{rs} = q_{sr}$,

$$W = \tfrac{1}{2}q_{11}V_1^2 + \tfrac{1}{2}q_{22}V_2^2 \ldots\ldots + \tfrac{1}{2}q_{nn}V_n^2 \tag{8}$$
$$+ q_{12}V_1V_2 + q_{13}V_1V_3 + \ldots\ldots + q_{23}V_2V_3 + \ldots\ldots.$$

That is, the energy is a homogeneous quadratic function of the potentials of the n conductors, the coefficients being the coefficients of capacity and induction. The energy expressed in this form will be denoted by W_V.

We have

$$\frac{\partial W_V}{\partial V_s} = q_{1s}V_1 + q_{2s}V_2 + \ldots\ldots + q_{ns}V_n = e_s, \tag{9}$$

or the charge of any conductor is obtained by differentiating the energy-function expressed in terms of the potentials partially with respect to the potential of the given conductor.

138. Coefficients of Potential. Solving the linear equations (4) we get n equations

$$\begin{aligned} V_1 &= p_{11}e_1 + p_{21}e_2 \ldots\ldots + p_{n1}e_n, \\ V_2 &= p_{12}e_1 + p_{22}e_2 \ldots\ldots + p_{n2}e_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ V_n &= p_{1n}e_1 + p_{2n}e_2 \ldots\ldots + p_{nn}e_n, \end{aligned} \tag{10}$$

where any coefficient p_{rs} is the minor of q_{rs} divided by Δ in the determinant

$$\Delta = \begin{vmatrix} q_{11} & q_{21} \cdots q_{n1} \\ q_{12} & q_{22} \cdots q_{n2} \\ \cdots & \cdots\cdots \\ q_{1n} & q_{2n} \cdots q_{nn} \end{vmatrix}.$$

The coefficients p are called *coefficients of potential*. Their dimensions are $\left[\frac{V}{e}\right]=\left[\frac{1}{L}\right]$. Since the determinant Δ is unchanged by interchanging columns and rows, the determinant of the p's must have the same property, or $p_{rs}=p_{sr}$. We may prove this directly as we did for the q's. Let V be the value of the potential when K_r has the charge e and all other conductors charge 0. Let V' be the value of the potential when K_s has charge e and the others charge 0. Then as in (5)

$$(11) \qquad \iint_K \left(V\frac{\partial V'}{\partial n} - V'\frac{\partial V}{\partial n}\right] dS = 0,$$

and since on any conductor K_i, V and V' are constant and respectively equal to V_i, V_i',

$$(12) \qquad \Sigma_i\left[V_i\iint_{K_i}\frac{\partial V'}{\partial n}dS - V_i'\iint_{K_i}\frac{\partial V}{\partial n}dS\right]=0.$$

Now since the potential V' is due to a distribution in which only K_s is charged, all the integrals

$$\iint_{K_i}\frac{\partial V'}{\partial n}dS = 0$$

vanish except for $i=s$, for which the integral is $-4\pi e$, likewise all the integrals

$$\iint_{K_i}\frac{\partial V}{\partial n}dS$$

vanish except for $i=r$, when the value is $-4\pi e$. Consequently we have

$$(13) \qquad -4\pi e\,(V_s - V_r') = 0.$$

Now from the equations (10), putting $e_r=e$, the other e's zero,

$$V_s = p_{rs}e.$$

Again putting $e_s=e$, the others zero,

$$V_r' = p_{sr}e.$$

Whence

$$(14) \qquad p_{rs}=p_{sr},$$

and we have the reciprocal theorem:

If a conductor A receive a certain charge, e, all the other conductors of the system being uncharged, the potential of any other conductor B is the same as would be attained by A if B should receive the charge e, all the other conductors being uncharged.

Making use of the equations (10) and the condition $p_{rs} = p_{sr}$, the energy

$$W = \tfrac{1}{2}\Sigma_s e_s V_s$$

becomes

$$(15)\quad W = \tfrac{1}{2}p_{11}e_1^2 + \tfrac{1}{2}p_{22}e_2^2 \ldots\ldots + \tfrac{1}{2}p_{nn}e_n^2 \\ + p_{12}e_1e_2 + p_{13}e_1e_3 + \ldots\ldots + p_{23}e_2e_3 + \ldots\ldots,$$

or W is a homogeneous quadratic function of the charges of the n conductors, the coefficients being the coefficients of potential p. This form will be denoted by W_e. If we differentiate partially by any charge e_s we get

$$(16)\qquad \frac{\partial W_e}{\partial e_s} = p_{1s}e_1 + p_{2s}e_2 \ldots\ldots + p_{ns}e_n = V_s,$$

or the potential of any conductor is the partial derivative of the energy of the system as a quadratic function of the charges, by the corresponding charge.

139. Properties of the Coefficients. As the energy of an electrified system is intrinsically positive, the values of the coefficients q and p must be such that the functions W_V and W_e shall be positive for all possible values of the V's and e's. We may deduce certain properties of the coefficients from the elementary properties of the tubes of force and equipotential surfaces. Let one conductor K_s receive a positive unit of charge, all the others being uncharged, its potential is then p_{ss}, and the energy

$$W_e = \tfrac{1}{2}p_{ss}e^2 = \tfrac{1}{2}p_{ss},$$

and since this must be positive p_{ss} is positive, or: *Any coefficient of potential with double suffix is positive.*

Any conductor K_r completely enclosed by K_s has the same potential, so that for these two $p_{rs} = p_{ss}$. Any conductor K_r outside of K_s has a potential of the same sign but of less absolute value. For the charge of a conductor is proportional to the excess of unit tubes issuing from it over that entering. An uncharged conductor accordingly has as many leaving as entering. Accordingly all tubes have one end on K_s and the other at infinity (Fig. 57), and the potential of K_r, p_{rs} is consequently intermediate between that of K_s and that at infinity,

$$p_{ss} \geqq p_{rs} > 0.$$

All coefficients of potential are positive, and those with double suffixes are not greater than those with single suffixes.

Secondly, let all the conductors be at potential 0, except K_s, which is at potential unity. The energy is

$$W = \tfrac{1}{2} q_{ss} V^2 = \tfrac{1}{2} q_{ss},$$

so that the capacity of any conductor is positive. The number of unit tubes of force issuing from K_s is proportional to the charge q_{ss}. Some of these extend to infinity, while others end on the other conductors. At the latter the charges will be negative, but the sum of all such charges is not as great as q_{ss}. Accordingly, *every q with double suffixes is negative, and*

$$(17) \qquad q_{ss} > -(q_{1s} + q_{2s} \cdots + q_{s-1,s} + q_{s+1,s} \cdots + q_{ns}).$$

If, however, K_s is completely surrounded by a conductor K_r,

$$-q_{ss} = q_{rs}.$$

If a new conductor be introduced into the field, the coefficient of potential with the double suffix for any conductor is diminished. For if any portion of the field be made suddenly conducting, electricity will move in it so as to make the energy less than before. If K_s was the only charged body, the energy $\frac{1}{2} p_{ss} e^2$ must be diminished, but as the charge e has not changed, p_{ss} must be diminished.

Introducing a new body into the field increases the capacity of any conductor, and diminishes the absolute value of every coefficient of induction. For if the new conductor and all the others be at potential 0, while K_s is at potential unity, some of the tubes of force which before ended on the other conductors, now end on the new conductor, which receives a negative charge. This induces a positive charge on K_s, increasing its charge q_{ss}, and positive charges on the other conductors K_r, diminishing their negative charges q_{rs}.

140. Work done during displacement of conductors. Suppose that we deform or displace the conductors of the system, thus changing the geometrical coefficients p and q. Suppose the configuration of the system is specified by m parameters

$$\phi_1, \phi_2, \ldots \phi_m,$$

so that if the conductors are displaced as rigid bodies $m = 6n$. Let the mechanical forces due to the electrification be denoted by Φ so that the force tending to change the parameter ϕ_i is Φ_i.

Then the work done in a displacement $\delta\phi_1, \dots \delta\phi_m$ is

(1) $$\Phi_1\delta\phi_1 + \Phi_2\delta\phi_2 \dots\dots + \Phi_m\delta\phi_m,$$

and if no energy is furnished to the system this work must be done at the expense of the electrical energy W and

(2) $$-\delta W = \Sigma_i \Phi_i \delta\phi_i.$$

In the differential δW we may use according to circumstances either of the three forms

$$W_{eV} = \tfrac{1}{2}\Sigma_s e_s V_s, \quad W_e \text{ or } W_V,$$

which are of course identical, though expressed in terms of different variables. If we choose W_{eV} the total differential

(3) $$\delta W_{eV} = \tfrac{1}{2}\Sigma_s \{e_s \delta V_s + V_s \delta e_s\}$$

does not contain the $\delta\phi$'s explicitly. For neither the coefficients p nor q appear in W_{eV}. However, the δe's and δV's are not independent, being connected by either set of equivalent linear relations (4) or (10) above, which in the coefficients q or p involve the parameters ϕ, consequently we may eliminate either the δV's or δe's, and replace them by $\delta\phi$'s.

Now we see by the relations

$$\frac{\partial W_e}{\partial e_s} = V_s, \quad \frac{\partial W_V}{\partial V_s} = e_s$$

that the functions W_e and W_V are *reciprocal functions* (§ 63) with respect to either set of independent variables $e_1, \dots e_n$, or $V_1, \dots V_n$, containing also the independent variables ϕ, corresponding to the variables z of § 63. Accordingly by the last of equations (5), § 63,

(4) $$\frac{\partial W_e}{\partial \phi_s} = -\frac{\partial W_V}{\partial \phi_s}.$$

If the conductors are insulated, so that all the charges are constant, we use the form W_e, so that any force Φ_s has the value, from (2),

(5) $$\Phi_s = -\frac{\partial W_e}{\partial \phi_s}.$$

The system tends to move so as to diminish the energy.

If on the other hand the potentials are maintained constant we must use the form W_V. In this case we must supply energy from

outside and the equation (5) can no longer be used, but in place of it we have, by (4),

$$(6) \qquad \Phi_s = \frac{\partial W_V}{\partial \phi_s}.$$

The system now tends to move so as to *increase* the energy, and the increase of energy is exactly equal to the work done by the electrical forces. For

$$(7) \qquad \delta W_V = \Sigma \frac{\partial W_V}{\partial \phi_s} \delta\phi_s = \Sigma \Phi_s \delta\phi_s.$$

We accordingly see that the system is analogous to a cyclic system. The forces Φ_s correspond to the *negative* values of the positional forces P_s, for the latter are defined as the forces that must be applied from outside in order to equilibrate the reactions of the system. Comparing equations (5) and (6) with (1) and (3) of § 69, we obtain the analogous results

$$P_s = -\frac{\partial T_q'}{\partial q_s}, \qquad -\Phi_s = -\frac{\partial W_V}{\partial \phi_s},$$

$$P_s = \frac{\partial T_p}{\partial q_s}, \qquad -\Phi_s = \frac{\partial W_e}{\partial \phi_s}.$$

The electrical energy W plays the role of the kinetic energy T in the cyclic system. In order to determine which of the variables e or V are to be assimilated to velocities and which to momenta, we must recall that in an adiabatic motion work is done through the positional coordinates at the expense of the energy T. This corresponds to the case of constant charges (2). The charges are accordingly the analogues of the momenta, and the potentials of the velocities. Accordingly to an isocyclic motion will correspond a motion in which the potentials are maintained constant. We have already seen that in this case electrical energy must be supplied from without, and since this must not only do work but also *increase* the energy of the system by an equal amount we have the analogue of the Theorem I of § 70:—In any motion of a system of conductors in which the potentials of the conductors are maintained constant, an amount of electrical energy must be supplied from without equal to twice the amount of work done by the electrical forces during the motion.

The equations for the cyclic forces $\overline{P}_s = \frac{dp_s}{dt}$ are here not applicable.

Example. In the case of a single conductor, the coefficients q and p reduce in each case to a single one, the capacity and its reciprocal respectively, and

$$e = qV,$$
$$V = pe,$$
$$p = \frac{1}{q},$$
$$W = \tfrac{1}{2}eV = \tfrac{1}{2}qV^2 = \tfrac{1}{2}pe^2.$$

If the conductor is a sphere of radius r, we have

$$V = \frac{e}{r}, \; q = r, \; p = \frac{1}{r}, \; W = \frac{1}{2}rV^2 = \frac{1}{2}\frac{e^2}{r}.$$

The only geometrical parameter ϕ is here r, and since W_e tends to decrease, W_V to increase, r tends to increase.

If the sphere is elastic, as in the case of a soap-bubble, the electrical forces tending to enlarge the sphere may be held in equilibrium by a greater pressure of the air on the outside than on the inside, or by the surface tension of the film. If P denote this excess of pressure, that is, the force acting normally on a unit of surface, the work done by the whole surface S in increasing the radius by dr is $PSdr$. If T is the *surface tension* of the film, or the elastic force tangential to the surface exerted normally across a curve on the surface per unit of length, in increasing the surface by dS we must do work TdS. Hence we have

$$\Phi d\phi = PSdr + TdS = -dW_e = dW_V,$$

$$S = 4\pi r^2, \; dS = 8\pi r dr, \; W_e = \frac{1}{2}\frac{e^2}{r},$$

$$4\pi\{Pr^2 + 2Tr\}\,dr = \frac{1}{2}\frac{e^2}{r^2}dr = \frac{1}{2}V^2 dr,$$

$$e^2 = 8\pi r^3\{Pr + 2T\},$$
$$V^2 = 8\pi r\{Pr + 2T\}.$$

If the soap-bubble be blown on a tube connected with a manometer, the difference in pressure P may be observed. T may be determined by an observation when the bubble is unelectrified. Calling r_0 the radius under these circumstances, P_0 the pressure,

$$P_0 r_0 + 2T = 0,$$

$$T = -\frac{P_0 r_0}{2}, \quad P_0 \text{ being negative,}$$

and using this value of T,

$$V^2 = 8\pi r \{Pr - P_0 r_0\}.$$

Accordingly a potential may be measured in this simple manner by a measurement of P and r, P_0 and r_0 having been observed. If the tube on which the bubble is blown is open to the air, $P=0$, and

$$V^2 = 16\pi r T.$$

141. Distribution on an Ellipsoid. We have found the potential due to an equipotential layer of amount e distributed on an ellipsoid of semi-axes a, b, c to be, § 109 (12),

$$V = \frac{e}{2}\int_\lambda^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}},$$

where λ is the greatest root of the cubic

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

At the surface of the ellipsoid $\lambda = 0$

$$V_S = \frac{e}{2}\int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = \frac{e}{q},$$

so that the capacity q is the elliptic integral

$$q = \frac{2}{\displaystyle\int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}}.$$

The surface density of the charge is given by

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n_e} = -\frac{1}{4\pi}\left(\frac{dV}{d\lambda}h_\lambda\right)_{\lambda=0},$$

which by § 110 gives

$$= \frac{e}{4\pi abc}\delta_\lambda = \frac{e}{4\pi abc}\cdot\frac{1}{\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}}}.$$

Ellipsoids of Rotation.

If $a = b$, the ellipsoid is one of rotation, and the elliptic integral simplifies into

$$\int_0^\infty \frac{ds}{(a^2+s)\sqrt{c^2+s}} = \frac{2}{\sqrt{a^2-c^2}}\sin^{-1}\frac{\sqrt{a^2-c^2}}{a}, \text{ if } a > c,$$

i.e., when the ellipsoid is oblate, and

$$\int_0^\infty \frac{ds}{(a^2+s)\sqrt{c^2+s}} = \frac{2}{\sqrt{c^2-a^2}} \log\left\{\frac{\sqrt{c^2-a^2}+c}{a}\right\}, \text{ if } c>a,$$

when the ellipsoid is prolate. The capacity is

$$q = \frac{\sqrt{a^2-c^2}}{\sin^{-1}\dfrac{\sqrt{a^2-c^2}}{a}}, \text{ oblate ellipsoid,}$$

$$q = \frac{\sqrt{c^2-a^2}}{\log\left\{\dfrac{\sqrt{c^2-a^2}+c}{a}\right\}}, \text{ prolate ellipsoid.}$$

For a very long prolate ellipsoid, neglecting $\left(\frac{a}{c}\right)^2$,

$$q = \frac{c}{\log\dfrac{2c}{a}},$$

so that as a approaches zero, the capacity approaches zero, but more slowly, viz., logarithmically*.

In the limiting case of an oblate ellipsoid, for $c=0$, we have a circular disc, whose capacity is

$$q = \frac{2a}{\pi}.$$

If in the expression for the surface density we eliminate c by the equation of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{1}{c^2}=\frac{1}{z^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

$$\sigma = \frac{e}{4\pi abz}\,\frac{\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}}{\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{z^4}\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)^2}}.$$

* $\lim\limits_{x=0}\left[x:\left(1/\log\frac{1}{x}\right)\right]=0.$

If we now make $z=0$, we get the density on an elliptical disc

$$\sigma = \frac{e}{4\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}},$$

and if $a=b$,

$$\sigma = \frac{e}{4\pi a\sqrt{a^2-r^2}},$$

for a circular disc of radius a. At the edge of the disc,

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$$

and the density is infinite, so that this case is not physically possible. It is however of considerable theoretical importance.

For the case of the circular disc the potential at any point becomes

$$V=\frac{e}{2}\int_\lambda^\infty \frac{ds}{(a^2+s)\sqrt{s}}=\frac{e}{a}\left\{\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda}}{a}\right\}=\frac{e}{a}\tan^{-1}\frac{a}{\sqrt{\lambda}},$$

where λ is the greatest root of the quadratic

$$\frac{x^2+y^2}{a^2+\lambda}+\frac{z^2}{\lambda}=1.$$

142. Concentric Spheres. Suppose we have a sphere of radius R_1, surrounded by a concentric spherical shell of radii R_2 and R_3. In the space between the conductors and outside of the outer, V satisfies the equation, § 88 (7),

$$\Delta V=\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=0,$$

whose integral is

$$\frac{dV}{dr}=\frac{A}{r^2},$$

$$V=-\frac{A}{r}+B.$$

If V_1 is the potential of the inner sphere, V_2 of the shell,

$$V_1=-\frac{A}{R_1}+B,$$

$$V_2=-\frac{A}{R_2}+B,$$

which determines A:

$$A = -\frac{R_1R_2}{R_2 - R_1}(V_1 - V_2).$$

The surface density on the inner sphere is

$$\sigma_1 = -\frac{1}{4\pi}\frac{\partial V}{\partial n_e} = -\frac{1}{4\pi}\frac{\partial V}{\partial r} = -\frac{1}{4\pi}\frac{A}{r^2},$$

$$\sigma_1 = \frac{R_1R_2}{4\pi(R_2 - R_1)} \cdot \frac{V_1 - V_2}{R_1^2}.$$

The charge on the sphere is

$$e_1 = 4\pi R_1^2\sigma_1 = \frac{R_1R_2}{R_2 - R_1}(V_1 - V_2).$$

In like manner, at the surface $r = R_2$; differentiating by $-r$, we get for the charge e_2',

$$e_2' = \frac{R_1R_2}{R_2 - R_1}(V_2 - V_1) = -e_1.$$

To find the charge at the outer surface $r = R_3$, we must redetermine the constants A and B. Since $V_\infty = 0$

$$V_\infty = B = 0,$$

$$V = -\frac{A}{r},$$

$$V_2 = -\frac{A}{R_3},$$

$$A = -V_2R_3,$$

$$\frac{dV}{dr} = -\frac{V_2R_3}{r^2},$$

$$\sigma_3 = -\frac{1}{4\pi}\left(\frac{dV}{dr}\right)_{r=R_3} = \frac{V_2}{4\pi R_3},$$

and the charge at the outer surface

$$e_2'' = 4\pi R_3^2\sigma_3 = V_2R_3.$$

The whole charge of the conductor 2 is

$$e_2 = e_2' + e_2'' = \frac{R_1R_2}{R_2 - R_1}(V_2 - V_1) + V_2R_3.$$

We accordingly have for the coefficients q

$$q_{11} = \frac{R_1R_2}{R_2 - R_1}, \quad q_{22} = \frac{R_1R_2}{R_2 - R_1} + R_3,$$

$$q_{12} = -q_{11}.$$

143. Condensers. The capacity q_{11} decreases as R_2 increases, becoming equal to R_1 when $R_2 = \infty$. Accordingly by the presence of the envelope the capacity of the sphere is increased in the ratio $\frac{R_2}{R_2 - R_1}$ which may be made very large. Such an arrangement of two conductors, by which the presence of the second largely increases the capacity of the first, is called a *condenser*, for by it a larger quantity of electricity is condensed on the first by raising it to a given potential, the second being to earth. The coefficient q_{11}, or $-q_{12}$ which is here equal to it, is called the capacity of the condenser, and will be denoted by K. It is not necessary that one conductor shall surround the other. If it does not, we shall not have $q_{12} = -q_{11}$, but in any condenser we shall suppose the coefficients q_{11}, q_{22}, $-q_{12}$ to be nearly equal. In that case we need not distinguish between the two conductors, or *plates*, of the condenser.

The energy is

$$W_V = \tfrac{1}{2} q_{11} V_1^2 + \tfrac{1}{2} q_{22} V_2^2 + q_{12} V_1 V_2$$
$$= \tfrac{1}{2} q_{11} (V_1 - V_2)^2 + \tfrac{1}{2} (q_{22} - q_{11}) V_2^2 + (q_{12} + q_{11}) V_1 V_2.$$

In virtue of the supposition made regarding the q's, the last two terms are small compared to the first, and we may write

$$W_V = \tfrac{1}{2} K (V_1 - V_2)^2,$$

or the energy of a charged condenser is proportional to the square of the *difference of potentials* of the plates. If one of the plates is to earth this is accurately true, and this is generally the condition in practice.

Now $e_1 = q_{11} V_1 + q_{12} V_2 = q_{11} (V_1 - V_2) + (q_{11} + q_{12}) V_2$,

$$e_2 = q_{12} V_1 + q_{22} V_2 = q_{11} (V_2 - V_1) + (q_{22} - q_{11}) V_2 + (q_{12} + q_{11}) V_1,$$

or $$e_1 = -e_2 = K (V_1 - V_2),$$

to the same order of approximation, and the two plates of a condenser receive equal and opposite charges, proportional to the *difference* of their potentials. Using

$$V_1 - V_2 = \frac{e_1}{K} \text{ in } W,$$

we get

$$W_e = \frac{1}{2} \frac{e_1^2}{K} = \frac{1}{2} \frac{e_2^2}{K},$$

or the energy of a condenser is proportional to the square of the *charge of either plate.* Since the forces tend to cause W_V to increase, if the potentials remain constant, and W_e to decrease, if the charges remain constant, the capacity K tends in any case to increase.

In the case of the sphere, if $R_2 - R_1$ be denoted by τ, and if S be the area of the sphere whose radius R is the geometrical mean of R_1, R_2,

$$K = \frac{R_1 R_2}{\tau} = \frac{R^2}{\tau} = \frac{S}{4\pi\tau}.$$

If τ is small, S is approximately the surface of either plate.

144. Concentric Cylinders. If the internal conductor is a circular cylinder of radius R_1, the external a hollow cylinder of radii R_2 and R_3, both of very great length, at a sufficient distance from the ends we have V dependent only on r, and in the space between the conductors (§ 88 (9)),

$$\Delta V = \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0,$$

$$\frac{dV}{dr} = \frac{A}{r},$$

$$V = A \log r + B.$$

The potentials of the conductors being V_1 and V_2,

$$V_1 = A \log R_1 + B,$$

$$V_2 = A \log R_2 + B,$$

$$V_1 - V_2 = A (\log R_1 - \log R_2),$$

$$A = \frac{V_1 - V_2}{\log \dfrac{R_1}{R_2}}.$$

At the inner surface

$$\sigma_1 = -\frac{1}{4\pi}\frac{\partial V}{\partial n_e} = -\frac{1}{4\pi}\frac{\partial V}{\partial r} = \frac{V_1 - V_2}{4\pi \log \dfrac{R_2}{R_1}} \cdot \frac{1}{R_1}.$$

The charge on a length l of the cylinder is

$$e_1 = 2\pi R_1 l \sigma_1 = l \frac{(V_1 - V_2)}{2 \log \dfrac{R_2}{R_1}}.$$

At the surface $r = R_2$, we get in like manner,

$$e_2 = l \frac{(V_2 - V_1)}{2 \log \frac{R_2}{R_1}} = - e_1 .$$

Accordingly the capacity of the condenser formed of l units of length of two conductors is

$$K = \frac{l}{2 \log \frac{R_2}{R_1}} .$$

If we put $R_2 - R_1 = \tau$, and consider τ small,

$$\log \frac{R_2}{R_1} = \log \left(1 + \frac{\tau}{R_1} \right) = \frac{\tau}{R_1} ,$$

$$K = \frac{l R_1}{2 \tau} = \frac{2 \pi R_1 l}{4 \pi \tau} = \frac{S}{4 \pi \tau} ,$$

where S is the surface of one condenser plate.

145. Parallel Planes. If the conductors are two parallel planes, of great extent, parallel to the plane of XY, at a sufficiently great distance from the edges in the space between the plates, V is independent of x and y, and

$$\Delta V = \frac{d^2 V}{dz^2} = 0,$$

$$\frac{dV}{dz} = A,$$

$$V = Az + B.$$

If z is measured from the plate whose potential is V_1, and τ is the distance between the plates,

$$V_1 = B,$$

$$V_2 = A \tau + B,$$

$$\frac{V_1 - V_2}{\tau} = - A.$$

The surface density on the plate 1 is

$$\sigma_1 = - \frac{1}{4 \pi} \frac{\partial V}{\partial n} = - \frac{1}{4 \pi} \left(\frac{\partial V}{\partial z} \right)_{z=0} = \frac{V_1 - V_2}{4 \pi \tau} ,$$

and on the plate 2,

$$\sigma_2 = \frac{1}{4\pi}\left(\frac{\partial V}{\partial z}\right)_{z=\tau} = \frac{V_2 - V_1}{4\pi\tau}.$$

The charge on an area S of either plate is

$$e_1 = S\sigma_1 = \frac{S}{4\pi\tau}(V_1 - V_2),$$

$$e_2 = S\sigma_2 = \frac{S}{4\pi\tau}(V_2 - V_1) = -e_1.$$

The capacity of the condenser is

$$K = \frac{S}{4\pi\tau},$$

agreeing with the results in the two preceding cases.

In fact, for any condenser in which the two plates are separated by a small distance, which is the same over the whole of their opposed surfaces, we may use the above value for K.

146. Standard Condensers. For the purpose of furnishing a standard of capacity or for measuring quantities of electricity when their potentials are known, condensers of one of the three forms just described, viz., plates, cylinders, or spheres, are nearly always used. The spherical condenser is the only one for which our formulas are exact, for in the other two cases some of the dimensions have been supposed infinite, and we have disregarded the charges on the backs of the plane plates, or on the outside of the outer cylinder. This difficulty is surmounted in the

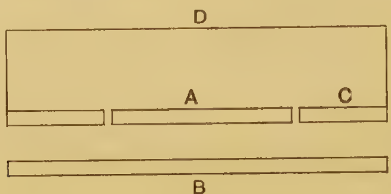

FIG. 58.

following manner. If a portion of the plate A (Fig. 58) be separated from the surrounding portion C by a narrow cut, but be placed in conducting connection with C, the charge on A, if the edge of the outer portion C is sufficiently remote, will be that calculated, for

between A and B the field is uniform. A being then disconnected from C, its charge may be used by itself. The outer part C is called the guard-ring, and its function is to render the field uniform all over the working conductor A. In order that there may be no charge on the back of A, the guard-ring C is made part of a closed conducting box D, which has no charge in its inner surface, hence none on the back of C and A. The principle of the guard-ring is due to Lord Kelvin*. It may equally well be applied to the cylindrical condenser, by separating a portion of the outer cylinder from the ends, which are connected with an enveloping annular box.

147. Absolute Electrometers. The potentials of the plate of a plane condenser being V_1 and V_2, the energy is

$$W_V = \frac{S}{8\pi\tau}(V_1 - V_2)^2.$$

The force tending to increase τ is

$$\Phi = \frac{\partial W_V}{\partial \tau} = -\frac{S}{8\pi\tau^2}(V_1 - V_2)^2.$$

The negative sign shows the force to be an attraction. If the working plate be hung from a balance, and counterbalanced by the weight of a mass M,

$$Mg = \frac{S}{8\pi\tau^2}(V_1 - V_2)^2,$$

$$V_1 - V_2 = \tau\sqrt{\frac{8\pi g M}{S}}.$$

We thus have an *electrometer*, or instrument for the purpose of measuring differences of potential. Lord Kelvin's† original instrument has the plate B carried by a micrometer screw, so that τ can be varied, while A is hung from a system of springs, whose tension, replacing Mg, is constant. In this case $V_1 - V_2$ is directly proportional to τ. In the balance form, used by Rowland and others, τ is constant, and $V_1 - V_2$ is proportional to $\sqrt{M}$. We have in this case a practical difficulty, in that if the upper plate

* Electrometers and Electrostatic Instruments. *B. A. Report*, 1855. *Papers on Electrostatics and Magnetism*, p. 263.

† *loc. cit.* § 358.

approach too near, the force becomes greater and the plate is attracted still nearer, and is accordingly in unstable equilibrium.

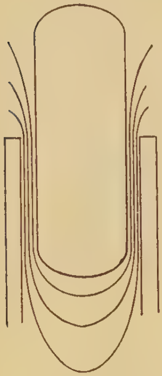

FIG. 59.

The electrometer of Bichat and Blondlot* consists of a cylinder with rounded ends inserted concentrically into a hollow cylinder and supported by a balance. If the ends of the internal cylinder are far enough from the end of the external cylinder, the distributions upon them will be independent of the depth to which the internal cylinder enters the other. For a certain distance the field between the cylinders (whose equipotential surfaces are shown in Fig. 59) will be the same as if the cylinders were of infinite length. Let the length of this portion be l. Then we have

$$K = K_0 + \frac{l}{2 \log \frac{R_2}{R_1}},$$

$$W_V = \frac{1}{2}\left\{K_0 + \frac{l}{2 \log \frac{R_2}{R_1}}\right\}(V_2 - V_1)^2,$$

and the force tending to increase l, that is, to draw the inner cylinder into the outer, is

$$\Phi = \frac{\partial W_V}{\partial l} = \frac{(V_2 - V_1)^2}{4 \log \frac{R_2}{R_1}}.$$

The difference of potential is proportional to the square root of the force, and independent of the position of the inner cylinder, provided only that the cylinders be long enough. This electrometer is of course less entitled to the designation *absolute* than Lord Kelvin's, on account of the assumptions made.

We have now seen that in any absolute measurement of potential, we must measure a *force* and certain geometrical quantities of the nature of lengths.

* *Journal de Physique*, 2me. Série, t. v.

148. Symmetrical Electrometers. The last described electrometer forms an intermediate type to the Kelvin absolute electrometer and the class of symmetrical electrometers, of which Lord Kelvin's quadrant electrometer is the commonest example.

Suppose a conductor C, Fig. 60, in the form either of a thin plate or a cylinder, to be surrounded by two conductors A and B,

Fig. 60.

composed in part respectively of parallel planes or cylinders, and together forming a box enclosing C. We must then consider the coefficients

$$q_{11}, \; q_{22}, \; q_{33}, \; q_{12}, \; q_{13}, \; q_{23},$$

where the suffixes 1, 2, 3, refer to A, B, C. As in the last example, the distributions on the edges or ends of C will be unaffected by a slight change in its position. Besides this there will be charges on portions where the field is uniform, and proportional to $\frac{S}{4\pi\tau}$, where S is that part of the surface of C on which the field is uniform. If we displace C from the symmetrical position by changing a coordinate θ, we shall change S by an amount proportional to θ.

Accordingly, if B and C are at potential 0, A at potential unity has the charge

$$q_{11} = a_{11} + c_1\theta,$$

where a_{11} and c_1 are positive constants.

If A and C are at potential 0, B at potential unity has the charge

$$q_{22} = a_{22} - c_2\theta.$$

If A and B are at potential 0, the charge of C at potential unity is

$$q_{33} = a_{33} + (c_1 - c_2)\,\theta.$$

If A is at potential unity, B at potential 0 is not affected by the position of C at potential 0 whereas the negative charge on C contains a part proportional to θ. Accordingly

$$q_{12} = a_{12}, \quad q_{13} = a_{13} - c_1\theta, \quad q_{23} = a_{23} + c_2\theta.$$

If the apparatus is symmetrical

$$c_1 = c_2 = \frac{1}{4\pi\tau}\frac{\partial S}{\partial \theta}.$$

In the quadrant electrometer, the box AB is a flat circular box divided into four quadrantal sectors, connected alternately to form the two conductors A and B.

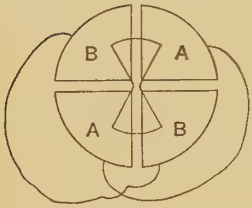

FIG. 61.

The conductor C, called the needle, is of the shape shown in the figure, and rotates about its center, the angle of rotation being the coordinate θ. The couple tending to produce rotation is

$$\Phi = \frac{\partial W_V}{\partial \theta},$$

where

$$W_V = \tfrac{1}{2}(a_{11} + c_1\theta) V_1^2 + \tfrac{1}{2}(a_{22} - c_2\theta) V_2^2 + \tfrac{1}{2}\{a_{33} + (c_1 - c_2)\,\theta\} V_3^2$$
$$+ a_{12} V_1 V_2 + (a_{13} - c_1\theta) V_1 V_3 + (a_{23} + c_2\theta) V_2 V_3,$$

giving

$$\Phi = \tfrac{1}{2} c_1 V_1^2 - \tfrac{1}{2} c_2 V_2^2 + \tfrac{1}{2}(c_1 - c_2) V_3^2 - c_1 V_1 V_3 + c_2 V_2 V_3.$$

If the electrometer is correctly constructed and adjusted $c_1 = c_2$, and

$$\Phi = c(V_2 - V_1)\{V_3 - \tfrac{1}{2}(V_1 + V_2)\}.$$

The needle is usually suspended either by a torsion fibre, or by a bifilar suspension, so that the force of restitution Φ is proportional to the deflexion, the factor of proportionality being denoted by A.

In the usual method of use, the potential V_3 of the needle is made large in comparison with V_1 and V_2. We may then neglect the second term in the brackets, and the deflection is proportional to $V_2 - V_1$. This is called the *heterostatic* method of use, the needle being charged by an extraneous source of potential.

In the *idiostatic* method, the needle is put into connection with one pair of quadrants, which are put at the potential to be measured, the second pair of quadrants being to earth. Then

$$V_2 = V_3 = V, \quad V_1 = 0,$$

$$\Phi = \frac{c}{2} V^2,$$

and the deflection is proportional to the square, instead of to the first power of the potential. This method does not show the sign of V like the former.

If the electrometer is not in perfect adjustment, we use the more general form

$$\Phi = A\theta = \tfrac{1}{2}c_1 V_1^2 - \tfrac{1}{2}c_2 V_2^2 + \tfrac{1}{2}(c_1 - c_2)V_3^2 - c_1 V_1 V_3 + c_2 V_2 V_3.$$

In order to be able to adjust c_1 and c_2 to equality, two of the quadrants should be capable of motion toward or from the center, one roughly, the other micrometrically, so that the amount of surface of the needle covered by them may be varied. In order to make the adjustment, we may first put both pairs of quadrants to earth and observe the deflection when the needle, originally to earth, is charged. Calling this θ_0, we have

$$A\theta_0 = \tfrac{1}{2}(c_1 - c_2)V_3^2,$$

which shows whether c_1 or c_2 is greater. We may then adjust until there is no deflection, however the needle is charged.

If a high potential is not available for V_3, we may conveniently proceed as follows:

By means of a voltaic battery and two commutators, we may charge either of the quadrant-pairs to a given potential V either positive or negative, the other quadrant-pair being to earth. We thus have four combinations, as follows:

θ	V_1	V_2	
θ_1	V	0	reverse commutator A
θ_2	$-V$	0	reverse „ B
θ_3	0	$-V$	reverse „ A
θ_4	0	V	

The deflections are given by

$$A\theta_1 = \tfrac{1}{2}c_1 V^2 + \tfrac{1}{2}(c_1 - c_2)V_3^2 - c_1 V V_3,$$

$$A\theta_2 = \tfrac{1}{2}c_1 V^2 + \tfrac{1}{2}(c_1 - c_2)V_3^2 + c_1 V V_3,$$

$$A\theta_3 = -\tfrac{1}{2}c_2 V^2 + \tfrac{1}{2}(c_1 - c_2)V_3^2 - c_2 V V_3,$$

$$A\theta_4 = -\tfrac{1}{2}c_2 V^2 + \tfrac{1}{2}(c_1 - c_2)V_3^2 + c_2 V V_3;$$

Taking differences we obtain

$$A(\theta_2 - \theta_1) = 2c_1 V V_3,$$

$$A(\theta_4 - \theta_3) = 2c_2 V V_3,$$

$$\frac{c_1}{c_2} = \frac{\theta_2 - \theta_1}{\theta_4 - \theta_3}.$$

In this manner we can accurately bring the ratio $\frac{c_1}{c_2}$ to unity.

Whether the adjustment be made or not, and without the necessity of making V_3 large, if we can reverse the sign of V we may by observing θ_1 and θ_2 get the correct value, since

$$A(\theta_2 - \theta_1) = 2c_1 V V_3,$$

so that V is directly proportional to the difference of the two deflections, or to the arithmetical mean of their absolute values.

149. Induction Electrical Machines. As a further example of induction in a system of conductors, we shall consider the action of a class of electrical machines typified by Lord Kelvin's Replenisher.

This consists essentially of two semi-cylindrical conductors A and B called the inductors, and two smaller conductors C and D called the carriers, which may be rotated as a rigid system about the axis of symmetry. If V_1 be the (positive) potential of A at any time, V_2 that of B, supposed negative, then if C and D be put in conducting communication with the earth while in the position shown, C will have a negative, and D a positive charge induced upon it. Now on insulating C and D, and turning them until C is opposite B and D opposite A, if C be put into communication with B, being nearly surrounded by B, it will give up its charge, thereby increasing the absolute value of the negative potential of B. D being put into communication with A gives up its positive charge, and increases the positive potential of A. The connections of C and D with each other and with A and B are made automatically by contact springs once in each half revolution.

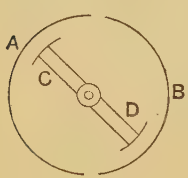

Fig. 62.

If $V_1^{(n)}$ and $V_2^{(n)}$ are the potentials of A and B after n half-revolutions, K_1 and K_2 the capacities of A and B and whatever

conductors they are respectively connected with, their charges are

$$e_1^{(n)} = K_1 V_1^{(n)}, \quad e_2^{(n)} = K_2 V_2^{(n)}.$$

If q_1 and q_2 be the absolute values of the coefficients of induction of A and B respectively upon the carriers when they are connected to earth, the charges induced on the carriers are

$$-q_1 V_1^{(n)} \text{ on } C \text{ and } -q_2 V_2^{(n)} \text{ on } D.$$

These are given up after a half-turn to B and A respectively, so that

$$e_1^{(n+1)} = e_1^{(n)} - q_2 V_2^{(n)},$$
$$e_2^{(n+1)} = e_2^{(n)} - q_1 V_1^{(n)},$$

or in terms of the potentials,

$$V_1^{(n+1)} = V_1^{(n)} - \frac{q_2}{K_1} V_2^{(n)},$$

$$V_2^{(n+1)} = V_2^{(n)} - \frac{q_1}{K_2} V_1^{(n)}.$$

If we write q^2 and p^2 for the positive constants $\frac{q_2}{K_1}$ and $\frac{q_1}{K_2}$, we have

$$V_1^{(n+1)} = V_1^{(n)} - q^2 V_2^{(n)},$$
$$V_2^{(n+1)} = V_2^{(n)} - p^2 V_1^{(n)}.$$

Multiplying these equations respectively by p and q, adding and subtracting, gives

$$pV_1^{(n+1)} + qV_2^{(n+1)} = (pV_1^{(n)} + qV_2^{(n)})(1 - pq),$$
$$pV_1^{(n+1)} - qV_2^{(n+1)} = (pV_1^{(n)} - qV_2^{(n)})(1 + pq).$$

Consequently the linear functions

$$pV_1 + qV_2 \text{ and } pV_1 - qV_2$$

decrease and increase respectively in constant ratios for each half-turn, and

$$pV_1^{(n)} + qV_2^{(n)} = (pV_1^{(0)} + qV_2^{(0)})(1 - pq)^n,$$
$$pV_1^{(n)} - qV_2^{(n)} = (pV_1^{(0)} - qV_2^{(0)})(1 + pq)^n.$$

As n increases, no matter what the original signs or values of V_1 and V_2, $pV_1 + qV_2$ tends toward zero, so that ultimately V_1 and V_2 become opposite in sign, and since $pV_1 - qV_2$ increases, the values of the potentials may be made as large as we please, and increase very rapidly. If the replenisher be turned in the reverse direction V_1 and V_2 are rapidly reduced to equality.

In Lord Kelvin's form of the quadrant electrometer, a replenisher is used to bring V_3, the potential of the needle, to a definite value, which is controlled by a small guard-ring attracted disc electrometer, called the gauge. Descriptions of the complete instrument are found in Lord Kelvin's Reprint of Papers on Electrostatics and Magnetism, and in the usual treatises on Electrical Measurements.

CHAPTER VII.

METHODS FOR THE SOLUTION OF PROBLEMS IN ELECTROSTATICS.

150. **Equipotential Layers as Screens.** The theorems of Green, given in § 84, have an important electrostatic application. By the first theorem we may produce at all points outside of a closed surface S the same field as is produced by any electrification within S, whose potential is V, by distributing over the surface S a certain surface distribution, and if S is an equipotential surface, the surface density must be

$$\sigma = \frac{1}{4\pi}\frac{\partial V}{\partial n_i}.$$

If now we place on the surface S a distribution whose surface density is $-\sigma$ instead of σ, its effect on outside bodies will be the negative of that of the internal electrifications. Accordingly if a closed equipotential surface completely surrounding electrified bodies be made conducting, we may cover it with such a charge as to completely screen external bodies from the electrical action of the internal charges. By the same theorem the magnitude of the surface charge is equal and opposite to that of the internal electrification. By the theorem for an interior point, we see that upon such an equipotential surface made conducting we may place a charge that shall shield internal points from the action of the external electrifications, the magnitude of the shielding surface charge being now equal to that of the internal charges.

151. Green's Function. As a means of solving certain problems in electrostatics Green introduced a certain function*, whose properties we shall next consider. Green's problem for a portion of space τ bounded by a closed surface S may be stated as follows:

It is required to find a function G satisfying the following conditions:

1°. G is harmonic in the whole space considered with the exception of a single point P.

2°. G becomes infinite at P, but in such a manner that the function $G - \frac{1}{r}$ is harmonic, r being the distance from the pole P.

3°. The value of any function V harmonic in τ is given at the pole P by the surface integral

$$(1) \qquad V_P = \frac{1}{4\pi}\iint_S V \frac{\partial G}{\partial n_i} dS.$$

A function satisfying these conditions is called Green's function for the space τ and pole P.

The problem is unique, if it has a solution. For if there are two solutions G_1 and G_2, by 3°,

$$(2) \qquad V_P = \frac{1}{4\pi}\iint V \frac{\partial G_1}{\partial n_i} dS,$$

$$(3) \qquad V_P = \frac{1}{4\pi}\iint V \frac{\partial G_2}{\partial n_i} dS,$$

so that by subtraction

$$(4) \qquad \iint V \frac{\partial (G_1 - G_2)}{\partial n_i} dS = 0,$$

for any harmonic function V. But by 2°,

$$G_1 - \frac{1}{r} \text{ and } G_2 - \frac{1}{r}$$

are harmonic, so that their difference $G_1 - G_2$ is also harmonic. Applying the above result to the harmonic function $G_1 - G_2$,

$$(5) \qquad \iint (G_1 - G_2) \frac{\partial (G_1 - G_2)}{\partial n_i} dS = 0.$$

* Green, *Essay*, § 5. The name *Green's Function* is due to C. Neumann, who applies it, however, as does Maxwell, to the function $G - \frac{1}{r}$.

But by Green's theorem this is equal to the volume integral

$$-\iiint\left\{\left(\frac{\partial(G_1-G_2)}{\partial x}\right)^2+\left(\frac{\partial(G_1-G_2)}{\partial y}\right)^2+\left(\frac{\partial(G_1-G_2)}{\partial z}\right)^2\right\}d\tau$$

which, as in § 86 can vanish only if $G_1 - G_2 = \text{const}$. That is, with the exception of a constant, Green's function is unique. But as in the employment of the function only its derivative is used, the constant makes no difference.

Since the function $G - \frac{1}{r}$ is harmonic, we have by § 33 (2)

$$(6) \qquad \iint\left\{V\frac{\partial}{\partial n_i}\left(G-\frac{1}{r}\right)-\left(G-\frac{1}{r}\right)\frac{\partial V}{\partial n_i}\right\}dS=0,$$

or transposing,

$$(7) \qquad \iint\left(V\frac{\partial G}{\partial n_i}-G\frac{\partial V}{\partial n_i}\right)dS$$

$$=\iint\left(V\frac{\partial\left(\frac{1}{r}\right)}{\partial n_i}-\frac{1}{r}\frac{\partial V}{\partial n_i}\right)dS=4\pi V_P,$$

by § 83 (6). If on the surface $G=0$ we obtain

$$V_P=\frac{1}{4\pi}\iint V\frac{\partial G}{\partial n_i}dS.$$

Consequently if we can solve Dirichlet's problem for the given space, obtaining a harmonic function Γ which takes at the surface S the values

$$\Gamma_S=-\frac{1}{r},$$

then the function $$G=\Gamma+\frac{1}{r}$$

solves Green's problem. Conversely if we can solve Green's problem for the space and for any pole P, the equation (1) enables us to find any harmonic function V from its values at the surface, solving Dirichlet's problem.

The problems of Green and Dirichlet are thus exactly equivalent.

In physical language, Green's function is the potential due to a positive unit of electricity placed at the pole P together with that of the charge which it induces on the surface S made con-

ducting and connected to earth. If σ_G is the density of the induced charge,

$$\sigma_G = -\frac{1}{4\pi}\frac{\partial G}{\partial n_i},$$

and (1) is

$$V_P = -\iint_S V\sigma_G dS.$$

Suppose that G is Green's function for a certain space, with the pole P, whose co-ordinates are a, b, c, and that G' is Green's function for the same space, but a different pole P' whose co-ordinates are a', b', c'. Then there exists the reciprocal relation that the values of either function at the pole of the other are equal. For

$$G = \Gamma + \frac{1}{r_P}, \quad G' = \Gamma' + \frac{1}{r_{P'}},$$

where the suffixes indicate from what point the distance is measured. Now since Γ and Γ' are harmonic, by the property of the two Green's functions G and G',

$$(8) \qquad \begin{aligned} 4\pi\Gamma_{P'} &= \iint \Gamma \frac{\partial G'}{\partial n} dS, \\ 4\pi\Gamma'_P &= \iint \Gamma' \frac{\partial G}{\partial n} dS, \end{aligned}$$

so that

$$(9) \qquad 4\pi\left[\Gamma_{P'} - \Gamma'_P\right] = \iint\left(\Gamma\frac{\partial G'}{\partial n} - \Gamma'\frac{\partial G}{\partial n}\right) dS$$

$$= \iint\left(\Gamma\frac{\partial \Gamma'}{\partial n} - \Gamma'\frac{\partial \Gamma}{\partial n}\right) dS + \iint\left(\Gamma\frac{\partial\left(\frac{1}{r_{P'}}\right)}{\partial n} - \Gamma'\frac{\partial\left(\frac{1}{r_P}\right)}{\partial n}\right) dS.$$

The last integral but one vanishes because Γ and Γ' are harmonic functions, while on account of the surface values of Γ and Γ', the last becomes

$$\iint\left(\frac{1}{r_{P'}}\frac{\partial\left(\frac{1}{r_P}\right)}{\partial n} - \frac{1}{r_P}\frac{\partial\left(\frac{1}{r_{P'}}\right)}{\partial n}\right) dS.$$

Since both the functions $1/r_P$ and $1/r_{P'}$ are harmonic except at their poles P and P', by constructing small spheres about the points P and P' and proceeding as in § 83, we find that the two parts of the last integral destroy each other (each being equal to

$4\pi/r_{PP'}$), so that $\Gamma_{P'} = \Gamma'_P$. Accordingly we have for the two points P and P',

(10) $$G_{P'} = \Gamma_{P'} + \frac{1}{r_{PP'}} = \Gamma'_P + \frac{1}{r_{PP'}} = G'_P.$$

In order to show the dependence of the function G on the coordinates of its pole P let us write it

(11) $$G(x, y, z) = g(x, y, z, a, b, c),$$

and $$G'(x, y, z) = g(x, y, z, a', b', c').$$

Then by the above theorem

$$G(a', b', c') = G'(a, b, c),$$

(12) $$g(a', b', c', a, b, c) = g(a, b, c, a', b', c'),$$

or Green's function is a symmetric function of its variables a, b, c and a', b', c'.

152. Examples of Green's Function. Plane. Let us seek Green's function for all that portion of space lying on one side of a given plane. Let A be the given pole, at a distance a from the

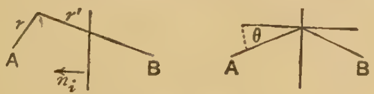

FIG. 62 *a*.

plane, on the left, and let B be its geometrical image in the plane. Let the distances of any point at the left of the plane from A and B be r and r' respectively. Now for every point at the left of the plane the function $-\frac{1}{r'}$ is harmonic, and for points on the plane, where $r = r'$, it assumes the value $-\frac{1}{r}$. It is therefore the function Γ of the preceding article. We have then

$$G = \frac{1}{r} - \frac{1}{r'},$$

$$\frac{\partial G}{\partial n_i} = -\frac{\cos(n_i r)}{r^2} + \frac{\cos(n_i r')}{r'^2} = \frac{2\cos\theta}{r^2},$$

where θ is the acute angle included between the radius r and the normal to the plane. Consequently, the equation

$$V_A = \frac{1}{4\pi}\iint V\frac{\partial G}{\partial n_i}dS = \frac{1}{2\pi}\iint V\frac{\cos\theta}{r^2}dS = \frac{a}{2\pi}\iint\frac{V}{r^3}dS$$

solves Dirichlet's problem for the left-hand side of the plane. If we suppose a charge of a positive unit placed at A, and a negative unit at B, the plane of symmetry will be an equipotential surface of zero potential, and we may apply the theorem of equipotential layers. If the plane of symmetry is made conducting, and the charge B removed, the conducting plane receives a charge -1 which screens the space on the right from the action of A. The surface density on the plane is $\sigma_G = -\cos\theta/2\pi r^2$, so that the whole charge on the plane is, applying Gauss's theorem,

$$e = \frac{-1}{2\pi}\iint \frac{\cos\theta}{r^2}\, dS = -1.$$

This is an example of the second theorem of § 150, the space on the right being considered internal.

The charge -1 at B is said to be the *electrical image* in the plane of the charge $+1$ at A.

Two point-charges A and B are said to be electrical images of each other in a certain closed surface separating them if either one, say B, produces in the portion of space in which the other, A, lies, the same effect as would be produced there by the charge induced on the surface made conducting and connected to earth, by the point A alone, the image B, being removed.

153. Planes intersecting in a sub-multiple of two right angles.

Let us seek Green's function for a portion of space lying in the acute angle between two planes intersecting in an angle which is equal to two right angles divided by an integer. Let the planes be denoted by 1 and 2, let the pole be P, and let P_1 be the geometrical image of P in 1, P_2 that of P_1 in 2, P_3 that of P_2 in 1, and so on alternately in the two planes. Let Q_1 be the image of P in 2, Q_2 that of Q_1 in 1, Q_3 that of Q_2 in 2, and so on. Since the angle is a submultiple of π it is easily seen that the series of images will be finite, the Q's and P's finally coinciding. Let the distance of any point from P be denoted by r, from any P_s by r_s, and from any Q_s by r_s'. Then the reciprocal of any distance r_s or r_s' is a harmonic function in the space between the planes since none of the images lie in that space. Also for all points lying on the plane 1,

$$\frac{1}{r} - \frac{1}{r_1} = 0, \quad \frac{1}{r_1'} - \frac{1}{r_2'} = 0, \quad \frac{1}{r_2} - \frac{1}{r_3} = 0, \quad \frac{1}{r_3'} - \frac{1}{r_4'} = 0 \;\ldots\ldots,$$

and for all points lying on the plane 2,

$$\frac{1}{r}-\frac{1}{r_1'}=0, \quad \frac{1}{r_1}-\frac{1}{r_2}=0, \quad \frac{1}{r_2'}-\frac{1}{r_3'}=0, \quad \frac{1}{r_3}-\frac{1}{r_4}=0 \,\ldots\ldots$$

Consequently the function

$$G=\frac{1}{r}-\left(\frac{1}{r_1}+\frac{1}{r_1'}\right)+\left(\frac{1}{r_2}+\frac{1}{r_2'}\right)-\left(\frac{1}{r_3}+\frac{1}{r_3'}\right)+\ldots\ldots,$$

vanishes for points on either plane, and being harmonic except at P, is Green's function.

154. Two parallel Planes. If P lie in the space between two parallel planes the successive images will all lie in a straight line, and will be infinite in number. Using the same notation as in the last example, we have the same equations, and the same form of Green's function, except that we shall have an infinite series.

$$G=\frac{1}{r}+\sum_{1}^{\infty}(-1)^s\left(\frac{1}{r_s}+\frac{1}{r_s'}\right).$$

155. Sphere. Let A be the given pole, at a distance a from the center of the sphere of radius R. Take a point B lying on the same radius as A, at a distance from the center b such that $ab=R^2$. Then A and B are said to be inverse points with respect to the sphere. If M be any point on the surface of the sphere, the triangles OMB and MAO are similar, for they have a common angle at O, and the sides including it are proportional, for by hypothesis,

(1) $$\frac{b}{R}=\frac{R}{a}.$$

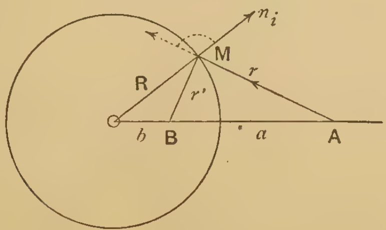

FIG. 63.

Accordingly, for points on the surface

(2) $$\frac{r'}{R}=\frac{r}{a} \text{ and } \frac{1}{r}=\frac{R}{a}\frac{1}{r'},$$

r and r' being the distances of any point from A and B respectively. Therefore, since $1/r'$ is harmonic in the space containing A, Green's function is for that space

(3) $$G = \frac{1}{r} - \frac{R}{a}\frac{1}{r'},$$

(4) $$\frac{\partial G}{\partial n_i} = -\frac{1}{r^2}\frac{\partial r}{\partial n_i} + \frac{R}{a}\frac{1}{r'^2}\frac{\partial r'}{\partial n_i} = -\frac{\cos(n_i r)}{r^2} + \frac{R}{a}\frac{\cos(n_i r')}{r'^2},$$

(5) $$V_A = \frac{1}{4\pi}\iint V\left\{-\frac{\cos(n_i r)}{r^2} + \frac{R}{a}\frac{\cos(n_i r')}{r'^2}\right\} dS,$$

so that the density of an equipotential layer induced by a unit charge at A on the sphere made conducting is

(6) $$\sigma_G = \frac{1}{4\pi}\left\{\frac{\cos(n_i r)}{r^2} - \frac{R}{a}\frac{\cos(n_i r')}{r'^2}\right\}.$$

Now in the triangles OMB and MAO we have

(7) $$\begin{aligned} a^2 &= R^2 + r^2 - 2Rr\cos(n_i r), \\ b^2 &= R^2 + r'^2 - 2Rr'\cos(n_i r'), \end{aligned}$$

so that

(8) $$\frac{\cos(n_i r)}{r^2} - \frac{R}{a}\frac{\cos(n_i r')}{r'^2} = -\frac{a^2 - (R^2 + r^2)}{2Rr^3} + \frac{R}{a}\frac{(b^2 - (R^2 + r'^2))}{2Rr'^3},$$

which by (1) and (2) gives $\dfrac{R^2 - a^2}{Rr^3}$, and

(9) $$\sigma_G = \frac{1}{4\pi}\frac{R^2 - a^2}{Rr^3},$$

(10) $$V_A = \frac{a^2 - R^2}{4\pi R}\iint \frac{V}{r^3}\, dS.$$

The whole induced charge is

(11) $$e' = \frac{1}{4\pi}\iint\left(\frac{\cos(n_i r)}{r^2} - \frac{R}{a}\frac{\cos(n_i r')}{r'^2}\right) dS,$$

and if A is an outside point by Gauss's theorem

$$\iint \frac{\cos(n_i r)}{r^2}\, dS = 0, \qquad \iint \frac{\cos(n_i r')}{r'^2}\, dS = 4\pi,$$

so that

(12) $$e' = -\frac{R}{a}.$$

If we should place a charge e at the outside point A, and a charge e' at the inside B, the potential on the surface of the sphere would be $V=\frac{e}{r}+\frac{e'}{r'}$ which, if we make $e'=-Re/a$ becomes zero. The action of the charge e' at B in portions of space outside of the sphere may thus be exactly replaced by making the sphere conducting and removing the charge B. Accordingly the charges at A and B are electrical images of each other in the sphere.

Suppose now that the sphere, instead of being connected to earth, is insulated and charged to a potential V, then beside the induced charge it will have a uniformly distributed charge VR of density $V/4\pi R$, so that the whole charge of the sphere is now

$$E=VR-\frac{eR}{a}, \text{ and } V=\frac{E}{R}+\frac{e}{a}. \tag{13}$$

The surface density

$$\sigma=\frac{1}{4\pi}\left\{\frac{V}{R}+\frac{R^2-a^2}{Rr^3}\right\}, \tag{14}$$

vanishes along the circle $Vr^3=a^2-R^2$, which divides the surface into two parts oppositely electrified. If however

$$V>\frac{a^2-R^2}{(a-R)^3} \text{ or } V<\frac{a^2-R^2}{(a+R)^3},$$

the surface density is of the same sign all over the sphere. Since the action of the induced charge on external points is the same as would be that of a charge e' at B, and the action of the uniform charge is the same as that of a charge VR at the center, the repulsion of the whole charge of the sphere on the charge e at A is

$$\frac{VRe}{a^2}+\frac{e'e}{(a-b)^2}=eR\left\{\frac{V}{a^2}-\frac{ea}{(a^2-R^2)^2}\right\} \tag{15}$$

$$=\frac{e}{a^2}\left\{E-e\frac{R^3(2a^2-R^2)}{a(a^2-R^2)^2}\right\}.$$

This is negative, so that there is an attraction, when $V=0$, or $E=0$, or $a-R$ is small; that is if the sphere is connected to earth, if it is insulated without charge, or in any case if the charged point A is very near to the sphere. On the other hand, by making V or E of the same sign as e and great enough in absolute

value, we have a repulsion, when

$$|V| > \left|\frac{ea^3}{(a^2-R^2)^2}\right| \text{ or } |E| > \left|\frac{eR^3(2a^2-R^2)}{a(a^2-R^2)^2}\right|.$$

156. Electrical Images in a Sphere. Points which are electrical images of each other, besides having the properties connected with equipotential layers described above, possess peculiar reciprocal properties with respect to the portions of space in which they are respectively situated. There thus arises a method of finding from the known solutions of electrostatic problems a new class of problems whose solutions can be found. This method of electrical images was discovered by Lord Kelvin in 1848*. Suppose as before that A and B are inverse points with reference to the sphere of radius R, A being outside. Let M and M' (M outside) be two other inverse points situated at distances l and l' from the center, and at distances r and r' respectively from A and B. Then the triangles OAM and $OM'B$ are similar, since $ab = ll' = R^2$. Suppose a charge e placed at A, and a charge $e' = -eR/a$ placed at B. If we call V the potential at M due to the charge e, and V' the potential at M' due to the charge e', we have

$$(1) \qquad \frac{V'}{V} = \frac{e'}{r'} \cdot \frac{r}{e} = \frac{e'}{e} \cdot \frac{r}{r'} = -\frac{R}{a}\frac{l}{b} = -\frac{l}{R} = -\frac{R}{l'}.$$

If then we have any number of electrified points such as A, and find their images B, and if V be the potential of the system A at any external point, M, then

$$(2) \qquad V' = -\frac{RV}{l'} = -\frac{lV}{R},$$

will be the potential at M' the inverse point to M, of the system B which is the *electrical image* of the system A.

We shall give an analytical proof of the same proposition, based on the method of curvilinear coordinates. If x, y, z are the coordinates of the point M, x', y', z', those of the point M', we have $x'/x = y'/y = z'/z$ and since $ll' = R^2$,

$$(3) \qquad x' = \frac{R^2x}{l^2}, \quad y' = \frac{R^2y}{l^2}, \quad z' = \frac{R^2z}{l^2},$$

$$l^2 = x^2 + y^2 + z^2, \quad l'^2 = x'^2 + y'^2 + z'^2.$$

* *Papers on Electrostatics and Magnetism*, p. 144.

If x', y', z' are given, we know x, y, z and the position of M, so that we may consider x', y', z' given by the above equations, as *curvilinear* coordinates of the point M, disregarding for the present their relation as *rectangular* coordinates of the inverse point M'. Forming their differential parameters,

$$(4)\quad \begin{aligned}
&\frac{\partial x'}{\partial x} = R^2\left\{\frac{1}{l^2} - \frac{2x^2}{l^4}\right\}, && \frac{\partial x'}{\partial y} = -2R^2\frac{xy}{l^4}, && \frac{\partial x'}{\partial z} = -2R^2\frac{xz}{l^4},\\
&\frac{\partial y'}{\partial x} = -2R^2\frac{xy}{l^4}, && \frac{\partial y'}{\partial y} = R^2\left\{\frac{1}{l^2} - \frac{2y^2}{l^4}\right\}, && \frac{\partial y'}{\partial z} = -2R^2\frac{yz}{l^4},\\
&\frac{\partial z'}{\partial x} = -2R^2\frac{zx}{l^4}, && \frac{\partial z'}{\partial y} = -2R^2\frac{yz}{l^4}, && \frac{\partial z'}{\partial z} = R^2\left\{\frac{1}{l^2} - \frac{2z^2}{l^4}\right\},
\end{aligned}$$

$$h^2 = h_{x'}^2 = h_{y'}^2 = h_{z'}^2 = \frac{4R^4x^2}{l^8}(x^2+y^2+z^2) + \frac{R^4}{l^4} - \frac{4R^2x^2}{l^6} = \frac{R^4}{l^4},$$

$$(5)\qquad h = \frac{R^2}{l^2} = \frac{l'^2}{R^2}.$$

It is easily seen that the surfaces, $x' = \text{const.}$, $y' = \text{const.}$, $z' = \text{const.}$, cut each other orthogonally, for example the cosine of the angle between the normals of $x' = \text{const.}$ and $y' = \text{const.}$ is proportional to

$$\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial x'}{\partial z}\frac{\partial y'}{\partial z} = R^4\left\{\frac{4(x^3y + xy^3 + xyz^2)}{l^8} - \frac{4xy}{l^6}\right\} = 0.$$

We have then by § 87, (5)

$$(6)\qquad \Delta V = \frac{l'^6}{R^4}\left\{\frac{\partial}{\partial x'}\left(\frac{1}{l'^2}\frac{\partial V}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\frac{1}{l'^2}\frac{\partial V}{\partial y'}\right) + \frac{\partial}{\partial z'}\left(\frac{1}{l'^2}\frac{\partial V}{\partial z'}\right)\right\},$$

and performing the differentiations

$$\frac{\partial}{\partial x'}\left(\frac{1}{l'^2}\frac{\partial V}{\partial x'}\right) = \frac{1}{l'^2}\frac{\partial^2 V}{\partial x'^2} + \frac{2}{l'}\frac{\partial\left(\frac{1}{l'}\right)}{\partial x'}\frac{\partial V}{\partial x'}, \text{ etc.}$$

Now we have

$$\frac{\partial}{\partial x'}\left\{\frac{V}{l'}\right\} = \frac{1}{l'}\frac{\partial V}{\partial x'} + V\frac{\partial\left(\frac{1}{l'}\right)}{dx'},$$

$$\frac{\partial^2}{\partial x'^2}\left\{\frac{V}{l'}\right\} = \frac{1}{l'}\frac{\partial^2 V}{\partial x'^2} + 2\frac{\partial V}{\partial x'}\frac{\partial\left(\frac{1}{l'}\right)}{\partial x'} + V\frac{\partial^2\left(\frac{1}{l'}\right)}{\partial x'^2}.$$

Forming the derivatives for y and z, writing

$$\Delta' \text{ for } \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2},$$

and comparing with the expression for ΔV above, we get

(7) $$\frac{1}{l'}\Delta'\left\{\frac{V}{l'}\right\}=\frac{R^4}{l'^6}\Delta V+\frac{V}{l'}\Delta\left(\frac{1}{l'}\right).$$

But $1/l'$ is harmonic except at 0, and therefore

(8) $$\Delta'\left\{\frac{V}{l'}\right\}=\frac{R^4}{l'^5}\Delta V.$$

If we put

$$V'=\frac{RV}{l'},\text{ then}$$

(9) $$\Delta' V'=\frac{R^5}{l'^5}\Delta V,$$

and we get the proposition that if V is a harmonic function of the point M, then V' is a harmonic function of the corresponding point M'. If the distribution causing V is distributed continuously in three dimensions, the density is $\rho=-\Delta V/4\pi$ and in the image the density is $\rho'=-\Delta' V'/4\pi$ so that

(10) $$\frac{\rho'}{\rho}=\frac{R^5}{l'^5}=\frac{l^5}{R^5}.$$

If ds and ds' be corresponding infinitesimal arcs, expressing ds in terms of the curvilinear coordinates x', y', z'

$$ds^2=\frac{dx'^2}{h_{x'}^2}+\frac{dy'^2}{h_{y'}^2}+\frac{dz'^2}{h_{z'}^2}=\frac{l^4}{R^4}(dx'^2+dy'^2+dz'^2)=\frac{l^4 ds'^2}{R^4},$$

so that we have for the ratios of corresponding infinitesimal arcs, surfaces, and volumes

(11) $$\frac{ds'}{ds}=\frac{R^2}{l^2}=\frac{l'^2}{R^2},\quad \frac{dS'}{dS}=\frac{R^4}{l^4}=\frac{l'^4}{R^4},\quad \frac{d\tau'}{d\tau}=\frac{R^6}{l^6}=\frac{l'^6}{R^6}.$$

The ratio of charges of corresponding infinitesimal volumes is

(12) $$\frac{de'}{de}=\frac{\rho' d\tau'}{\rho d\tau}=\frac{R}{l}=\frac{l'}{R'},$$

and of the surface densities

(13) $$\frac{\sigma'}{\sigma}=\frac{de'}{dS'}\frac{dS}{de}=\frac{l^3}{R^3}=\frac{R^3}{l'^3}.$$

It is to be noticed that if the original distribution is an equipotential one, which is the case if it is on the surface of a conductor, the image will not be equipotential, on account of the variable factor $1/l'$, but if we place at 0, the center of inversion, a charge $-RV$, its potential $-RV/l'$ added to the potential due to the image gives zero, an equipotential distribution. Consequently any problem of electrical equilibrium whose solution is known gives us the solution of the problem of induction by any point-charge on a conductor whose surface is the inverse of the given conductor with respect to the point at which the inducing charge is placed. Conversely the solution of a problem of induction by a point-charge gives us the solution of a problem of undisturbed equilibrium. Thus the solutions of the problems treated above furnish us new solutions.

The image of a sphere is a sphere (including a plane as a special case) for the equation of a sphere

$$A(x^2+y^2+z^2)+Bx+Cy+Dz+E=0,$$

becomes, using the equations (3),

$$\frac{AR^4}{x'^2+y'^2+z'^2}+\frac{BR^2x'}{x'^2+y'^2+z'^2}+\frac{CR^2y'}{x'^2+y'^2+z'^2}+\frac{DR^2z'}{x'^2+y'^2+z'^2}+E=0,$$

that is,

$$E(x'^2+y'^2+z'^2)+BR^2x'+CR^2y'+DR^2z'+AR^4=0.$$

If A is zero, we have originally a plane, which inverts into a sphere passing through the origin, while if E is zero, the sphere through the origin inverts into a plane.

As an example of the method, let us invert a sphere of radius $a/2$ charged to potential V about a point on its surface, with radius of inversion, $R=a$. The sphere inverts into a plane tangent to the sphere at the point diametrically opposite the center of inversion. The charge of the sphere being $Va/2$, the surface density is

$$\sigma=\frac{V}{2\pi a}.$$

Consequently

$$\sigma'=\frac{a^3}{l'^3}\sigma=\frac{Va^2}{2\pi l'^3},$$

and if we put at the center of inversion a charge $e = -Va$,

$$\sigma' = -\frac{ea}{2\pi l'^3}$$

is the density of the charge induced on the plane, agreeing with the result of § 152. Inverting the distribution induced by a point-charge on the two parallel planes gives us the equilibrium distribution on two spheres tangent to each other, and inverting the distribution on two intersecting planes gives the equilibrium distribution on two spheres intersecting at an angle which is a sub-multiple of two right angles. For the full treatment of these and other examples the reader may be referred to Lord Kelvin's Reprint of Papers on Electrostatics and Magnetism, XIV, XV, and to Maxwell, Treatise, Vol. I, Chapter XI.

157. Distribution on Spherical Bowl. As a final example we shall work out the solution of the most remarkable problem that has been treated by this method, namely the distribution of electricity on an open spherical bowl, or segment of a sphere. This is the only case in which the distribution on a *portion* of a geometrical surface has been solved, except in the case of the distribution on a circular plate, the inversion of which gives the circular bowl. We shall not follow the method of Lord Kelvin, but that given by Lipschitz*, who solved the problem independently, being unacquainted with the existence of a previous solution.

Let R be the radius of the sphere of which the bowl is a segment, Fig. 64, and let the radius of the opening be a. Let the surface of the bowl be denoted by S, and let the plane surface which closes the bowl, of radius a and distance c from the center of the sphere, be denoted by Σ. Inverting the figure with respect to the center of the bowl and radius R, let the spherical segment into which the plane Σ inverts be denoted by Σ'. Let the space enclosed between S and Σ be denoted by T, that between Σ and Σ' by T'', and the remaining portion of space by T'. Let

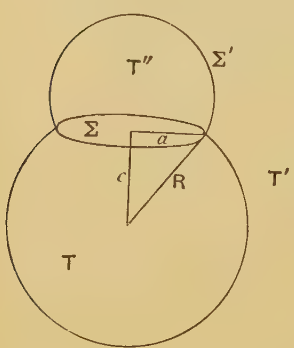

FIG. 64.

* Lipschitz, *Borchardt's Journal*, Bd. LVIII., p. 162, 1861.

us now form a function W which behaves at infinity like a potential function, and is harmonic in all space, except that it is discontinuous everywhere on the spherical segment S. (If it were not for the discontinuity, such a function would vanish everywhere.) Let us also form a function W' defined by

$$(1)\qquad W'(x, y, z) = \frac{R}{l} W\left(\frac{R^2 x}{l^2}, \frac{R^2 y}{l^2}, \frac{R^2 z}{l^2}\right)$$
$$= \frac{R}{l} W(x', y', z') = \frac{l'}{R} W(x', y', z').$$

Multiplying by $l/R = R/l'$,

$$(2)\qquad W(x', y', z') = \frac{R}{l'} W'(x, y, z) = \frac{R}{l'} W'\left(\frac{R^2 x'}{l'^2}, \frac{R^2 y'}{l'^2}, \frac{R^2 z'}{l'^2}\right).$$

If the values of the functions for points internal and external to the sphere S be distinguished by the suffixes i and e, on the surface S, since $x = x'$, $y = y'$, $z = z'$,

$$(3)\qquad W_i' = W_e, \quad W_e' = W_i.$$

$W'(x, y, z)$ vanishes for $l = \infty$ and is finite for $l = 0$ since

$$(4)\qquad W'(x, y, z) = \frac{l'}{R} W(x', y', z') \text{ and}$$

$$\lim_{l=0} W' = \frac{1}{R} \lim_{l'=\infty} l' W(x', y', z') = \text{const.}$$

Let us now put

$$(5)\qquad V(x, y, z) = W(x, y, z) + W'(x, y, z),$$

and we shall show that the function W may be so defined that V will be the potential of an equilibrium distribution on the spherical segment.

We have seen in § 141 that the potential at any point due to an equilibrium distribution on a circular disc of radius a is

$$\frac{e}{a}\left\{\frac{\pi}{2} - \tan^{-1}\frac{\sqrt{\lambda}}{a}\right\},$$

where λ is the greater root of the quadratic

$$\frac{x^2 + y^2}{a^2 + \lambda} + \frac{z^2}{\lambda} = 1.$$

The derivative of this function according to the normal of course has a discontinuity by changing sign on crossing the disc. If we consider the disc placed in the mouth of the bowl, on

account of the change of z-coordinate we must take λ as the root of the equation

(6) $$\frac{x^2+y^2}{a^2+\lambda}+\frac{(z-c)^2}{\lambda}=1,$$

and if λ' be the same function of x', y', z' that λ is of x, y, z we must have

(7) $$\frac{x'^2+y'^2}{a^2+\lambda'}+\frac{(z'-c)^2}{\lambda'}=1, \text{ or}$$

(8) $$\frac{x^2+y^2}{a^2+\lambda'}+\frac{\left(z-\frac{cl^2}{R^2}\right)^2}{\lambda'}=\frac{l^4}{R^4}.$$

We will now define our function W by two different analytic expressions.

In the space T we take

$$W=\frac{\pi}{2}+\tan^{-1}\frac{\sqrt{\lambda}}{a},$$

and in the space T' and T''

$$W=\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda}}{a}.$$

This makes W continuous at Σ as required, since on the disc $\lambda=0$, and the change of sign in the second term makes the normal derivative continuous in crossing the disc Σ. By the definition of W' we have in T and T'' (since the inverse of T is T', and of T'' is itself)

$$W'=\frac{R}{l}\left(\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda'}}{a}\right),$$

and in T'

$$W'=\frac{R}{l}\left(\frac{\pi}{2}+\tan^{-1}\frac{\sqrt{\lambda'}}{a}\right).$$

Accordingly we have for the values of V in

$$T,\quad V=\frac{\pi}{2}+\tan^{-1}\frac{\sqrt{\lambda}}{a}+\frac{R}{l}\left(\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda'}}{a}\right),$$

(9) $$T',\quad V'=\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda}}{a}+\frac{R}{l}\left(\frac{\pi}{2}+\tan^{-1}\frac{\sqrt{\lambda'}}{a}\right),$$

$$T'',\quad V''=\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda}}{a}+\frac{R}{l}\left(\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda'}}{a}\right).$$

The function V is everywhere continuous, for W and W' are continuous except at S and there, by (3),

(10) $$V_i = W_i + W_i' = W_e' + W_e = V_e,$$

so that V is continuous.

We have already seen that the derivative of W is continuous in crossing Σ, and accordingly that of W' is continuous in crossing Σ'. Now the derivative of W is continuous in crossing Σ', since W is defined by the same continuous analytic expression in T' and T'', and the derivative of W' is continuous in crossing Σ, since W' is defined by the same continuous analytic expression in T and T''. Accordingly the derivatives of V as well as V itself satisfy the required conditions of continuity, and on S, since

$$x = x', \; y = y', \; z = z', \; l = R, \; \lambda' = \lambda,$$

we have $V = \pi$ and V, the function assumed, is therefore the potential of an equilibrium distribution on the bowl. At the center of the bowl we have on the one hand

(11) $$V(0) = \iint \frac{\sigma dS}{R} = \frac{e}{R},$$

while in order to employ the formula (9) we have $x = y = z = 0$, $\lambda = c^2$.

But when l is infinitely small, λ' must be infinite of the second order, as we see by making l infinitesimal in the equation

(8) $$x^2 + y^2 + \left(z - \frac{cl^2}{R^2}\right)^2 \left(\frac{a^2}{\lambda'} + 1\right) = (a^2 + \lambda') \frac{l^4}{R^4}.$$

The terms of the lowest order are

$$x^2 + y^2 + z^2 = \frac{\lambda' l^4}{R^4}.$$

Hence approximately

$$\lambda' = \frac{R^4}{l^2},$$

and

$$W'(0) = \lim_{l=0} \frac{R}{l} \left(\frac{\pi}{2} - \tan^{-1} \frac{R^2}{la}\right) = \lim_{l=0} \frac{R}{l} \tan^{-1} \frac{la}{R^2} = \frac{a}{R}.$$

Therefore we have finally

(12) $$V(0) = \frac{\pi}{2} + \tan^{-1} \frac{c}{a} + \frac{a}{R} = \frac{e}{R},$$

(13) $$e = R\left(\frac{\pi}{2} + \tan^{-1} \frac{c}{a}\right) + a.$$

If we call γ the angular half-opening of the bowl, $a/R = \sin\gamma$, $a/c = \tan\gamma$ and the charge of the bowl is

$$e = R(\pi - \gamma) + R\sin\gamma, \tag{14}$$

giving as its capacity

$$K = \frac{e}{V} = R\frac{\pi - \gamma + \sin\gamma}{\pi}. \tag{15}$$

To complete the problem we have to find the surface density. We find

$$\frac{\partial V}{\partial n_i} = \frac{1}{1 + \frac{\lambda}{a^2}} \cdot \frac{1}{2a\sqrt{\lambda}}\frac{\partial\lambda}{\partial n_i} - \frac{R}{l}\frac{1}{1 + \frac{\lambda'}{a^2}} \cdot \frac{1}{2a\sqrt{\lambda'}}\frac{\partial\lambda'}{\partial n_i} - \frac{R}{l^2}\frac{\partial l}{\partial n_i}\left(\frac{\pi}{2} - \tan^{-1}\frac{\sqrt{\lambda'}}{a}\right). \tag{16}$$

$$\frac{\partial V'}{\partial n_e} = -\frac{1}{1 + \frac{\lambda}{a^2}} \cdot \frac{1}{2a\sqrt{\lambda}}\frac{\partial\lambda}{\partial n_e} + \frac{R}{l}\frac{1}{1 + \frac{\lambda'}{a^2}} \cdot \frac{1}{2a\sqrt{\lambda'}}\frac{\partial\lambda'}{\partial n_e} - \frac{R}{l^2}\frac{\partial l}{\partial n_e}\left(\frac{\pi}{2} + \tan^{-1}\frac{\sqrt{\lambda'}}{a}\right). \tag{17}$$

Now on the surface S

$$\lambda' = \lambda, \quad \frac{\partial\lambda'}{\partial n} = -\frac{\partial\lambda}{\partial n}, \quad l = R, \quad \frac{\partial l}{\partial n_e} = -\frac{\partial l}{\partial n_i} = 1,$$

and therefore

$$\frac{\partial V}{\partial n_i} = \frac{1}{R}\left(\frac{\pi}{2} - \tan^{-1}\frac{\sqrt{\lambda}}{a}\right) + \frac{a}{(a^2 + \lambda)\sqrt{\lambda}}\frac{\partial\lambda}{\partial n_i}. \tag{18}$$

$$\frac{\partial V'}{\partial n_e} = -\frac{1}{R}\left(\frac{\pi}{2} + \tan^{-1}\frac{\sqrt{\lambda}}{a}\right) - \frac{a}{(a^2 + \lambda)\sqrt{\lambda}}\frac{\partial\lambda}{\partial n_e}. \tag{19}$$

Now the direction cosines of the normal n_e being $x/R, y/R, z/R$, we have

$$\frac{\partial\lambda}{\partial n_e} = \frac{x}{R}\frac{\partial\lambda}{\partial x} + \frac{y}{R}\frac{\partial\lambda}{\partial y} + \frac{z}{R}\frac{\partial\lambda}{\partial z}. \tag{20}$$

The quadratic for λ is, cleared of fractions,

$$\lambda(a^2 + \lambda) - (x^2 + y^2)\lambda - (z - c)^2(a^2 + \lambda) = 0, \tag{21}$$

which being partially differentiated gives

$$[a^2+2\lambda-(x^2+y^2)-(z-c)^2]\frac{\partial\lambda}{\partial x}=2\lambda x,$$

(22) $$[a^2+2\lambda-(x^2+y^2)-(z-c)^2]\frac{\partial\lambda}{\partial y}=2\lambda y,$$

$$[a^2+2\lambda-(x^2+y^2)-(z-c)^2]\frac{\partial\lambda}{\partial z}=2(a^2+\lambda)(z-c).$$

Multiplying these respectively by x/R, y/R, z/R and adding gives

(23) $$\frac{\partial\lambda}{\partial n_e}=\frac{2\lambda[(x^2+y^2+z^2)-cz]+2a^2z(z-c)}{R[a^2+2\lambda-(x^2+y^2)-(z-c)^2]}.$$

Putting now $x^2+y^2+z^2=R^2=a^2+c^2$ and using the quadratic (21), the numerator $2\{\lambda(R^2-cz)+a^2z^2-a^2cz\}$, becomes equal to the denominator, $2R(\lambda+cz-c^2)$, multiplied by $(a^2+\lambda)/R$ so that finally

(24) $$\frac{\partial\lambda}{\partial n_e}=-\frac{\partial\lambda}{\partial n_i}=\frac{a^2+\lambda}{R},$$

and for the density within and without we obtain

$$\sigma_i=-\frac{1}{4\pi}\frac{\partial V}{\partial n_i}=-\frac{1}{4\pi R}\left\{\frac{\pi}{2}-\tan^{-1}\frac{\sqrt{\lambda}}{a}-\frac{a}{\sqrt{\lambda}}\right\}$$
$$=\frac{V}{4\pi^2R}\left\{\frac{a}{\sqrt{\lambda}}-\tan^{-1}\frac{a}{\sqrt{\lambda}}\right\},$$

(25)

$$\sigma_e=-\frac{1}{4\pi}\frac{\partial V'}{\partial n_e}=\frac{1}{4\pi R}\left\{\frac{\pi}{2}+\tan^{-1}\frac{\sqrt{\lambda}}{a}+\frac{a}{\sqrt{\lambda}}\right\}$$
$$=\frac{V}{4\pi^2R}\left\{\pi+\frac{a}{\sqrt{\lambda}}-\tan^{-1}\frac{a}{\sqrt{\lambda}}\right\}.$$

The difference of the densities within and without is $V/4\pi R$. The smaller the opening of the bowl, the smaller is the density within. At the edges of the bowl, where λ becomes zero, the density is infinite, as in the case of the circular disc, but the capacity in either case remains finite.

158. Application of the Conformal Representation to two-dimensional problems. In cases where the densities of a distribution are the same at all points situated on the same line parallel to a given direction, as for instance, in the case of the

electrification of very long cylindrical conductors at a distance from the ends, the potential is independent of the coordinate whose axis is parallel to the given direction, and Laplace's equation reduces to two terms.

$$(1)\qquad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$

Such problems are called two-dimensional, or uniplanar, since the distribution is the same in all planes parallel to a single one. For the solution of two-dimensional problems the method of the conformal representation by means of functions of a complex variable gives a powerful method. Let V be a function of x, y, holomorphic in a certain region of the plane. Let us make a conformal transformation of the plane by means of two conjugate functions u, v. V then becomes a function of u, v in the transformed plane. That is, u, v are to be taken as rectangular coordinates in the transformed plane. Then to every point P in the original plane, having the coordinates x, y, there correspond definite values of the functions u, v, and the point P' in the transformed plane having the coordinates u, v corresponds to the point P. To corresponding values points P and P' belong the same values of V. Thus level lines of V in the XY-plane correspond to level lines of different form in the UV-plane. We shall first show that a certain function of the derivatives of V remains unchanged if we replace in it one set of rectangular coordinates x, y by the other u, v.

Considering V as a function of u, v

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial V}{\partial v}\frac{\partial v}{\partial x},\quad \frac{\partial V}{\partial y}=\frac{\partial V}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial V}{\partial v}\frac{\partial v}{\partial y},$$

$$(2)\qquad \frac{\partial^2 V}{\partial x^2}=\frac{\partial^2 V}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial^2 V}{\partial v\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial V}{\partial u}\frac{\partial^2 u}{\partial x^2}$$

$$+\frac{\partial^2 V}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 V}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial V}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial^2 V}{\partial v\partial u}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial V}{\partial u}\frac{\partial^2 u}{\partial y^2}$$

$$+\frac{\partial^2 V}{\partial u\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial^2 V}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial V}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Adding the second derivatives, and making use of the fundamental equations of conjugate functions, § 42 (A), § 44,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0,$$

we obtain

$$(3)\qquad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\left(\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right).$$

If we call

$$\Delta' V=\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2},$$

we see that in a conformal transformation the second differential parameter $\Delta' V$ of any function V in the transformed plane is equal to the second differential parameter ΔV for the corresponding point in the original plane divided by the square of the ratio of linear magnification $h=\left|\frac{dw}{dz}\right|$ at the point P' (§ 43). Consequently a harmonic function of x, y is transformed into a harmonic function of u, v.

In like manner squaring the first derivatives and adding, we obtain

$$(4)\qquad \left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2=\left[\left(\frac{\partial V}{\partial u}\right)^2+\left(\frac{\partial V}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right].$$

Calling

$$\left(\frac{\partial V}{\partial u}\right)^2+\left(\frac{\partial V}{\partial v}\right)^2=h_V'^2,$$

we see that the square of the first differential parameter h_V possesses the same property with regard to the transformation.

Dividing equation (3) by (4) obtain

$$(5)\qquad \frac{\Delta V}{h_V^2}=\frac{\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}}{\left(\dfrac{\partial V}{\partial x}\right)^2+\left(\dfrac{\partial^2 V}{\partial y}\right)^2}=\frac{\dfrac{\partial^2 V}{\partial u^2}+\dfrac{\partial^2 V}{\partial v^2}}{\left(\dfrac{\partial V}{\partial u}\right)^2+\left(\dfrac{\partial V}{\partial v}\right)^2}=\frac{\Delta' V}{h_V'^2},$$

or the ratio of the second differential parameter of any function to the square of the first is unchanged by a conformal transformation. We may call such a quantity an *invariant* of the transformation.

We now require the condition that an equation $\phi(x, y) = C$ represents an equipotential family of curves. In this case we shall have for the potential function V, $V = f(\phi)$ and as in § 108, (2)

$$\text{(6)} \qquad \Delta V = \frac{dV}{d\phi}\Delta\phi + \frac{d^2V}{d\phi^2}h_\phi^2 = 0,$$

so that

$$\text{(7)} \qquad \frac{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}}{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2} = -\frac{\dfrac{d^2V}{d\phi^2}}{\dfrac{dV}{d\phi}} = -\frac{f''(\phi)}{f'(\phi)}.$$

The right-hand member depends on ϕ alone, consequently the left hand must also. Consequently in order that $\phi(x, y) = C$ shall represent an equipotential family the ratio of the second to the square of the first differential parameter of ϕ must be a function of ϕ alone. Let now $\phi(x, y) = C$ represent an equipotential family, and let $\Phi(u, v) = C'$ be the transformed family. Since by (5)

$$\frac{\Delta'\Phi}{h_\Phi'^2} = \frac{\Delta\phi}{h_\phi^2},$$

and since $\Delta\phi/h_\phi^2$ depends only on ϕ, $\Delta'\Phi/h_\Phi'^2$ will depend only on Φ, for ϕ and Φ are constant together.

Accordingly a conformal transformation leaves every equipotential family equipotential. It is upon this property that the application to electrostatical and other physical problems depends.

If we integrate the second parameter of V over a portion of the XY-plane where it does not vanish, using the element of area in curvilinear coordinates

$$dS = \frac{du\,dv}{h^2}.$$

$$\text{(8)} \qquad \iint\left(\frac{\partial^2V}{\partial x^2} + \frac{\partial^2V}{\partial y^2}\right)dx\,dy = \iint h^2\left(\frac{\partial^2V}{\partial u^2} + \frac{\partial^2V}{\partial v^2}\right)\frac{du\,dv}{h^2},$$

and now considering the second integral to refer to the transformed plane, and e and e' to be charges of corresponding regions,

$$\text{(9)} \qquad e = \iint\rho\,dx\,dy = -\frac{1}{4\pi}\iint\Delta V\,dx\,dy = -\frac{1}{4\pi}\iint\Delta'V\,du\,dv$$

$$= \iint\rho'\,du\,dv = e',$$

or corresponding regions in the two planes have equal charges (the densities being different).

If Ψ be the conjugate function to V, we have for the charge upon any conductor $V=C$ between points A and B,

$$e=\int_A^B \sigma ds = -\frac{1}{4\pi}\int_A^B \frac{\partial V}{\partial n_e} ds$$

$$= -\frac{1}{4\pi}\int_A^B \left\{\frac{\partial V}{\partial x}\cos(n_e x)+\frac{\partial V}{\partial y}\cos(n_e y)\right\} ds,$$

or since $\cos(nx)\,ds = dy$, $\cos(ny)\,ds = -dx$,

$$e=-\frac{1}{4\pi}\int_A^B\left(\frac{\partial V}{\partial x}dy-\frac{\partial V}{\partial y}dx\right)$$

$$=-\frac{1}{4\pi}\int_A^B\left(\frac{\partial \Psi}{\partial x}dx+\frac{\partial \Psi}{\partial y}dy\right)=\frac{1}{4\pi}(\Psi_A-\Psi_B),$$

so that the flux of force of any tube of force is measured by $1/4\pi$ times the difference of the values at its two sides of the conjugate function to the potential, as in § 103.

159. Examples. Eccentric Cylinders. Let us transform by means of the function $w=\log z$ giving (§ 45),

(1) $$u=\log r=\log\sqrt{x^2+y^2},\quad v=\phi=\tan^{-1}\frac{y}{x}.$$

A pair of parallel planes $u=V_1$, $u=V_2$, transforms into a pair of concentric circular cylinders $r=r_1$, $r=r_2$. To the potential function $V=u$ we have the conjugate function $\Psi=v$ so that for the charge of the cylinders we have between $\phi=0$ and 2π,

(2) $$e=\frac{1}{4\pi}\{\Psi_{2\pi}-\Psi_0\}=\frac{1}{4\pi}\cdot 2\pi=\frac{1}{2},$$

and the capacity is

(3) $$K=\frac{e}{\pm(V_2-V_1)}=\frac{e}{\pm\log\frac{r_2}{r_1}}=\pm 1/2\log\frac{r_2}{r_1},\text{ as in § 144.}$$

The use of the fractional linear function $w=(z+a)/(z-a)$, gives us an important new result. Replacing i in

(4) $$u+iv=\frac{x+iy+a}{x+iy-a}$$

by $-i$, as a reference to the theory of the complex variable shows is always possible, gives

(5) $$u-iv=\frac{x-iy+a}{x-iy-a},$$

and multiplying together,

$$(6) \qquad u^2 + v^2 = \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}.$$

We will now make use of the results of the preceding example, denoting, however, by our present u and v what were there denoted by x and y. The cylinders

$$u^2 + v^2 = r_1^2, \quad u^2 + v^2 = r_2^2$$

transform into

$$(7) \qquad \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = r_1^2, \quad \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = r_2^2,$$

which on clearing of fractions,

$$(8) \qquad \begin{aligned} (x^2+y^2)(1-r_1^2) + 2ax(1+r_1^2) + a^2(1-r_1^2) = 0, \\ (x^2+y^2)(1-r_2^2) + 2ax(1+r_2^2) + a^2(1-r_2^2) = 0, \end{aligned}$$

are seen to be eccentric circular cylinders. Their trace on the new XY-plane is shown in Fig. 65, which represents the transformation of the right-hand part of Fig. 24 by means of the function

$$w = \log \frac{z+a}{z-a}.$$

If we denote for either cylinder the radius by R and the distance of the center from the origin by d, since we may write (8)

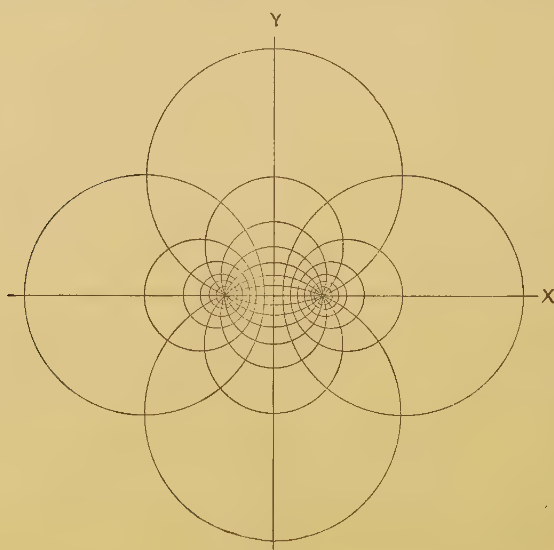

FIG. 65.

$$(9)\qquad x^2+y^2+2ax\frac{1+r^2}{1-r^2}+a^2\left(\frac{1+r^2}{1-r^2}\right)^2 = a^2\left\{\left(\frac{1+r^2}{1-r^2}\right)^2-1\right\}=\frac{4a^2r^2}{(1-r^2)^2},$$

we have

$$(10)\qquad d=a\frac{r^2+1}{r^2-1}=a\frac{r+1/r}{r-1/r},\quad R=\frac{\pm 2ar}{r^2-1}=\frac{\pm 2a}{r-1/r}.$$

from which

$$(11)\qquad r=\frac{d+\sqrt{d^2-R^2}}{\pm R}.$$

Since R and r must be positive, we take the upper signs in (10) and (11) for $r>1$, which makes $d>0$, and gives the circles on the right, the lower for $r<1$, which makes $d<0$, and gives the circles on the left.

Now making use of the results of the last example, we have for the functions V and Ψ

$$(12)\qquad \begin{aligned} V&=\log r=\log\sqrt{(u^2+v^2)},\\ \Psi&=\tan^{-1}\frac{v}{u},\end{aligned}$$

so that in the transformation, for $r=r_1$, $r=r_2$,

$$(13)\qquad \begin{aligned} V_1&=\log r_1=\log\frac{d_1+\sqrt{d_1^2-R_1^2}}{R_1},\\ V_2&=\log r_2=\log\frac{d_2+\sqrt{d_2^2-R_2^2}}{R_2},\end{aligned}$$

and the capacity of the pair of eccentric cylinders is

$$(14)\qquad K=\frac{\pm 1}{2\log\dfrac{(d_1+\sqrt{d_1^2-R_1^2})\,R_2}{(d_2+\sqrt{d_2^2-R_2^2})\,R_1}}.$$

In case $r_2=1$, d_2 and R_2 become infinite, and we have for the capacity of a single cylinder in presence of the infinite conducting plane $x=0$,

$$(15)\qquad K=\frac{1}{2\log\left(\dfrac{d+\sqrt{d^2-R^2}}{R}\right)}.$$

The formula given above for the capacity of a pair of cylinders of which one is internal to the other is not convenient in practice, since we are given not the distances d_1, d_2, but only their difference

$d = d_2 - d_1$, the distance apart of the lines of centers, together with the radii R_1, R_2. We must therefore solve the equations

$$(16) \qquad d_1 = a\frac{r_1^2+1}{r_1^2-1}, \qquad (17) \qquad d_2 = a\frac{r_2^2+1}{r_2^2-1},$$

$$(18) \qquad R_1 = 2a\frac{r_1}{r_1^2-1}, \qquad (19) \qquad R_2 = 2a\frac{r_2}{r_2^2-1},$$

$$(20) \qquad d_2 - d_1 = d,$$

so as to obtain r_1, r_2, a, d_1, d_2, in terms of R_1, R_2, d. We need for use only the ratio r_1/r_2. Eliminating d_1, d_2 from (20) by (16) and (18), (17) and (19),

$$(21) \qquad R_2\frac{r_2^2+1}{2r_2} - R_1\frac{r_1^2+1}{2r_1} = d,$$

and a by (18), (19),

$$(22) \qquad R_2\frac{r_2^2-1}{2r_2} - R_1\frac{r_1^2-1}{2r_1} = 0.$$

Taking the sum and difference of these two equations we obtain

$$(23) \qquad R_2 r_2 - R_1 r_1 = d,$$

$$(24) \qquad \frac{R_2}{r_2} - \frac{R_1}{r_1} = d.$$

Multiplying these equations together

$$R_2^2 + R_1^2 - R_1 R_2\left\{\frac{r_1}{r_2} + \frac{r_2}{r_1}\right\} = d^2,$$

or

$$(25) \qquad \frac{r_1}{r_2} + \frac{r_2}{r_1} = \frac{R_1^2 + R_2^2 - d^2}{R_1 R_2},$$

a quadratic for r_1/r_2 or r_2/r_1. Solving we obtain

$$(26) \qquad \frac{r_2}{r_1} = \frac{1}{2R_1R_2}\{R_1^2 + R_2^2 - d^2 \\ \pm\sqrt{[(R_1+R_2)^2 - d^2][(R_2-R_1)^2 - d^2]}\}.$$

It is easily seen that taking the square root with one sign makes the whole expression the reciprocal of its value with the other sign. Consequently we use the upper or lower sign according as r_2 is greater or less than r_1. The capacity is accordingly

$$(27) \quad K = \frac{1}{2\log\left\{\dfrac{R_1^2 + R_2^2 - d^2 \pm\sqrt{[(R_1+R_2)^2-d^2][(R_2-R_1)^2-d^2]}}{2R_1R_2}\right\}}.$$

which for $d = 0$ becomes $\pm 1/2\log(R_2/R_1)$ as in § 144.

If the two cylinders are external to each other, we must insert the minus sign on the right of (18), so that the equations are the same as before, with R_1 replaced by its negative. Accordingly we obtain

$$(28)\qquad \frac{r_2}{r_1} = \frac{1}{2R_1R_2}\{d^2 - (R_1^2 + R_2^2) \pm \sqrt{[(R_1+R_2)^2 - d^2]\,[(R_2-R_1)^2 - d^2]}\},$$

$$(29)\quad K = \frac{1}{2\log\left\{\dfrac{d^2-(R_1^2+R_2^2) \pm \sqrt{[d^2-(R_1+R_2)^2][d^2-(R_2-R_1)^2]}}{2R_1R_2}\right\}}.$$

The formulae (15), (27) and (29) are important in calculating the capacities of telegraph wires.

160. Elliptic and Hyperbolic Cylinders. In the preceding examples we were given a function of a complex variable, and from that obtained a conformal representation. We will now consider a case in which we are given a set of orthogonal curves, and we shall seek a function of a complex variable which will make them the conformal representation of orthogonal straight lines. The functions λ and μ defined by the equations

$$(1)\qquad \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \qquad \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} = 1,$$

are a pair of orthogonal coordinates. Solving for x and y we obtain

$$(2)\qquad x = \sqrt{\frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2}}, \qquad y = \sqrt{\frac{(b^2+\lambda)(b^2+\mu)}{b-a^2}},$$

and differentiating logarithmically

$$(3)\qquad \frac{dx}{x} = \frac{1}{2}\left(\frac{d\lambda}{a^2+\lambda} + \frac{d\mu}{a^2+\mu}\right), \qquad \frac{dy}{y} = \frac{1}{2}\left(\frac{d\lambda}{b^2+\lambda} + \frac{d\mu}{b^2+\mu}\right),$$

so that

$$(4)\qquad \begin{aligned} dx &= \frac{1}{2}\left\{d\lambda\sqrt{\frac{a^2+\mu}{(a^2+\lambda)(a^2-b^2)}}+d\mu\sqrt{\frac{a^2+\lambda}{(a^2+\mu)(a^2-b^2)}}\right\}, \\ dy &= \frac{1}{2}\left\{d\lambda\sqrt{\frac{b^2+\mu}{(b^2+\lambda)(b^2-a^2)}}+d\mu\sqrt{\frac{b^2+\lambda}{(b^2+\mu)(b^2-a^2)}}\right\}, \end{aligned}$$

$$(5)\qquad ds^2 = dx^2+dy^2 = \frac{1}{4}(\lambda-\mu)\left\{\frac{d\lambda^2}{(a^2+\lambda)(b^2+\lambda)} - \frac{d\mu^2}{(a^2+\mu)(b^2+\mu)}\right\}.$$

For a conformal relation we must have

$$ds^2 = (du^2+dv^2)/h^2.$$

Consequently if we put

$$(6)\qquad \begin{aligned} du &= \frac{d\lambda}{2\sqrt{(a^2+\lambda)(b^2+\lambda)}}, \\ dv &= -i\,\frac{d\mu}{2\sqrt{(a^2+\mu)(b^2+\mu)}}, \end{aligned}$$

the functions u, v will give us a conformal relation in which the straight lines $u=$ const., $v=$ const. in the UV-plane correspond to the ellipses and hyperbolas λ, μ, in the XY-plane.

Integrating the differential equations (6)

$$(7)\qquad \begin{aligned} u &= \log\{\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\}, \\ v &= \frac{1}{2}\cos^{-1}\left\{\frac{2\mu+a^2+b^2}{a^2-b^2}\right\}. \end{aligned}$$

Taking the antilogarithm and its reciprocal, of the first equation,

$$(8)\qquad \begin{aligned} e^u &= \sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}, \\ e^{-u} &= \frac{\sqrt{a^2+\lambda}-\sqrt{b^2+\lambda}}{a^2-b^2}. \end{aligned}$$

Solving these for $\sqrt{a^2+\lambda}$ and $\sqrt{b^2+\lambda}$,

$$(9)\qquad \begin{aligned} \sqrt{a^2+\lambda} &= \tfrac{1}{2}\{e^u+(a^2-b^2)e^{-u}\}, \\ \sqrt{b^2+\lambda} &= \tfrac{1}{2}\{e^u-(a^2-b^2)e^{-u}\}. \end{aligned}$$

From the integral for v we get

$$(10)\qquad \begin{aligned} \frac{1+\cos 2v}{2} &= \cos^2 v = \frac{a^2+\mu}{a^2-b^2}, \\ \frac{1-\cos 2v}{2} &= \sin^2 v = \frac{b^2+\mu}{b^2-a^2}, \end{aligned}$$

which give

$$\sqrt{a^2+\mu} = \cos v \,.\, \sqrt{a^2-b^2}, \quad \sqrt{b^2+\mu} = \sin v \,.\, \sqrt{b^2-a^2}.$$

Inserting the values of the four square roots in the values of x and y in (2)

$$\text{(11)} \qquad \begin{aligned} x &= \tfrac{1}{2}\{e^u + (a^2-b^2)\, e^{-u}\} \cos v, \\ y &= \tfrac{1}{2}\{e^u - (a^2-b^2)\, e^{-u}\} \sin v, \end{aligned}$$

$$\text{(12)} \qquad \begin{aligned} x + iy &= \tfrac{1}{2}[e^u(\cos v + i \sin v) + (a^2-b^2)\, e^{-u}(\cos v - i \sin v)] \\ &= \tfrac{1}{2}[e^{u+iv} + (a^2-b^2)\, e^{-(u+iv)}], \end{aligned}$$

which gives the form of the function sought,

$$\text{(13)} \qquad z = \tfrac{1}{2}\{e^w + (a^2-b^2)\, e^{-w}\},$$

or

$$\text{(14)} \qquad w = \log\{z \pm \sqrt{z^2-(a^2-b^2)}\}.$$

We may now conveniently change our unit so that the focal distance $\sqrt{a^2-b^2}$ shall equal unity. Then the function z becomes the hyperbolic cosine of w. A table of comparison of the principal properties of the hyperbolic and circular functions is appended*.

* 160A. **Hyperbolic and Circular Functions.**

$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$ $\qquad$ $\sinh x = \tfrac{1}{2}(e^x - e^{-x})$

$\cosh^{-1} x = \log(x \pm \sqrt{x^2-1})$ $\qquad$ $\sinh^{-1} x = \log(x \pm \sqrt{x^2+1})$

$\tanh x = \sinh x / \cosh x$

$\operatorname{sech} x = 1/\cosh x.$

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(1) $\sinh(-x) = -\sinh x.$

(2) $\cosh(-x) = \cosh x.$

(3) $\cosh^2 x - \sinh^2 x = 1.$

(4) $1 - \tanh^2 x = \operatorname{sech}^2 x.$

(5) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$

(6) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$

(7) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$

(8) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}.$

(9) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}.$

(10) $\dfrac{d}{dx} \sinh x = \cosh x.$

(1′) $\sin(-x) = -\sin x.$

(2′) $\cos(-x) = \cos x.$

(3′) $\cos^2 x + \sin^2 x = 1.$

(4′) $1 + \tan^2 x = \sec^2 x.$

(5′) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$

(6′) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$

(7′) $\tan(x \pm y) \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$

(8′) $\sin^2 x = \dfrac{1 - \cos 2x}{2}.$

(9′) $\cos^2 x = \dfrac{1 + \cos 2x}{2}.$

(10′) $\dfrac{d}{dx} \sin x = \cos x.$

By means of it we may show that the four functions

$$\cosh z, \quad \sinh z, \quad \cos z, \quad \sin z$$

give conformal representations which are identical except for interchanges of the axes. By means of the equations § 160A, (15), (16), (15′), (16′) we obtain the pairs of functions u, v for the four cases, and by the use of equations (3′), (3) after division by one of the factors on the right in the values of u, v we get

I. $w = \cosh z$,

$$u = \cosh x \cos y, \qquad v = \sinh x \sin y;$$

$$(a) \quad \frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1, \qquad (b) \quad \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1.$$

II. $w = \sinh z$,

$$u = \sinh x \cos y, \qquad v = \cosh x \sin y;$$

$$(a) \quad \frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1, \qquad (b) \quad \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = -1.$$

III. $w = \cos z$,

$$u = \cos x \cosh y, \qquad v = -\sin x \sinh y;$$

$$(a) \quad \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1, \qquad (b) \quad \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1.$$

IV. $w = \sin z$,

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y;$$

$$(a) \quad \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1, \qquad (b) \quad \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1.$$

(11) $\dfrac{d}{dx} \cosh x = \sinh x.$

(12) $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x.$

(13) $\sinh (ix) = i \sin x.$

(14) $\cosh (ix) = \cos x.$

(15) $\sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y.$

(16) $\cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y.$

(17) $\tanh (x + iy) = \dfrac{\tanh x + i \tan y}{1 + i \tanh x \tan y} = \dfrac{\sinh x \cosh x + i \sin y \cos y}{\cos^2 y \cosh^2 x + \sin^2 y \sinh^2 x}.$

(11′) $\dfrac{d}{dx} \cos x = -\sin x.$

(12′) $\dfrac{d}{dx} \tan x = \sec^2 x.$

(13′) $\sin (ix) = i \sinh x.$

(14′) $\cos (ix) = \cosh x.$

(15′) $\sin (x + iy) = \sin x \cosh y + i \cos x \sinh y.$

(16′) $\cos (x + iy) = \cos x \cosh y - i \sin x \sinh y.$

(17′) $\tan (x + iy) = \dfrac{\tan x + i \tanh y}{1 - \tan x \tanh y} = \dfrac{\sin x \cos x + i \sinh y \cosh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}.$

We see that the straight lines $x = \text{const.}$ correspond to confocal ellipses, in (I) and (II), while the lines $y = \text{const.}$ correspond to confocal hyperbolas, while in (III) and (IV), the lines $y = \text{const.}$ correspond to ellipses and $x = \text{const.}$ to hyperbolas. The geometrical character of all four transformations being therefore the same, we shall consider only case I, Fig. 66. (It is to be noticed that we have interchanged z and w in (13).) To any line $x = \text{const.}$ corresponds the ellipse whose semi-axes are $\cosh x$ and $\sinh x$. When $x = 0$ the ellipse reduces to the

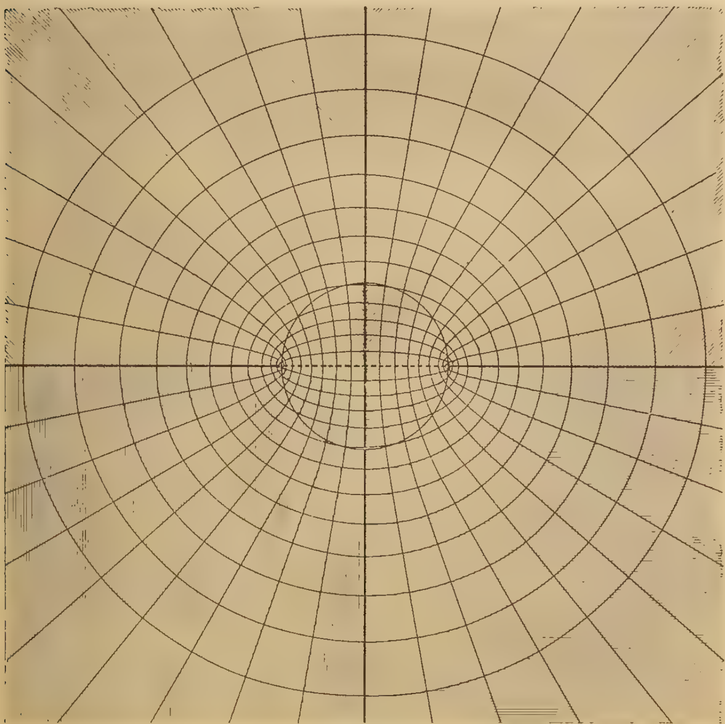

FIG. 66.

straight line between the focal points $u = 1$, $v = 0$ and $u = -1$, $v = 0$. As x increases the ellipses become continually larger, until for x infinite they become infinite circles. For any negative x we get the same ellipse as for the corresponding positive. The lines $y = \text{const.}$ correspond to the hyperbolas whose semi-axes are $\cos y$ and $\sin y$. When $y = 0$ the hyperbola reduces to those portions of

the U-axis outside the focal points and as y increases the hyperbolas become continually less sharp, until when $y = \pi/2$ the hyperbola reduces to the V-axis. As y goes on increasing we obtain the same hyperbolas in inverse order. Accordingly the whole of the UV-plane is represented by the portion of the XY-plane lying to the right of the Y-axis and between the lines $y = 0$ and $y = 2\pi$. Other regions of the XY-plane of similar dimensions correspond repeatedly to the UV-plane. The point $x = 0$, $y = 0$ corresponds to the two focal points. We may get a good idea of the correspondence of the two planes by describing any path composed of portions of horizontal and vertical lines in the XY-plane, and noticing that when we turn to the right or left through a right angle in the XY-plane we turn in the same direction in the UV-plane. Such a path is represented in Figs. 66 a and b, where the correspondence of the various regions is indicated by the shading.

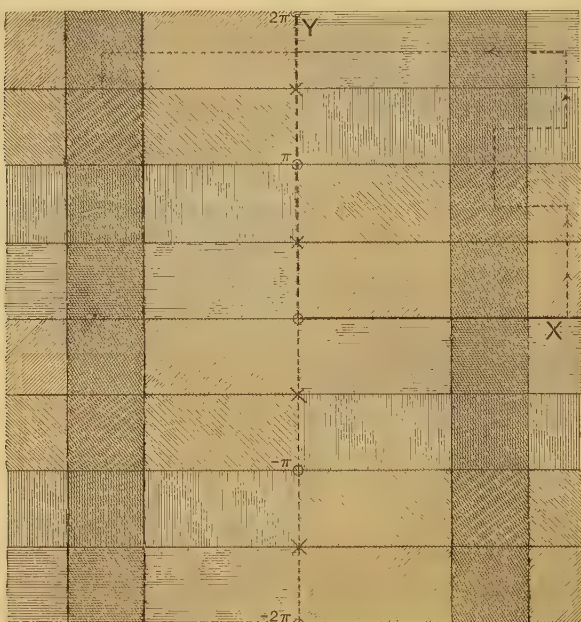

FIG. 66 a. XY-plane.

The whole of the space between two ellipses corresponds to a rectangle of altitude 2π in the right-hand upper quadrant of the XY-plane.

As we have interchanged z and w since equation (13) we must interchange x, y with u, v throughout, so that instead of (7) we use, putting $a^2 - b^2 = 1$,

$$x = \log\{\sqrt{a^2+\lambda} + \sqrt{a^2+\lambda-1}\}.$$

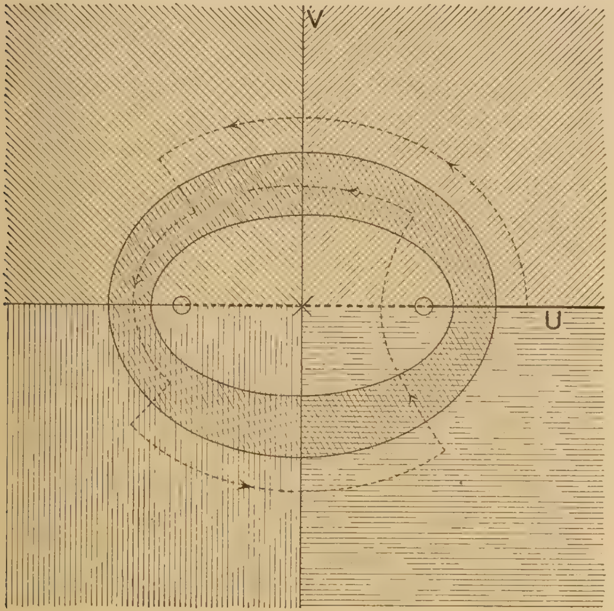

Fig. 66 *b*. *UV*-plane.

Now choosing for the functions V and Ψ, $V = x$, $\Psi = y$, we obtain on the cylinders whose semi-axes are $a_1 = \sqrt{a^2+\lambda_1}$ and $a_2 = \sqrt{a^2+\lambda_2}$ the whole charge $e = \frac{1}{2}$, with the potentials

$$V_1 = \log(a_1 + \sqrt{a_1^2 - 1}), \quad V_2 = \log(a_2 + \sqrt{a_2^2 - 1}).$$

The capacity of the pair of cylinders is accordingly

$$K = \pm \frac{1}{2\log\dfrac{a_2 + \sqrt{a_2^2 - 1}}{a_1 + \sqrt{a_1^2 - 1}}}.$$

By interchanging the functions V and Ψ we may find the distribution on the hyperbolic cylinders, and in particular by putting $y = 0$, $y = \pi/2$ we may find the distribution on the edge of an infinite plate in presence of a second infinite plate at right angles with it beyond the edge.

161. Logarithmic Transformation of last case. If we transform the last figure by means of the function $w' = \log w$, we obtain the whole of the UV-plane represented on a strip of the $U'V'$-plane of width 2π, so that the transformation

$$(1) \qquad w' = \log \cosh z$$

transforms the right-hand half of a horizontal strip of width 2π in the XY-plane into the whole of the same strip in the $U'V'$-plane. Taking the antilogarithms we have

$$(2) \qquad e^{w'} = \cosh z = \cosh x \cos y + i \sinh x \sin y,$$

that is,

$$(3) \qquad e^{u'}(\cos v' + i \sin v') = \cosh x \cos y + i \sinh x \sin y,$$

so that

$$(4) \qquad e^{u'} \cos v' = \cosh x \cos y, \quad e^{u'} \sin v' = \sinh x \sin y,$$

from which, in the same manner as above,

$$(5) \qquad \frac{\cos^2 v'}{\cosh^2 x} + \frac{\sin^2 v'}{\sinh^2 x} = \frac{1}{e^{2u'}}, \quad \frac{\cos^2 v'}{\cos^2 y} - \frac{\sin^2 v}{\sin^2 y} = \frac{1}{e^{2u'}},$$

and taking logarithms,

$$(6) \qquad \begin{aligned} u' &= -\frac{1}{2} \log \left\{ \frac{\cos^2 v'}{\cosh^2 x} + \frac{\sin^2 v'}{\sinh^2 x} \right\}, \\ u' &= -\frac{1}{2} \log \left\{ \frac{\cos^2 v'}{\cos^2 y} - \frac{\sin^2 v'}{\sin^2 y} \right\}. \end{aligned}$$

From these equations the curves corresponding to $x = \text{const.}$ and $y = \text{const.}$ may be immediately plotted by the aid of tables of logarithms and hyperbolic functions. They are shown in Fig. 67. It is at once seen that u' is a periodic function of v', the period being π. The figure is the same for negative x and y as for positive. In order to represent the whole of the UV-plane corresponding to the half strip in the XY-plane, we must however let v' vary from 0 to 2π. The curves $x = \text{const.}$ are sinuous curves, u' having maxima for $v' = 0, \pi, 2\pi, \ldots$ and minima for $v = \pi/2, 3\pi/2, \ldots$. The maxima $u' = \log \cosh x$ and minima $u' = \log \sinh x$ differ but little for large values of x, since then approximately $\cosh x = \sinh x = e^x/2$ so that we may then take out this factor from u', obtaining $u' = x - \log 2$ for all values of v, so that the curves $x = \text{const.}$ are nearly straight lines.

As x diminishes the maxima and minima both diminish, but get farther apart, the maxima being always positive, while the

minima eventually become negative. The curves all cut the axis of u' to the right of the origin, but stretch out farther and farther

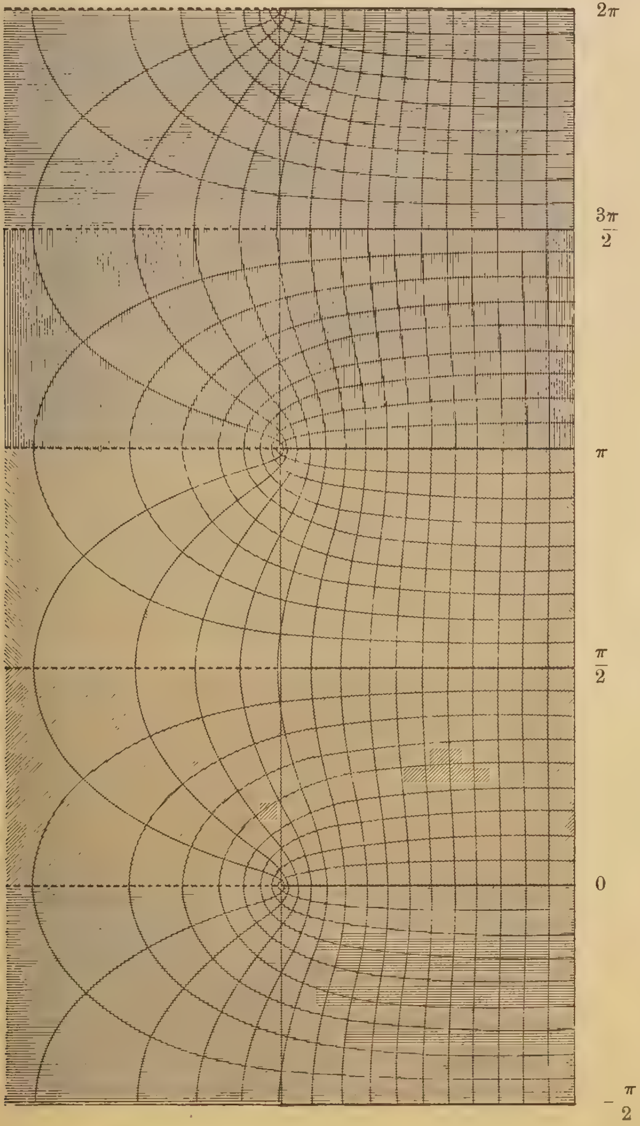

FIG. 67. $U'V'$-plane.

toward the left, so that for $x=0$ the curve reaches from 0 to minus infinity, coinciding with the left-hand half of the u' axis.

In fact we see from the equation that for any finite u', v' must be zero when x is zero.

The curves $y = \text{const.}$ are different in appearance, on account of the minus sign. u' has minima for $v' = 0, \pi, 2\pi\ldots$ having the values $u' = \log\cos y$ which are all negative, and decrease more and more rapidly as y increases to $\pi/2$. The maxima of u' are however infinite. In fact while u' increases continuously as v' varies from 0 to $\pi/2$, as soon as $v' > y$ the parenthesis becomes negative and u' is imaginary. The curves $y = \text{const.}$ accordingly approach horizontal asymptotes $v' = y$. These curves correspond to the hyperbolas of the last figure, the sinuous curves corresponding to the ellipses. Corresponding regions in the three figures are similarly shaded. The circle in Fig. 66 corresponds to the vertical V'-axis in Fig. 67.

If we choose for the functions V and Ψ the values $V = y$, $\Psi = x$, and consider the strip between $v' = \pi/2$ and $v' = -\pi/2$, we have the case of the electrification of an infinite plane with a free edge, lying between two infinite planes parallel with it at distances $\pi/2$ from it, and extending to infinity on all sides. Since at a distance from the edge x is equal to $u' + \log 2$, the field is straight, but the charge from the edge to the point u' is greater by $(\log 2)/4\pi$ than if the plate extended to infinity instead of stopping at the edge. Thus the edge increases the capacity of the upper or lower side of a portion of the plate of any width by the amount $K = (\log 2)/4\pi\,(V_2 - V_1) = (\log 2)/2\pi^2$. This result may be used to find the capacity of a circular plate between two infinite parallel plates at a distance from it d so small that the edge of the circular disc may be considered straight*. The effect of the edge is the same as that of increasing the radius by $(\log 2)\,d/(\pi/2)$, so that the capacity would be, counting both sides,

$$K = \frac{S + 2\pi R\,.\,2d\,(\log 2)/\pi}{2\pi d} = \frac{R^2}{2d} + \frac{2R\log 2}{\pi}.$$

* Maxwell, *Treatise*, Vol. I. Art. 196.

CHAPTER VIII.

ELECTROKINETICS. STEADY FLOW IN CONDUCTORS.

162. Ohm's Law. The condition of equilibrium of electricity in homogeneous conductors has been found to be that in each conductor the potential has a constant value. If this condition is not fulfilled in any conductor, the electrification changes with the time, if the conductor be left to itself, or in ordinary terms electricity moves from one place to another in the conductor. The laws of this flow of electricity were enunciated in 1827 by Georg Simon Ohm*, although the notion of the potential was unknown to him. If at any point in the conductor we construct an element of surface dS, the quantity of electricity q crossing the surface in the unit of time per unit of area will vary according to the direction of the normal to dS at the point. That direction of normal for which the quantity per unit of time is greatest is called the direction of the *current* at the point, and the quantity q is called the current density. The current density is a vector quantity, and its components according to the axes will be denoted by u, v, w. If the quantities u, v, w are independent of the time, we call the state of the conductor a state of *steady flow*. We shall now consider the properties of the steady state. If we consider any portion of a conductor in which there is no electricity created nor destroyed, as much electricity must enter the space during any interval as leaves it, or the whole flow resolved along the inward normal must be zero. Accordingly

$$(1) \qquad 0 = \iint q \cos(qn)\, dS = \iint \{u \cos(nx) + v \cos(ny) + w \cos(nz)\}\, dS$$
$$= -\iiint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) d\tau,$$

* G. S. Ohm, *Die galvanische Kette mathematisch bearbeitet.* Berlin, 1827.

and as this must be true for any portion of space fulfilling the above conditions, we must have everywhere in such regions

$$(2) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

This is called the equation of continuity, and shows that the current density is a solenoidal vector. Current lines and tubes accordingly possess properties similar to those of tubes of force in the case of equilibrium.

The law of Ohm is identical with that stated by Fourier* for the conduction of heat, and connects the current density with the potential, the corresponding quantity for heat being the temperature. If the conductor be isotropic, that is if its properties are at each point the same for all directions, the direction of the current is the same as that of the electrostatic field, and their magnitudes are proportional, the factor of proportionality depending on the physical properties of the conductor at each point. If n is the normal to an equipotential surface at the point in question drawn in the direction of the current, we have

$$(3) \qquad q = \lambda F = -\lambda \frac{\partial V}{\partial n}$$

as the mathematical statement of Ohm's Law. The factor of proportionality λ is called the *conductivity*, and its reciprocal the specific resistance, or *resistivity* of the conductor. If λ is the same at all points of the conductor, the conductor is said to be homogeneous, if λ is variable, the conductor is heterogeneous.

The above equation is equivalent to the three

$$(4) \qquad u = -\lambda \frac{\partial V}{\partial x}, \quad v = -\lambda \frac{\partial V}{\partial y}, \quad w = -\lambda \frac{\partial V}{\partial z}.$$

Inserting these in the equation of continuity,

$$(5) \qquad \frac{\partial}{\partial x}\left(\lambda \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial V}{\partial z}\right) = 0.$$

If the conductor is homogeneous, this becomes $\Delta V = 0$. Hence the density ρ is zero, or there is no free electricity in any portion of a homogeneous conductor in the state of steady flow†.

* Fourier, *Théorie analytique de la chaleur*, 1822.

† Kirchhoff. "Ueber eine Ableitung der Ohm'schen Gesetze, welche sich an die Theorie der Elektrostatik anschliesst." *Pogg. Ann.*, Bd. 78, 1849. *Ges. Abh.*, p. 49.

163. Boundary Condition. Refraction of Lines of Flow. In passing from one homogeneous conductor to another, λ may be discontinuous, and since the current must be continuous, we must have at the surface

$$q_1 \cos(q_1 n_1) + q_2 \cos(q_2 n_2) = 0,$$

(6) $$\lambda_1 F_1 \cos(F_1 n_1) + \lambda_2 F_2 \cos(F_2 n_2) = 0,$$

or $$\lambda_1 \frac{\partial V}{\partial n_1} + \lambda_2 \frac{\partial V}{\partial n_2} = 0.$$

The boundary condition (6) has a simple geometrical meaning. Since the derivatives of V are discontinuous only on crossing the surface, we have the derivatives in any direction t tangent to the surface, $\partial V/\partial t$ the same on both sides of the surface. If θ_1 be the acute angle made by the current line with the normal on one side of the surface, θ_2 the acute angle on the other, resolving along the normal,

(7) $$\lambda_1 F_1 \cos\theta_1 = \lambda_2 F_2 \cos\theta_2.$$

Resolving along the tangent plane, since this component is continuous,

(8) $$F_1 \sin\theta_1 = F_2 \sin\theta_2.$$

Dividing the second of these equations by the first, we obtain

(9) $$\frac{\tan\theta_1}{\lambda_1} = \frac{\tan\theta_2}{\lambda_2},$$

or the line of flow is refracted on passing the surface, so that the tangents of the angles of incidence and refraction are in the ratio λ_1/λ_2 dependent only on the media. The law of refraction is different from the optical law, in which we have the sine instead of the tangent, and in the case of the tangent law we do not have the phenomenon of total reflection, since the tangent takes all values from zero to infinity.

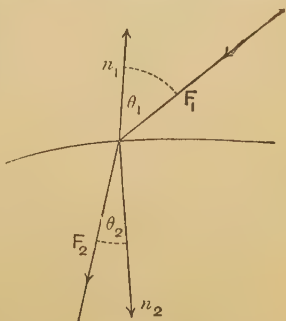

FIG. 67*a*.

164. Systems of Conductors. All the statements heretofore made are true for the flow of heat, if V represent the temperature, but whereas in the case of heat in passing from one conductor to another the temperature

is continuous, in the case of electricity, in passing from the conductor 1 to the conductor 2, we have at the surface of separation

$$V_2 - V_1 = E_{12}, \tag{10}$$

where E_{12} is a quantity depending on the nature of the two conducting substances.

In the theory of heat, if we have a chain of conductors in contact with each other, surrounded by a non-conductor, we may have equilibrium, but in the case of electricity this is the case only if the sum of the discontinuities of potential is zero,

$$E = E_{12} + E_{23} + \ldots + E_{n1} = 0. \tag{11}$$

Conductors may be divided into two classes. Those of such a nature that for any number of them an equation of this sort holds constitute the first class. To it belong all metals (their temperatures being the same). To the second class, for which in general such equations do not hold, belong solutions of salts and dilute acids.

If we have a set of conductors of either class, the constants $E_{r,\, r+1}$ being given, and also the conductivity λ as a point-function, we shall show that the problem of flow is determined as soon as we are given any two equipotential surfaces.

Let V_A be the potential at one of the surfaces A, V_B that at the other B.

Let Φ be a function holomorphic in the whole space occupied by the conductors, satisfying the differential equation

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial \Phi}{\partial z}\right) = 0, \tag{12}$$

and the boundary condition

$$\lambda_1 \frac{\partial \Phi}{\partial n_1} + \lambda_2 \frac{\partial \Phi}{\partial n_2} = 0, \tag{13}$$

at surfaces of separation of two conductors, taking the value unity for all points of the surface A, and the value zero for all points of the surface B, while $\partial\Phi/\partial n = 0$ for all points of surfaces separating the conductors from the surrounding insulators, or at infinity, if the conductors reach so far. Then if v_1 be the potential function in the conductor 1 (in which lies the surface A), v_2 that in the conductor 2, ... v_n that in the conductor n (in which

lies the surface B), we may show that in the different conductors the potential is given by the functions

$$
\begin{aligned}
v_1 \;\; &= (E + V_A - V_B)\,\Phi + V_B - E,\\
v_2 \;\; &= (E + V_A - V_B)\,\Phi + V_B - E + E_{12},\\
v_3 \;\; &= (E + V_A - V_B)\,\Phi + V_B - E + E_{12} + E_{23},\\
(14) \qquad & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
v_{n-1} &= (E + V_A - V_B)\,\Phi + V_B - E + E_{12} + E_{23} \ldots E_{n-2,\, n-1},\\
v_n \;\; &= (E + V_A - V_B)\,\Phi + V_B.
\end{aligned}
$$

For since the function Φ satisfies the differential equation that is satisfied by the potential, any v_s, which is a linear function of Φ, must also satisfy the same equation. Also at any surface separating the conductors r and $r+1$,

$$v_{r+1} - v_r = E_{r,\, r+1},$$

and

$$\lambda_r \frac{\partial v_r}{\partial n_r} + \lambda_{r+1} \frac{\partial v_{r+1}}{\partial n_{r+1}} = 0,$$

from the definition of the function Φ. At the insulating boundary of any conductor

$$\frac{\partial v}{\partial n} = (E + V_A - V_B)\frac{\partial \Phi}{\partial n} = 0,$$

that is, there is no flow across the boundary. The function v_1 takes at the surface A the value V_A, and the function v_n at the surface B the value V_B. But these are all the conditions satisfied by the potential function. It remains to show that the function Φ is uniquely determined by the conditions that have been imposed upon it. The problem of finding the function Φ is of the same nature as Dirichlet's problem, differing from it in that while the values of Φ are given over part of the bounding surface, over the remainder instead of Φ the values of $\partial\Phi/\partial n$ are given.

Suppose that there are two functions Φ both satisfying the conditions of definition. Let them be denoted by Φ_1 and Φ_2. Then let us form the integral taken throughout the conductors considered

$$(15) \quad J = \iiint \lambda \left\{ \left(\frac{\partial(\Phi_1 - \Phi_2)}{\partial x}\right)^2 + \left(\frac{\partial(\Phi_1 - \Phi_2)}{\partial y}\right)^2 + \left(\frac{\partial(\Phi_1 - \Phi_2)}{\partial z}\right)^2 \right\} d\tau.$$

By Green's theorem this is equal to

$$(16)\qquad J = -\iint\left\{(\Phi_1 - \Phi_2)\,\lambda\frac{\partial(\Phi_1 - \Phi_2)}{\partial n}\right\} dS$$
$$-\iiint(\Phi_1 - \Phi_2)\left\{\frac{\partial}{\partial x}\left(\lambda\frac{\partial(\Phi_1 - \Phi_2)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial(\Phi_1 - \Phi_2)}{\partial y}\right)\right.$$
$$\left. + \frac{\partial}{\partial z}\left(\lambda\frac{\partial(\Phi_1 - \Phi_2)}{\partial z}\right)\right\} d\tau.$$

The surface integral is taken over the surfaces A and B, and the surfaces bounding the composite conductor, the integrals over the surfaces separating two conductors vanishing in virtue of (13).

But at the surface A, Φ_1 and Φ_2 are both equal to 1, hence

$$\Phi_1 - \Phi_2 = 0,$$

and at the surface B, Φ_1 and Φ_2 are equal to 0, while on the remaining surfaces $\partial\Phi_1/\partial n = \partial\Phi_2/\partial n = 0$. Consequently the surface integrals vanish. But the integrand in the volume integral vanishes in virtue of the differential equation satisfied by both functions. Consequently the integral J vanishes, but as in Dirichlet's demonstration this can only be if $\Phi_1 - \Phi_2$ is constant. But since Φ_1 and Φ_2 are equal on the surfaces A and B, they must be everywhere equal. Consequently the solution is unique.

165. Properties of Vectors obeying Fourier-Ohm Law. The vectors F, the electrostatic force, and q the electric current-density are typical of a class of pairs of vector-functions of frequent occurrence in all parts of mathematical physics, distinguished by the following properties. The first vector is lamellar, the second is solenoidal. In isotropic bodies the vectors have the same direction, and their ratio depends only on the physical nature of the body at each point. When two vector-functions have these properties we shall say that they satisfy the law of Fourier-Ohm. The study of the properties of such vectors is of great importance. We shall in general call the solenoidal vector the *flux*-density, and the surface integral of its normal component over any surface the flux through that surface.

It is remarkable that the characteristic properties of such vector-functions are embodied in the single statement that if V, the potential function of the lamellar vector, is uniform, finite, and continuous, in a certain region τ, its first derivatives possessing

the same properties with the possible exception of certain surfaces Σ at which they may be discontinuous, then if the values of V are given on parts of the surface S bounding the region τ, and the value of $\partial V/\partial n$ is zero on the remainder, the integral J throughout the region τ,

$$(1)\qquad J(V)=\iiint\lambda\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}d\tau,$$

is a *minimum** for that function V which makes the vector q solenoidal, where

$$(2)\qquad u=q\cos(qx)=\lambda\frac{\partial V}{\partial x},\quad v=q\cos(qy)=\lambda\frac{\partial V}{\partial y},$$
$$w=q\cos(qz)=\lambda\frac{\partial V}{\partial z}.$$

For if we change the form of the function V by the arbitrary amount δV,

$$J+\delta J=J(V+\delta V)$$
$$=\iiint\lambda\left\{\left(\frac{\partial(V+\delta V)}{\partial x}\right)^2+\left(\frac{\partial(V+\delta V)}{\partial y}\right)^2+\left(\frac{\partial(V+\delta V)}{\partial z}\right)^2\right\}d\tau$$
$$(3)\qquad =J(V)+2\iiint\lambda\left\{\frac{\partial V}{\partial x}\frac{\partial\delta V}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial\delta V}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial\delta V}{\partial z}\right\}d\tau$$
$$+\iiint\lambda\left\{\left(\frac{\partial\delta V}{\partial x}\right)^2+\left(\frac{\partial\delta V}{\partial y}\right)^2+\left(\frac{\partial\delta V}{\partial z}\right)^2\right\}d\tau.$$

The integral with the coefficient 2 is equal, by Green's theorem, to

$$-\iint_S\delta V.\lambda\frac{\partial V}{\partial n}dS-\iint_\Sigma\delta V.\left\{\lambda_1\frac{\partial V}{\partial n_1}+\lambda_2\frac{\partial V}{\partial n_2}\right\}d\Sigma$$
$$-\iiint\delta V\left\{\frac{\partial}{\partial x}\left(\lambda\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial V}{\partial z}\right)\right\}d\tau,$$

where n_1 and n_2 are the normals on opposite sides of a surface Σ of discontinuity of the derivatives. On those portions of the bounding surfaces for which V is given $\delta V=0$, and for the re-

* Kirchhoff, *Ges. Abh.* p. 44.

maining parts $\lambda \partial V/\partial n = 0$. Consequently the integrals over the bounding surfaces disappear, and we have

$$(4) \quad \delta J = -2\iint_{\Sigma} \delta V \left\{ \lambda_1 \frac{\partial V}{\partial n_1} + \lambda_2 \frac{\partial V}{\partial n_2} \right\} d\Sigma$$

$$- \iiint \delta V \left\{ \frac{\partial}{\partial x}\left(\lambda \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial V}{\partial z}\right) \right\} d\tau$$

$$+ \iiint \lambda \left\{ \left(\frac{\partial \delta V}{\partial x}\right)^2 + \left(\frac{\partial \delta V}{\partial y}\right)^2 + \left(\frac{\partial \delta V}{\partial z}\right)^2 \right\} d\tau.$$

In order for $J(V)$ to be a minimum, this must be positive for all possible choices of the arbitrary function δV. This can be true only if we have everywhere in the region τ

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial V}{\partial z}\right) = 0,$$

and at every surface of discontinuity Σ,

$$\lambda_1 \frac{\partial V}{\partial n_1} + \lambda_2 \frac{\partial V}{\partial n_2} = 0.$$

Consequently the statement that J is a minimum is equivalent to stating that q is solenoidal.

166. Integral form of Ohm's Law. We have seen in § 35 that the solenoidal condition signifies that the flux,

$$I = \iint q \cos(qn)\, dS$$

across any surface bounded by the sides of a vector tube is the same for all parts of the tube. In the case of electrical flow, the flux is called the *current* (current-strength, or intensity) in the tube. Although V has discontinuities, the function Φ has not. Since we have between any equipotential surfaces A and B,

$$I = -\iint \lambda \frac{\partial V}{\partial n} dS = -(E + V_A - V_B)\iint \lambda \frac{\partial \Phi}{\partial n} dS,$$

the ratio of the flux to $E + V_A - V_B$, the difference of potential plus the sum of the sudden rises of potential as we go in the direction of flow, thus depends only on the function Φ, which depends only on the configuration of the space τ, and the values of the function λ. That is, the ratio of the flux in any tube of the

vector q to E plus the difference of potential between two equipotential surfaces depends only on the physical properties of the substance in the tube. This is the usual form of the statement of Ohm's Law, and is the integral form, whereas our previous statement was the differential form. In the case of electrical flow, the difference of potential $V_A - V_B$ is called the external or electrostatic *electromotive force* from A to B, and it is evidently the line integral of electrostatic force along any line from A to B. E is called the impressed, intrinsic, or internal electromotive force. The ratio C of current to total electromotive force is called the *conductance* of the tube. Its reciprocal R is called the *resistance* of the tube.

If we consider a closed tube of flow, the two surfaces A and B will coincide, and we shall have the ordinary expression of Ohm's Law,

$$I = CE = \frac{E}{R},$$

or:—For any closed tube of flow, the current is equal to the impressed electromotive force divided by the resistance of the tube.

167. Heat developed in Conductors. We shall now consider the physical meaning of the integral J in the case of electrical flow. In passing from a point where the potential is V_A to one where it is V_B a unit of electricity does $V_A - V_B$ units of work, and that quantity of electrostatic energy thus disappears. Also at every surface of discontinuity, $E_{r,\,r+1}$ units of work must be done *upon* it. But if we consider heat as a form of energy, if mechanical energy disappears, an equivalent amount of heat must make its appearance. If accordingly we find energy appearing in no other form, the electrostatic energy W that disappears, together with the work done by the impressed electromotive forces, must be converted into heat. In the case of steady flow we find this to be the case.

In unit time the quantity

$$I = \iint q \cos(qn)\, dS$$

crosses any section of a tube of flow, so that considering that part of the conductor between the equipotential surfaces A and B we

have I units entering at potential V_A and emerging at potential V_B. The energy converted into heat in that portion of the conductor will accordingly be

$$(1) \qquad -\frac{dW}{dt} + EI = H = (E + V_A - V_B) I = \\ -(E + V_A - V_B) \iint \lambda \frac{\partial V}{\partial n} dS.$$

But transforming the integral, § 165 (1), by Green's theorem, and taking the normal at A, B, and the surfaces of discontinuity always in the direction of the current,

$$(2) \qquad J = -\iint_A V\lambda \frac{\partial V}{\partial n} dS + \Sigma_r \iint \left(V_r \lambda_r \frac{\partial V}{\partial n} - V_{r+1} \lambda_{r+1} \frac{\partial V}{\partial n} \right) dS \\ + \iint_B V\lambda \frac{\partial V}{\partial n} dS \\ - \iiint V \left\{ \frac{\partial}{\partial x}\left(\lambda \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial V}{\partial z} \right) \right\} d\tau.$$

The volume integral vanishes by the equation § 162 (5), and in virtue of the surface conditions § 164 (10) and (13),

$$(3) \qquad J = -(E + V_A - V_B) \iint \lambda \frac{\partial V}{\partial n} dS = H.$$

The integral which is a minimum in the actual distribution of current accordingly represents the heat generated in the conductor in the unit of time.

The equation (1), written

$$(4) \qquad EI = \frac{dW}{dt} + H,$$

is the equation of activity for steady currents. It may serve for a definition of the magnitude of an impressed electromotive force, as the rate at which energy is taken into the system per unit of current in its direction. Combining with (1) the equation of Ohm's Law,

$$(5) \qquad RI = E + V_A - V_B,$$

we have

$$(6) \qquad H = RI^2.$$

This is the equation of Joule's Law*:—

The heat developed in any portion of conductor in unit time is equal to the resistance of that portion of the conductor multiplied by the square of the current traversing it. This law is universally true, whether the flow is steady or not.

168. Sources of Electromotive Force. Suppose we have a closed circuit of a number of different conductors. We have already seen that if all are of the first class there can be no current. Suppose that one only is of the second class, and let its suffix be 1. Then for all the others we have

$$E_{23} + E_{34} \ldots\ldots + E_{n-1,\,n} + E_{n2} = 0,$$

so that the total electromotive force around the circuit is

$$E = E_{12} - E_{n2} + E_{n1} = E_{12} + E_{2n} + E_{n1},$$

depending only on the conductor of the second class and the two of the first class in contact with it. Such an arrangement is called a galvanic or voltaic cell.

In a conductor of the second class traversed by a current, chemical actions go on, whose laws were discovered by Faraday and Helmholtz. Such actions belong to the subject of electrochemistry, which is a branch of thermodynamics, and will be treated by the author elsewhere. For the same reason the theory of thermoelectromotive forces will not be treated here.

We have so far considered impressed electromotive forces to exist only at certain surfaces, where the potential is discontinuous. If, starting at any equipotential surface in a closed conductor, we plot the potential as an ordinate, on a diagram in which the abscissa is the resistance from the initial to any other equipotential surface, the curve will be composed of portions of parallel straight lines, whose slope is proportional to the current. The total impressed electromotive force will be equal to the sum of sudden rises minus the sum of sudden falls as we pass in the direction of the downward slope. It is evident that the discontinuities may occur at as many points as we please, and that provided the algebraic sum is the same the current will be unchanged. It is evident, comparing the two figures in which this is the case, and

* Joule. "On the Heat evolved by Metallic Conductors of Electricity, and in the Cells of a Battery during Electrolysis." *Phil. Mag.* 19, p. 260, 1841.

the slope of the broken lines is the same, that the more evenly the discontinuities are distributed the less is the maximum difference of potential between various parts of the circuit. By making the discontinuities small enough, we may therefore, without changing the current, make the differences of potential in the circuit as small as we please. In the limit the electromotive forces would be continuously distributed, and there would be no difference of potential. In that case there would be no electrostatical electromotive force. Such a continuous distribution of electromotive force may be produced by electromagnetic induction, the theory of which will be given at length in Chapter XII. The existence of a current does not, therefore, imply differences of potential.

169. Conductors in Parallel and Series. By the definition of conductivity of a current tube, it is evident that the conductivity of any number of current tubes between the same two equipotential surfaces is the sum of their individual conductivities. Now those portions of the surface of any conductor which are in contact with an insulator are portions of the sides of a current tube, for there is no flux across them. If then two equipotential surfaces are given in such a conductor so that the current flows in at one and out at the other, these surfaces are known as *electrodes* for the conductor, and if the electrode surfaces of several conductors are brought into contact and kept equipotential, the conductivity of the system is the sum of the individual conductivities. The essential in this proposition is that the contact of the several conductors shall not change the form of the equipotential surfaces that have been called electrodes. This condition is sure to be fulfilled if the conductors are *linear*, that is if each conductor forms a tube of flow whose cross-section is small enough to be neglected in comparison with its length. The electrodes in this case reduce to surfaces of infinitesimal area, and may be regarded as points. Conductors having two common electrodes are said to be connected in parallel, or in multiple arc, and for such the resistance R of the system is given by the equation,

$$(1) \qquad \frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}\cdots\cdots+\frac{1}{R_n}.$$

The resistance of the system is evidently less than any of the separate resistances. If several conductors be placed in order so

that each is in contact only with the preceding and succeeding, the system forms a single current tube, and the current is the same through any cross-section. The conductors are then said to be connected in series. If the surfaces of contact are equipotential we may apply Ohm's Law to each conductor. The potential at the entering electrode of the rth conductor being V_r, and at the issuing electrode V_r', we have

(2)
$$\begin{aligned} V_1 - V_1' &= R_1 I, \\ V_2 - V_2' &= R_2 I, \\ &\dots\dots\dots \\ V_n - V_n' &= R_n I. \end{aligned}$$

Adding these equations we have

(3)
$$\begin{aligned} V_1 + E_{12} + E_{23} \dots\dots + E_{n-1,n} - V_n' &= E + V_1 - V_n' \\ &= I(R_1 + R_2 + \dots\dots + R_n), \end{aligned}$$

so that if R be the resistance of the system,

(4)
$$R = R_1 + R_2 \dots\dots + R_n,$$

or the resistance of conductors in series is the sum of their individual resistances. If the conductors are linear the conditions at the ends are sure to be fulfilled.

170. Networks of Conductors. Kirchhoff's Laws. We have so far considered conductors filling a singly-connected space. In order to treat a conductor filling a multiply-connected space we have only to reduce it to a singly-connected region by the insertion of cross-sections, and it is easily seen that if the difference of potential on the two sides of a cross-section is given the potential is determined. These cross-sections are most naturally taken as the surfaces of impressed electromotive force.

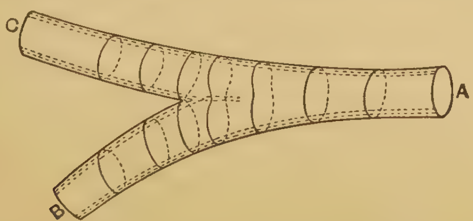

FIG. 68.

Suppose now that a conductor has in a certain region a forked or embranched form, as in Fig. 68. Then a portion of the tubes of

flow which fill the portion A of the conductor continue in the portion B, while the remainder leave them to traverse the portion C. Then if we consider successive equipotential cross-sections beginning in the portion A we shall finally reach an equipotential which is divided into two parts, one lying in B and the other in C. The last equipotential which does not break up into two consists of two parts touching each other and touching the surface of the fork of the conductor in a common point. This point, and this equipotential surface may be taken to define the branching of the conductors, and the surface will be taken for the common electrode for the three portions A, B, and C. In a similar manner we may have a conductor branching into any number of portions at a common equipotential surface. Consider now any network of conductors forming a figure of any degree of connectivity. The distribution of current and potential is determined when the impressed electromotive forces are given. If the equipotentials of embranchment are given, we may consider each conductor r between two successive surfaces of embranchment as a separate conductor, to which we may apply Ohm's Law,

$$(1) \qquad E_r + V_r - V_r' = R_r I_r,$$

for I_r, the total current in the branch, is perfectly defined.

At every surface of embranchment p the equation of continuity holds, so that if we call the currents in the s different branches positive if they all flow away from the embranchment,

$$(2) \qquad I_{p1} + I_{p2} + \ldots\ldots I_{ps} = 0.$$

For every conductor there is an equation of the form (1), and for every embranchment one of the form (2). The equations are all linear in the currents in the different branches and the potentials of the embranchments. They therefore suffice to determine all the currents and potentials, in terms of the resistances and impressed electromotive forces, except that the potentials may contain an arbitrary constant. This is determined if the potential at any one equipotential surface is given.

In the above we have assumed the equipotentials of embranchment given. It is easily seen however that these surfaces will vary in form as the impressed electromotive forces vary. Suppose for instance that an electromotive force be impressed in the branch C of Fig. 68 so as to make the total current in that branch

zero. Then all the tubes of flow in A pass into B and the equipotential surface of embranchment is as it were sucked up as

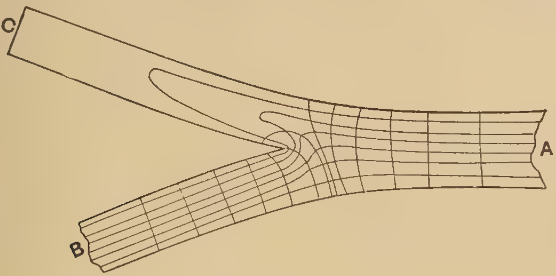

FIG. 68 a.

shown in Fig. 68 a. The conductor B is now longer than before, and we accordingly see that the resistance of a branch is not constant, but depends upon the electromotive forces. This difficulty immediately disappears if the conductors are linear, when the surfaces of embranchment reduce to points, where the several conductors join. The resistance is then between definite points, and the above linear equations determine the distribution of currents.

Kirchhoff, who first treated the general problem of a network of linear conductors*, eliminates the potentials by adding the equations of the first kind above for any group of conductors of the series forming a closed circuit. The potentials thus disappear, and for the circuit we have the equation

$$(3) \qquad E_1 + E_2 + \ldots\ldots + E_n = R_1 I_1 + R_2 I_2 \ldots\ldots + R_n I_n.$$

This and the equations (2) for the junctions are generally referred to as the equations of Kirchhoff's two Laws. Maxwell† treats the problem in the following more symmetrical form.

171. Maxwell's treatment of Networks. Consider n points of junction, each of which, in the most general case, is connected with each of the others by a conductor. The number of conductors in this case is $n(n-1)/2$. If some of the conductors are lacking this will be expressed by putting the conductivities

* Kirchhoff. "Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird." *Pogg. Ann.*, Bd. 72, 1847. *Ges. Abh.*, p. 22.

† Maxwell, *Treatise*, § 280.

between the corresponding points equal to zero. Let the current from the point p to the point q be I_{pq}, and let the conductivity of the conductor pq be C_{pq}, the impressed electromotive force E_{pq}. Then evidently

$$I_{pq} = -I_{qp},\ E_{pq} = -E_{qp},\ C_{pq} = C_{qp},\ I_{pp} = E_{pp} = 0.$$

The equation (1) may be written

$$I_{pq} = C_{pq}(E_{pq} + V_p - V_q). \tag{4}$$

Substituting the values of the currents in the equation of continuity (2) for the point p,

$$C_{p1}(E_{p1} + V_p - V_1) + C_{p2}(E_{p2} + V_p - V_2) \ldots\ldots + C_{pn}(E_{pn} + V_p - V_n) = 0. \tag{5}$$

Let us introduce a symbol C_{pp}, defined by the equation

$$C_{pp} = -(C_{p1} + C_{p2} \ldots\ldots + C_{pn}) \text{ or } \sum_{s=1}^{s=n} C_{ps} = 0.$$

The equations (5) may then be written in the symmetrical form

$$C_{p1}V_1 + C_{p2}V_2 \ldots\ldots + C_{pp}V_p + \ldots\ldots + C_{pn}V_n = C_{p1}E_{p1} + C_{p2}E_{p2} + \ldots + C_{pn}E_{pn}. \tag{6}$$

If we add these equations for all the points of junction, the result will be an identity, so that the equations are not all independent. The equations therefore suffice to determine the differences of potential between the junctions, but not the potentials themselves.

Since in the equations (4) only the differences of the potentials appear, it is evident that we may choose one of the potentials arbitrarily. Let us therefore put V_n equal to zero, and use the first $n-1$ of the equations (6), which are independent, to determine the potentials $V_1, V_2 \ldots\ldots V_{n-1}$. Calling Δ the determinant of the coefficients of conductivity,

$$\begin{vmatrix} C_{11}, & C_{12} & \ldots\ldots & C_{1,n-1} \\ C_{21}, & C_{22} & \ldots\ldots & C_{2,n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ C_{n-1,1} & C_{n-1,2} & \ldots\ldots & C_{n-1,n-1} \end{vmatrix}$$

and Δ_{rs} the minor of C_{rs}, we have Δ a symmetrical determinant, and $\Delta_{rs}=\Delta_{sr}$, since $C_{pq}=C_{qp}$. The solutions of the equations (6) are of the form

$$(7)\quad \Delta\,.\,V_t=\Delta_{1t}\,(C_{11}E_{11}+C_{12}E_{12}+\ldots\ldots+C_{1n}E_{1n})$$
$$+\Delta_{2t}\,(C_{21}E_{21}+C_{22}E_{22}+\ldots\ldots+C_{2n}E_{2n})\ldots\ldots$$
$$+\Delta_{n-1,\,t}\,(C_{n-1,\,1}\,E_{n-1,\,1}+C_{n-1,\,2}\,E_{n-1,\,2}\ldots\ldots$$
$$+C_{n-1,\,n}\,E_{n-1,\,n}).$$

Inserting these values of the potentials in the equations (4), we obtain the currents in all the branches as linear functions of the impressed electromotive forces in the branches. Picking out the terms containing E_{rs} or its negative E_{sr} in the current I_{pq} we obtain

$$(8)\quad \frac{\partial I_{pq}}{\partial E_{rs}}=\frac{C_{pq}(\Delta_{rp}C_{rs}-\Delta_{sp}C_{sr}-\Delta_{rq}C_{rs}+\Delta_{sq}C_{sr})}{\Delta}$$
$$=\frac{C_{pq}C_{rs}\,(\Delta_{rp}-\Delta_{sp}-\Delta_{rq}+\Delta_{sq})}{\Delta}.$$

In like manner the coefficient of E_{pq} in I_{rs} is

$$(9)\quad \frac{\partial I_{rs}}{\partial E_{pq}}=\frac{C_{rs}C_{pq}(\Delta_{pr}-\Delta_{qr}-\Delta_{ps}+\Delta_{qs})}{\Delta}.$$

But since $\Delta_{rs}=\Delta_{sr}$, etc., this is equal to $\dfrac{\partial I_{pq}}{\partial E_{rs}}$.

Consequently the current produced in a branch pq as a result of introducing an electromotive force E in a branch rs is the same as the current produced in the branch rs on introducing an equal electromotive force into the branch pq. This theorem is analogous to the reciprocal property of electrified conductors given in § 136. If

$$(10)\quad \Delta_{pr}+\Delta_{qs}=\Delta_{qr}+\Delta_{ps},$$

an electromotive force applied in one branch produces no current in the other, and the conductors are said to be conjugate.

172. Heat developed in the System. If we denote the coefficient

$$C_{pq}C_{rs}\,(\Delta_{pr}+\Delta_{qs}-\Delta_{qr}-\Delta_{ps})\ \text{by}\ C_{pqrs},$$

we have

$$(11)\quad I_{pq}=\tfrac{1}{2}\sum_{r=1}^{r=n}\sum_{s=1}^{r=n}C_{pqrs}E_{rs}.$$

Now the activity of the electromotive force E_{pq} is $E_{pq}I_{pq}$. Forming the products for all the branches and summing, bearing in mind that each branch appears twice, we obtain for the total activity

$$\tfrac{1}{2}\sum_{p=1}^{p=n}\sum_{q=1}^{q=n} E_{pq}I_{pq} = \tfrac{1}{4}\sum_p\sum_q\sum_r\sum_s C_{pqrs}E_{pq}E_{rs}.$$

But since there is supposed to be no electrostatic energy, this must be the heat developed in the system in unit time. The heat is accordingly a homogeneous quadratic function of the impressed electromotive forces. If we should solve the equations (11) we should obtain the electromotive forces as linear functions of the currents. Then forming the expression for the activity we should obtain a homogeneous quadratic function of the currents, and by our general theorem for the heating this must be equal to

$$\tfrac{1}{2}\sum_p\sum_q R_{pq}I_{pq}^2.$$

This might be obtained from the equations above by the aid of certain properties of determinants.

173. Wheatstone's Bridge. As an example of the above principles let us consider the case of Wheatstone's Parallelogram or Bridge. It consists of four points connected by six conductors, which may be represented by the sides and diagonals of a parallelogram, or more symmetrically as in Fig. 69.

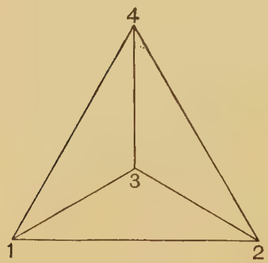

FIG. 69.

Suppose that the only impressed electromotive force is in the branch 12, and that we require the current in the branch 34. The equations (6) are

$$C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{14}V_4 = C_{12}E_{12},$$

$$C_{21}V_1 + C_{22}V_2 + C_{23}V_3 + C_{24}V_4 = C_{21}E_{21},$$

$$C_{31}V_1 + C_{32}V_2 + C_{33}V_3 + C_{34}V_4 = 0,$$

$$C_{41}V_1 + C_{42}V_2 + C_{43}V_3 + C_{44}V_4 = 0,$$

from which, putting $V_4=0$, and using the last three equations,

$$V_3 = E_{21}\frac{\begin{vmatrix} C_{21} & C_{22} & C_{21} \\ C_{31} & C_{32} & 0 \\ C_{41} & C_{42} & 0 \end{vmatrix}}{\Delta} = \frac{E_{21}C_{21}(C_{31}C_{42}-C_{41}C_{32})}{\Delta}.$$

From this we obtain the current in 34,

$$I_{34} = E_{21}\frac{C_{12}C_{34}(C_{31}C_{42}-C_{41}C_{32})}{\Delta}.$$

The current vanishes and the conductors 12, 34 are conjugate, if

$$C_{31}C_{42} = C_{41}C_{32}, \text{ that is } \frac{R_{31}}{R_{41}} = \frac{R_{32}}{R_{42}}.$$

This arrangement is used for measuring the resistance of a conductor in terms of three known resistances. A battery is inserted in one of the conductors 12, 34, and a galvanometer in the other, which is called the bridge wire. The resistances in the other branches being varied until the galvanometer shows no current, the condition of conjugacy is attained. In practice we wish to know how much current will pass when the condition of conjugacy is deviated from by a certain amount, in order to determine the accuracy with which a resistance can be measured. We therefore have to calculate the determinant Δ.

174. Resistance of Linear Conductor of Variable Section. If the cross-section of a conductor is infinitesimal, and equal to ω, we may write for the total current

$$I = -\iint \lambda \frac{\partial V}{\partial n} dS = -\lambda\omega \frac{dV}{ds},$$

s being the length of the conductor measured from a certain point. Integrating with respect to s from s_1 to s_2,

$$V_1 - V_2 = \int_1^2 \frac{Ids}{\lambda\omega} = I\int_1^2 \frac{ds}{\lambda\omega},$$

and the resistance is given by

$$R = \frac{V_1 - V_2}{I} = \int_1^2 \frac{ds}{\lambda\omega}.$$

This formula is important in the case of standards of resistance formed of tubes filled with mercury, the varying diameter of the tube being determined by a calibration. If the conductor is homogeneous, λ is constant, and if the cross-section is constant,

$$R=\frac{s_2-s_1}{\lambda\omega},$$

or the resistance of a uniform wire is proportional to its length and inversely to its cross-section.

175. Non-linear Homogeneous Conductors. In the case of homogeneous conductors, λ being constant, the equation of flow, § 162 (5), becomes

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

or the potential is harmonic. Consequently every theorem on harmonic functions applies to the potential in this case, and every method of solving problems of electrostatic distribution may be applied to the solution of problems of steady flow. We must have the electrodes of the conductor given. Now by the equation of Ohm's Law it is evident that the effect of increasing the conductivity of any portion of a conductor is to make the potential vary less rapidly there, the current being given. If then a portion of the conductor be made infinitely conducting its potential will become constant throughout. Accordingly if we introduce a thin plate of infinitely conducting material, this will form an equipotential surface and may be taken as an electrode for the conductor. This supposition will be made in the following examples. Since in the electrostatic problem the capacity is given by

$$K=\frac{\dfrac{1}{4\pi}\displaystyle\iint\frac{\partial V}{\partial n}\,dS}{V_2-V_1},$$

and in the problem of flow the conductance by

$$C=\frac{\displaystyle\iint\lambda\frac{\partial V}{\partial n}\,dS}{V_2-V_1}=\frac{1}{R},$$

we find that the conductance of a portion of a homogeneous conductor between two electrodes is equal to $4\pi\lambda$ times the capacity of a condenser whose plates have the geometrical form of the electrodes of the conductor, and whose dielectric occupies the

space corresponding to that occupied by the conductor. The case of a straight field, § 145, gives

$$K=\frac{S}{4\pi d}, \quad C=\frac{\lambda S}{d}, \quad R=\frac{d}{\lambda S},$$

as in the case of the uniform wire. The case of flow radially between concentric cylindrical electrodes gives, § 144,

$$K=\frac{l}{2\log(R_2/R_1)}, \quad C=\frac{2\lambda\pi l}{\log(R_2/R_1)}, \quad R=\frac{\log(R_2/R_1)}{2\lambda\pi l}.$$

This formula might be used for calculating the resistance of the liquid in galvanic cells where the plates are concentric cylinders. The case of radial flow in a sphere from a spherical electrode of radius R_0 (§ 142) gives, if the outer electrode is at an infinite distance,

$$K=R_0, \quad C=4\pi\lambda R_0, \quad R=\frac{1}{4\pi\lambda R_0}.$$

This formula may be used to find the resistance of the earth between two telegraphic earth-plates. If both earth-plates are equal spheres buried deeply in the earth at a distance apart so great that it may be considered infinite in comparison with their diameters, we may consider the resistance from one to the other as that of two conductors of the last case in series, so that

$$R=\frac{1}{2\pi\lambda R_0}.$$

If, as would more nearly represent the practical case, the conductors are hemispheres, with diametral planes in the surface of the earth, we may consider the space in the preceding problem split along the surface of flow formed by the plane through the centers of the spheres, and take the lower half, whose conductivity will be half of that just found, or

$$R=\frac{1}{\pi\lambda R_0}.$$

In like manner the problem of the ellipsoid and the circular disk will give us the resistance between earth-plates in the form of circular disks laid on the surface of the earth as $\pi/2$ times that for a hemisphere of the same radius. It is important to notice that in any case of geometrically similar electrodes, the resistance is inversely proportional to the linear dimensions of the earth-plate,

and not to its surface. This of course comes from the fact that the lines of flow diverge in all directions from the electrode instead of remaining parallel. It explains the necessity for large-sized plates for telegraphy or for the earth connection of a lightning rod. In practice, the conductivity of the earth varying from point to point, the conductivity of the portions near the electrode plays the most important part, so that it is important that the earth-plate be buried in good-conducting material. The problem of the spherical bowl shows that if such a bowl should be made an electrode immersed in an infinite conductor, the other electrode being at a great distance, nearly all the current would flow from the outside of the bowl, the current density being greatest at the lip.

The method of the conformal representation furnishes a means of solution for the case of two-dimensional problems, in particular for the flow of current in a thin plane sheet. Fig. 67 for instance shows the lines of flow in the case of a long ribbon of conductor slit along the axis of U'.

176. Correction for End of Wire. We shall conclude this subject with the consideration of the practical problem of finding the correction that must be made in the value of the resistance of a uniform wire when it ends in a conductor so large as to be capable of being considered infinite. This is of importance in the case of mercurial standards of resistance, for the tubes end in large cups of mercury. We shall consider a right circular cylindrical conductor ending in a conductor of in-

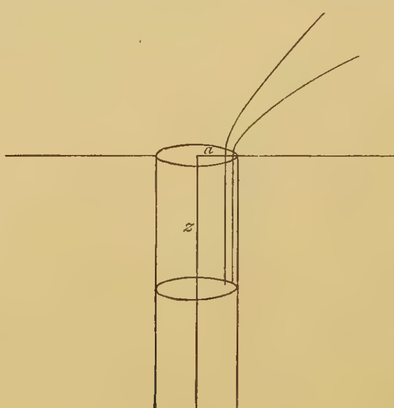

FIG. 70.

finite extent and bounded on one side by a plane perpendicular to the cylinder, Fig. 70. We may obtain an upper and lower limit for the desired correction by an artifice due to Lord Rayleigh*. It is evident that if we introduce anywhere a portion of conductor of greater conductivity we increase the conductance of the whole. Let us accordingly introduce in the mouth of the cylinder a plane sheet of infinite conductivity, thus rendering that circular section equipotential. The flow in this case will resemble the actual flow in that V will be continuous in crossing the plane, while it will differ from the actual case in that $\partial V/\partial n$ will be discontinuous, its integral over the section, or the total current being continuous. We may then use for the portion below the mouth the solution for a straight field, so that the resistance of a length l of radius a is

$$R_1 = \frac{l}{\lambda S} = \frac{l}{\lambda \pi a^2}.$$

Above the mouth of the cylinder we may use the formula for the flow from a circular disk of radius a to infinity, so that the resistance on the upper side is

$$R_2 = \frac{1}{4a\lambda}.$$

Consequently the lower limit of the resistance is

$$R_1 + R_2 = \frac{1}{\lambda \pi a^2}\left\{l + \frac{\pi a}{4}\right\}.$$

In a similar manner the resistance of the system will be increased if we introduce non-conducting surfaces not coincident with the walls of current tubes. Let us below the mouth of the cylinder suppose the cylinder split up into an infinite number of cylinders of infinitesimal cross-section, by means of cylindrical non-conducting surfaces introduced, and let the current density in these filaments be maintained constant, in the whole of the cylinder. Then below the mouth the equipotential surfaces will be planes, but on the upper side of the plane of the mouth the potential will not be constant, as we shall show. Consequently at the mouth of the cylinder V is discontinuous, while $\partial V/\partial z$ is in this case continuous. Since below the mouth

$$q = \lambda \frac{\partial V}{\partial z}$$

* Rayleigh, *Theory of Sound*, Vol. I. § 305.

is constant by hypothesis, we have

$$\partial V/\partial z = \text{const.},$$

and if $\partial V/\partial z$ is to have the same value on the upper side, V must there be the same as the potential due to a fictitious (non-equipotential) distribution on a disk of radius a of constant density

$$\sigma = -\frac{1}{2\pi}\frac{\partial V}{\partial z} = -\frac{1}{2\pi}\frac{q}{\lambda}.$$

The mass of such a distribution would be

$$m = \pi a^2 \sigma = -\frac{a^2 q}{2\lambda}.$$

The resistance of the upper side may be calculated by Joule's Law,

$$H = RI^2,$$

$$R = \frac{H}{I^2} = \frac{\iiint\left\{\lambda\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau}{\left[\iint_{z=0} \lambda \frac{\partial V}{\partial z} dS\right]^2}.$$

The integral in the numerator being through one-half of infinite space is $8\pi\lambda$ times one-half the energy of the distribution on the disk. The integral in the denominator is $4\pi\lambda$ times one-half the mass of the disk. Consequently

$$R_2 = \frac{4\pi\lambda W}{(2\pi\lambda m)^2} = \frac{W}{\pi\lambda m^2},$$

where W is the whole energy of the distribution of the disk. This energy is very easily calculated. The potential at the edge of a disk of radius ρ with constant surface density σ is

$$V_\rho = \sigma \iint \frac{dS}{r}.$$

If we introduce polar coordinates, the origin being the attracted point on the edge, and θ being the angle included between r, the radius to the point of integration and the diameter through the origin, this becomes

$$V_\rho = \sigma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=2\rho\cos\theta} \frac{r\,dr\,d\theta}{r} = 4\rho\sigma.$$

The work done in increasing the radius of the disk by $d\rho$ is, since the mass is increased by $2\pi\sigma\rho\, d\rho$,

$$dW = V_\rho \,.\, 2\pi\sigma\rho\, d\rho,$$

so that the whole energy of the distribution is

$$W = 2\pi\sigma \int_0^a V_\rho \rho\, d\rho = 8\pi\sigma^2 \int_0^a \rho^2 d\rho = \frac{8}{3}\pi\sigma^2 a^3 = \frac{8}{3}\frac{m^2}{\pi a}.$$

Inserting this in the value of R_2

$$R_2 = \frac{W}{\pi\lambda m^2} = \frac{8}{3\pi^2\lambda a},$$

we get

$$R = R_1 + R_2 = \frac{1}{\lambda\pi a^2}\left\{l + \frac{8a}{3\pi}\right\}.$$

Consequently the infinite conducting mass necessitates a correction equivalent in value to an increase in the length of the wire of between $\pi/4$ and $8/3\pi$, that is ·785 and ·849, times the radius of the wire. Lord Rayleigh has succeeded in bringing the limits still nearer together, and the results have been confirmed by experiment.

177. Current Sheets. The current-density being a solenoidal vector, all that has been said about lines and tubes of such vectors may be applied to current lines and tubes. The current tubes may be defined by the intersection of two families of surfaces. A current *sheet* will be defined as a portion of space bounded by two infinitely near parallel surfaces, in which currents flow, converging to or diverging from certain points called electrodes. If the equation of the surface is $q_3 =$ const. and q_1 and q_2 are two coordinates forming an orthogonal system, the flow may be defined by either the potential V or the current-function Ψ, which both satisfy the equation, § 104 (7),

$$\frac{\partial}{\partial q_1}\left\{\frac{h_1}{h_2}\frac{\partial V}{\partial q_1}\right\} + \frac{\partial}{\partial q_2}\left\{\frac{h_2}{h_1}\frac{\partial V}{\partial q_2}\right\} = 0.$$

Problems of plane current sheets may be at once solved by the method of functions of a complex variable, and from them any number of problems for other surfaces may be solved by finding conformal transformations. Such transformations may be found practically for a surface by constructing it of thin metal, intro-

ducing current from a battery to it by various electrodes, and finding by touching the surface with two sharp conducting points connected with a galvanometer, the locus of points of equal potential. Doing the same for any other surface with the same number of electrodes of the same sign will give a conformal transformation of the two surfaces*. If the whole of one surface is transformed upon only a part of the other, it is necessary to cut out corresponding parts of the two surfaces, making the whole of each equipotential edge an electrode. This can be done in practice by soldering the edge of the sheet to a massive three-dimensional conductor of great conductivity. Fig. 71 represents the flow in a spherical sheet corresponding by Mercator's projection to the flow in a plane sheet given in Fig. 22, and Fig. 47 represents Fig. 23 transformed to the sphere by stereographic projection.

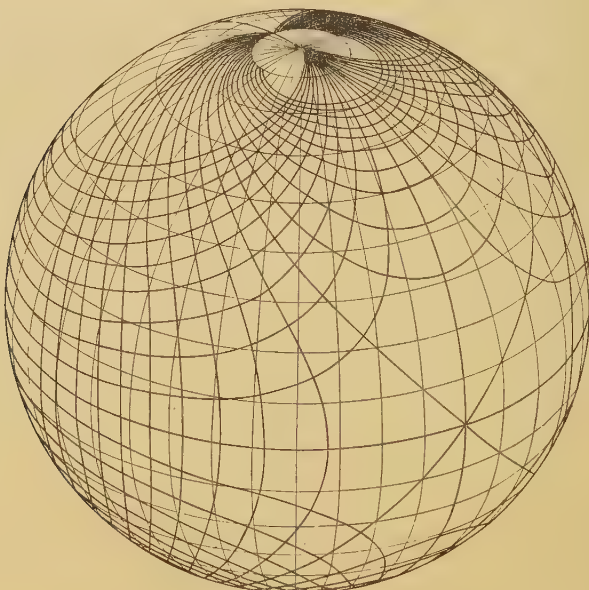

FIG. 71.

* Kirchhoff. "Ueber die stationären elektrischen Strömungen in einer gekrümmten leitenden Fläche." *Monatsber. der Berl. Akad.* 1875. *Ges. Abh.* p. 56. Kelvin. "Generalisation of Mercator's Projection performed by aid of Electrical Instruments." *Electrician*, p. 553, 1892.

CHAPTER IX.

PROPERTIES OF DIELECTRICS AND MAGNETIZABLE BODIES.

178. Magnetic Phenomena. In the previous treatment of electrostatics we have supposed all of space not occupied by conductors to be filled with a single uniform dielectric. In this chapter we shall remove this restriction and consider the distribution of the forces when any number of varying media are present besides the conductors. Inasmuch as all the phenomena here treated have exact analogues in the phenomena of magnetism, we shall first briefly describe magnetic phenomena. A magnet is a piece of loadstone, or of metal, generally iron or steel, possessing the property of attracting iron, and of attracting or repelling other magnets. The forces thus developed are called magnetic forces. A small magnet in the form of a filament or needle, under the action of any other magnet, tends to set itself in a certain direction at every point in space, and this direction is said to be the direction of the magnetic force at the point. A portion of space in which such forces are exerted on the magnetic needle is called a field of magnetic force, and may be represented by drawing lines of force in every portion of it. If the lines of force are straight and parallel, the field is said to be straight or uniform. Different parts of a magnet possess opposite properties with regard to attraction or repulsion, we may therefore consider them charged with matter of different signs. We make use of the term matter here precisely as in connection with electricity, not to denote something which has inertia, but simply something which attracts, and which is measured by its power of attraction. Experiment shows that any magnet placed in a uniform field experiences no resultant force, but only a couple. We therefore conclude

that a magnet is *polarized* (§ 120). The intensity of the polarization is called the intensity of magnetization. A magnet may be magnetized solenoidally, and will then appear to be charged only superficially. A long thin magnetic solenoid may be assimilated to two equal and opposite magnetic points. These are called the poles of the solenoid. It is a matter of indifference which one is taken as positive—in practice, as the earth is surrounded by a magnetic field whose lines run roughly north and south, the end of any magnetic solenoid which tends to move toward the north is called positive.

It is to be noticed that a magnetic point never exists alone, but is always accompanied by an equal and opposite point, just as when electricity is generated, equal and opposite amounts always appear simultaneously. In the case of electricity we often lose sight of one of the charges produced, but in magnetism we cannot do so, though we may remove one of the charges as far as we please by making the magnet long enough. It was in this manner that Coulomb, by experiments with a torsion-balance, was able to investigate the forces between magnetic poles, finding that they acted upon each other according to the Newtonian Law of the inverse square. The unit magnetic pole is then defined as the pole which will repel with unit force a similar pole placed at unit distance from it. This definition is the basis of the magnetic system of measurements, which stands in the same relation to magnetic quantities that the electrostatic system does to electric quantities. All the mathematical work that has been done for electricity (with one exception) is then directly applicable to magnetism. Magnetic potential, density, energy, and so on, are defined in a similar way to the corresponding electrical quantities, and their dimensions in the magnetic system are the same as those of the electrical quantities in the electric system. The exception noted is that phenomena of magnetic flow do not exist—there are no magnetic conductors, and no dissipation of magnetic energy into heat by flow. It may accordingly seem that the principal part of electrical phenomena, namely the distribution of charges on conductors, forming the subject of electrostatics, has no place in magnetism. While this is true, we shall find that a very important part of electrostatics, namely the consideration of the field in dielectrics, has exact analogues in magnetism, and these are yet to be treated.

If we perform an experiment analogous to Experiment VIII of Chapter I, namely, suspend two magnets so that two of their poles may repel each other in the air, and then surround them by another medium, for instance by a solution of a salt of iron, we shall find that the magnets fall together, or the system seems to lose energy. The experiment in this form would be difficult, but if we introduce into only a portion of the space a different medium, for instance by introducing a piece of iron between the magnets, the effect is unmistakeable. We are thus led to conclude that the energy of a magnetic or electric distribution depends not only on the distributions themselves but on the media which surround them. Many of the mathematical developments of the preceding chapters must therefore be abandoned, and all must be examined in the light of this conclusion.

179. Parallel treatment of Electrostatics and Magnetostatics. Inasmuch as all the phenomena to be considered in this chapter are exactly parallel, for electricity and for magnetism, we shall in general not distinguish which they may be, but shall consider in all cases the words *magnetic* or *electric* to be used interchangeably. We shall accordingly in this chapter not introduce different symbols for the two sorts of quantities, the necessity for so doing occurring only when both sorts of phenomena exist simultaneously.

Experiment shows that in the general case here considered the forces experienced by a point-charge are conservative, consequently a potential exists. The law of the inverse square however ceases to hold in general, and the potential is not harmonic in free space outside the acting distributions.

When the charges are given, since the forces are different from those previously calculated, the relation between the density and the potential must be different from that given by Poisson's equation. Since however we suppose the force due to any element to be proportional to the charge of the element, the differential equation must be linear. Let us examine what conclusions are true irrespective of the law of force. By the definition of potential as a quantity of work necessary to bring unit charge from infinity to any point, it follows, as was found in § 117, that the energy of any distribution is

(1) $$W_d = \frac{1}{2}\iint V\sigma dS + \frac{1}{2}\iiint V\rho d\tau.$$

The theorem on mutual potential energy of two distributions, § 117 (5), also holds,

$$(2) \qquad \iint V'\sigma dS + \iiint V'\rho d\tau = \iint V\sigma' dS' + \iiint V\rho' d\tau'.$$

By making use of these theorems, we find by the process of § 131, that the potential in conductors is constant. All the theorems on systems of conductors, §§ 136—140, also remain unchanged, except that the coefficients of capacity and potential receive different values from those there given.

180. New Law of Force. Action of Medium. We have found in § 118 that the energy of any distribution acting according to the Newtonian Law was exactly accounted for by supposing each element of volume in all space to contain a quantity of energy equal, per unit volume, to $1/8\pi$ times the square of the total force of the field. Since the phenomena now to be considered resemble the phenomena of Newtonian distribution to such an extent that it was long before any difference was discovered, this proposition must be nearly true. We have found however that the energy depends on the medium as well as on the distribution. We shall therefore, in order to explain the phenomena, make an assumption deviating as little as possible from the above proposition in regard to the energy, and containing it as a particular case, but allowing us to take account of the medium.

The assumption will be justified if its consequences accord with experiment. We shall assume merely that each element of volume contributes to the energy an amount per unit volume *proportional* to the square of the force of the field, the factor of proportionality being a property of the medium, which may vary from point to point. The whole energy is accordingly

$$(3) \qquad W_f = \frac{1}{8\pi}\iiint_\infty \mu\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,$$

where μ is a positive point-function, which is given as soon as the substances occupying each region of space are known. For a homogeneous medium μ is constant. We shall now insert this form of the energy in the expression $W = 2W_d - W_f$, and apply

the maximum theorem of § 119, which we shall also assume to hold.* We have now

$$(4)\qquad W=\iint\sigma V dS+\iiint_\infty \rho V d\tau$$
$$-\frac{1}{8\pi}\iiint_\infty \mu\left\{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right\}d\tau.$$

Letting V vary without changing the charges σ, ρ,

$$(5)\qquad W+\delta_V W=\iint\sigma(V+\delta V)\,dS+\iiint_\infty \rho(V+\delta V)\,d\tau$$
$$-\frac{1}{8\pi}\iiint_\infty \mu\left\{\left(\frac{\partial(V+\delta V)}{\partial x}\right)^2+\left(\frac{\partial(V+\delta V)}{\partial y}\right)^2+\left(\frac{\partial(V+\delta V)}{\partial z}\right)^2\right\}d\tau,$$

from which

$$(6)\qquad \delta_V W=\iint\sigma\delta V dS+\iiint_\infty \rho\delta V d\tau$$
$$-\frac{1}{4\pi}\iiint_\infty \mu\left\{\frac{\partial V}{\partial x}\frac{\partial\delta V}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial\delta V}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial\delta V}{\partial z}\right\}d\tau$$
$$-\frac{1}{8\pi}\iiint_\infty \mu\left\{\left(\frac{\partial\delta V}{\partial x}\right)^2+\left(\frac{\partial\delta V}{\partial y}\right)^2+\left(\frac{\partial\delta V}{\partial z}\right)^2\right\}d\tau,$$

and by Green's theorem applied to the third integral

$$(7)\qquad \delta_V W=\iint\left\{\sigma+\frac{1}{4\pi}\left(\mu_1\frac{\partial V}{\partial n_1}+\mu_2\frac{\partial V}{\partial n_2}\right)\right\}\delta V dS$$
$$+\iiint_\infty\left[\rho+\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right)\right\}\right]\delta V d\tau$$
$$-\frac{1}{8\pi}\iiint_\infty \mu\left\{\left(\frac{\partial\delta V}{\partial x}\right)^2+\left(\frac{\partial\delta V}{\partial y}\right)^2+\left(\frac{\partial\delta V}{\partial z}\right)^2\right\}d\tau.$$

If now the energy of the actual distribution of potential is to be a maximum for all possible values of δV, the first two integrals must vanish, which can be the case only if throughout space we have

$$(8)\qquad \rho=-\frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right)\right\},$$

* See Helmholtz, *Wiss. Abh.* Bd. I., p. 805.

and at every surface of discontinuity,

$$(9)\qquad \sigma = -\frac{1}{4\pi}\left\{\mu_1 \frac{\partial V}{\partial n_1} + \mu_2 \frac{\partial V}{\partial n_2}\right\}.$$

These equations will henceforth be known as the generalized Poisson's equations. They give us the law of distribution of force in the differential form, and contain the forms heretofore used as a special case, obtained by putting $\mu = 1$.

The form of the integral expressing the energy is the same as that of the integral J of § 165 (1). All the properties of the integral J are accordingly possessed by the integral W_f. In particular it follows that if the potential is given at certain surfaces the condition that the energy shall be a minimum requires that in the space between

$$(10)\qquad \frac{\partial}{\partial x}\left(\mu \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial V}{\partial z}\right) = 0,$$

and on surfaces of discontinuity

$$(11)\qquad \mu_1 \frac{\partial V}{\partial n_1} + \mu_2 \frac{\partial V}{\partial n_2} = 0.$$

We may call the problem of finding a function that shall satisfy these differential equations, and take the required surface-values, the generalized Dirichlet's Problem. The function V may be called quasi-harmonic.

It is evident, as in § 86, that the solution of the problem, if there be any, is unique.

We have heretofore said nothing regarding the localization of the energy of a distribution, which we have represented either by an integral W_d throughout the acting distribution, or by an integral W_f, which is expressed in terms of *the field at all points of space.* Whereas both representations are equivalent mathematically, it is a fundamental point in Maxwell's theory to regard the energy as localized in the medium wherever a field exists.

181. Induction. If we define a vector $\mathfrak{F}$ by the equations

$$(12)\qquad \begin{aligned} \mathfrak{X} &= \mathfrak{F}\cos(\mathfrak{F}x) = -\mu\frac{\partial V}{\partial x}, \\ \mathfrak{Y} &= \mathfrak{F}\cos(\mathfrak{F}y) = -\mu\frac{\partial V}{\partial y}, \\ \mathfrak{Z} &= \mathfrak{F}\cos(\mathfrak{F}z) = -\mu\frac{\partial V}{\partial z}, \end{aligned}$$

the vector has, by (8), the property of being solenoidal in all parts of space where there are no charges. That is

$$(13) \qquad \operatorname{div} \mathfrak{F} = \frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z} = 0.$$

The force, or field-strength F, no longer enjoys this property in general, but does so in a homogeneous medium, for which μ comes out as a constant factor. The vector $\mathfrak{F}$ is called the *induction*, and is connected with the force by the equations

$$(14) \qquad \mathfrak{X} = \mu X, \quad \mathfrak{Y} = \mu Y, \quad \mathfrak{Z} = \mu Z.$$

The induction accordingly satisfies everywhere the law of Fourier-Ohm.

The surface integral over any surface of the induction resolved normally to the surface is called the total induction, or induction-flux, through the surface. The quantity μ is called the *inductivity* of the medium. A more usual name for it is the specific inductive capacity or dielectric constant, in the electric case, magnetic permeability in the magnetic case. The latter name is due to Lord Kelvin, to whom the recognition of the analogy to the case of flow in electricity and heat is due.* The name permeability comes from the hydrokinetical analogy of water flowing through a porous medium.

The lines of induction suffer refraction in the manner described in § 163 when passing from one medium to another. In the

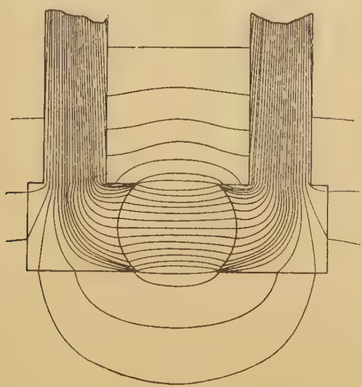

FIG. 71*a*.

* Magnetic Permeability, and Analogues in Electro-static Induction, Conduction of Heat, and Fluid Motion. *Papers on Electrostatics and Magnetism*, p. 487.

electric case, μ does not differ widely for different media, seldom reaching ten times the value for air or empty space, and never being less than for empty space, while in the magnetic case, μ may be, for iron, several thousand times as great as for air, and in some cases is slightly less than for air. Consequently lines of force emerging from iron into air are generally nearly normal to the surface in the air unless very nearly tangential in the iron. This is exemplified in Fig. 71*a*, showing the distribution of lines of force between the pole-pieces of the field magnet of a dynamo with the armature removed. In virtue of the analogy to electric conductivity, it is evident that the lines of force exhibit a tendency to crowd together into parts of the field where μ is large.

182. Relation of Charge to Induction. Since the force no longer possesses the solenoidal property, except in homogeneous media, while the induction does, it is more logical to speak of tubes of induction than of tubes of force, although geometrically the two coincide. The flux of force through various cross-sections of a tube, however, varies, while the flux of induction is constant for the tube.

The volume density is no longer determined by the divergence of the force, but of the induction, being equal to $1/4\pi$ times the divergence of the induction, § 180 (8),

$$(15) \qquad \rho = \frac{1}{4\pi}\left\{\frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z}\right\},$$

while the surface density is $1/4\pi$ times the discontinuity of its normal component, § 180 (9),

$$(16) \qquad \sigma = \frac{1}{4\pi}\{\mathfrak{F}_{1n_1} + \mathfrak{F}_{2n_2}\}.$$

Accordingly the charge of any portion of space τ,

$$(17) \qquad e = \iiint \rho d\tau = \frac{1}{4\pi}\iiint\left\{\frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z}\right\} d\tau$$

$$= -\frac{1}{4\pi}\iint\{\mathfrak{X}\cos(nx) + \mathfrak{Y}\cos(ny) + \mathfrak{Z}\cos(nz)\}\, dS,$$

is equal to $1/4\pi$ times the excess of the number of unit tubes issuing from the space over the number entering. We shall call the densities thus defined, for a reason to be presently explained, the densities of *true* electricity or magnetism.

183. Apparent Charge. Since on passing from one medium to another where the inductivity is different the force is discontinuous, the surface acts as a charged surface has previously been found to act, § 82. Also since the force is not solenoidal in a heterogeneous medium, there appear to be bodily charges. The magnitude of these *apparent* charges, whose densities are ρ', σ', are given by the usual equations

(18) $$\rho' = -\frac{1}{4\pi}\Delta V,$$

(19) $$\sigma' = -\frac{1}{4\pi}\left\{\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2}\right\} = \frac{1}{4\pi}(F_{1n_1} + F_{2n_2}),$$

and comparing these with the equations for the true densities we find

(20) $$\rho' = \frac{1}{\mu}\left\{\rho + \frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\frac{\partial \mu}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial \mu}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial \mu}{\partial z}\right)\right\},$$

or in a homogeneous medium

(20′) $$\rho' = \frac{\rho}{\mu}.$$

For the surface density

(21) $$\sigma' = \frac{\sigma}{\mu_1} + \frac{1}{4\pi}\left(\frac{\mu_1 - \mu_2}{\mu_1}\right)F_{2n_2} = \frac{\sigma}{\mu_2} + \frac{1}{4\pi}\left(\frac{\mu_2 - \mu_1}{\mu_2}\right)F_{1n_1}.$$

The potential is then determined by § 85 (18), as

(22) $$V = -\frac{1}{4\pi}\iint\frac{1}{r}\left(\frac{\partial V}{\partial n_1} + \frac{\partial V}{\partial n_2}\right)dS$$
$$-\frac{1}{4\pi}\iiint\frac{\Delta V}{r}\,d\tau = \iint\frac{\sigma'}{r}\,dS + \iiint\frac{\rho'}{r}\,d\tau;$$

that is:

A distribution of charges *acting according to the Newtonian Law*, of densities ρ' and σ', would produce everywhere exactly the same field as that actually produced by the true charges ρ and σ. The Newtonian Law thus reappears, and may be used to calculate the forces, only the true charges do not follow the law, but the apparent charges, which are known as soon as the true charges and the properties of the media are given.

184. Capacity. Reluctance. In the electric case we have for the capacity of a condenser whose dielectric is homogeneous, so that $\Delta V = 0$,

$$(23) \qquad K = \frac{1}{4\pi} \frac{\iint \mu \frac{\partial V}{\partial n} dS}{(V_1 - V_2)} = \frac{\mu}{4\pi} \frac{\iint \frac{\partial V}{\partial n} dS}{(V_1 - V_2)};$$

that is, the capacity equal to that found in Chapter VI multiplied by the inductivity of the medium. This is Faraday's capital discovery,* leading to the development of the theory according to which the energy resides in the medium, so that electrical actions are transmitted by means of the medium, and not by action at a distance. Faraday experimented with condensers in the form of concentric spheres, the intervening space being filled by the dielectric in question. The material of the dielectric outside the larger sphere was accordingly immaterial. Instead of capacity the term *permittance* has been proposed by Heaviside.

In the magnetic case, the value of the quantity analogous to the capacity has been called the *permeance* or *inductance*, while its reciprocal, corresponding to the resistance in the case of electric flow, was called *magnetic resistance* by Bosanquet, a name which has given way to that of *reluctance*.

185. Induced Charge. The apparent charges defined above minus the true charges are called the *induced* charges due to the action of the forces of the field. If we examine the amount of the induced charge in a body τ surrounded by a homogeneous medium, we shall obtain an important result. Let the constant inductivity of the external medium be μ_1, and let us denote the normal toward the interior of the body τ by n_i and the normal toward the outside by n_e. Then if we use the formula (21) for the apparent charge on the surface we find

$$(24) \qquad \iint \sigma' dS = \frac{1}{\mu_1} \iint \sigma dS - \frac{1}{4\pi} \iint \left(\frac{\mu_1 - \mu}{\mu_1} \right) \frac{\partial V}{\partial n_i} dS,$$

and transforming the second integral on the right by Green's theorem this becomes

* *Exp. Res.* § 1252 seq.

(25) $$\iint \sigma' dS = \frac{1}{\mu_1}\iint \sigma dS - \frac{1}{4\pi}\iint \frac{\partial V}{\partial n_i} dS + \frac{1}{4\pi\mu_1}\iint \mu \frac{\partial V}{\partial n_i} dS$$

$$= \frac{1}{\mu_1}\iint \sigma dS + \frac{1}{4\pi}\iiint \Delta V d\tau$$

$$- \frac{1}{4\pi\mu_1}\iiint \left\{\frac{\partial}{\partial x}\left(\mu \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial V}{\partial z}\right)\right\} d\tau.$$

But by the definitions of true and apparent volume density this is

(26) $$\iint \sigma' dS = \frac{1}{\mu_1}\iint \sigma dS - \iiint \rho' d\tau + \frac{1}{\mu_1}\iiint \rho d\tau,$$

so that, transposing,

(27) $$\iint \sigma' dS + \iiint \rho' d\tau = \frac{1}{\mu_1}\left[\iint \sigma dS + \iiint \rho d\tau\right],$$

or the total apparent charge of a body surrounded by a homogeneous medium is equal to the true charge of the body divided by the inductivity of the surrounding medium. In particular a body which has no true charge has a total apparent charge equal to zero, and since this remains true however the body may be subdivided, the body is polarized. In the magnetic case, the body is always found to be polarized, consequently we must conclude that the true magnetic charge of all bodies is zero, or in other words, true magnetism exists only as polarization. This is a second apparent difference between electricity and magnetism, but if we remember that whenever electrification is produced equal quantities of both signs appear, the difference disappears.

186. Polarizations. Since experiments on electrification and magnetization are almost always made on bodies surrounded by a homogeneous medium, namely air, it has become customary to regard their apparent charges as due to the polarizations of the bodies themselves, although it is evident by § 120 that the surface charges are due only to *differences* of polarization on the two sides of the surface. The surrounding medium might be uniformly polarized to any degree without producing any effect, consequently its absolute polarization cannot be determined, and is of no importance whatever. The apparent polarization of the body must produce the surface density, by § 120 (2),

(28) $$\sigma' = -I \cos(In_i).$$

But using the equation for the apparent surface density (21) when the true surface density is zero,

$$(29)\qquad \sigma' = \frac{1}{4\pi}\left\{\frac{\mu_1 - \mu}{\mu_1}\right\} F_i \cos(F_i n_i),$$

so that if we put

$$(30)\qquad I = \frac{1}{4\pi}\left\{\frac{\mu F}{\mu_1} - F\right\} = \frac{1}{4\pi}\left\{\frac{\mathfrak{F}}{\mu_1} - F\right\},$$

giving I in the direction of F and $\mathfrak{F}$, we obtain the proper surface density. From the components of I,

$$(31)\qquad \begin{aligned} A &= \frac{1}{4\pi}\left(\frac{\mathfrak{X}}{\mu_1} - X\right), \\ B &= \frac{1}{4\pi}\left(\frac{\mathfrak{Y}}{\mu_1} - Y\right), \\ C &= \frac{1}{4\pi}\left(\frac{\mathfrak{Z}}{\mu_1} - Z\right), \end{aligned}$$

we obtain for the volume density due to the polarization, § 120 (6)

$$(32)\qquad \begin{aligned} \rho' &= -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) \\ &= -\frac{1}{4\pi\mu_1}\left(\frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z}\right) + \frac{1}{4\pi}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right), \end{aligned}$$

and since by (18) we have

$$(33)\qquad \rho' = \frac{1}{4\pi}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = \frac{1}{4\pi}\,\mathrm{div}\, F,$$

we must have

$$(34)\qquad \frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z} = 0,$$

or the induction is solenoidal. The induction accordingly possesses the property of the vector called the induction in § 121, and by the equations (31) is equal to it if μ_1 is equal to unity.

187. Examples. Point-charge in Medium bounded by Plane Face. Suppose we have a point-charge e placed at P at a distance a from a plane face separating two media of inductivities μ_1, μ_2, their extent being infinite. We may solve the problem of induction by the method of images as in § 152. Suppose that e lies on the left of the dividing plane, and that at its geometrical

image P' we place a charge e'. Then we may determine the charge e' so that on the left the potential will be the same as that due to charges e and e' placed at P and P' in a uniform medium, while on the right we shall have V the same as would be produced by a charge $e+e'$ placed at P. For if we put

on the left $V = \dfrac{e}{r} + \dfrac{e'}{r'}$,

$$\frac{\partial V}{\partial n_1} = -\frac{e}{r^2}\frac{\partial r}{\partial n_1} - \frac{e'}{r'^2}\frac{\partial r'}{\partial n_1} = \frac{e-e'}{r^2}\cos\theta;$$

on the right $V = \dfrac{e+e'}{r}$,

$$\frac{\partial V}{\partial n_2} = -\frac{e+e'}{r^2}\frac{\partial r}{\partial n_2} = -\frac{e+e'}{r^2}\cos\theta.$$

But at the surface these must satisfy the equation

$$0 = \mu_1\frac{\partial V}{\partial n_1} + \mu_2\frac{\partial V}{\partial n_2} = \{\mu_1(e-e') - \mu_2(e+e')\}\frac{\cos\theta}{r^2}.$$

Consequently if we put

$$\mu_1(e-e') - \mu_2(e+e') = 0,$$

$$e' = e\frac{\mu_1-\mu_2}{\mu_1+\mu_2},$$

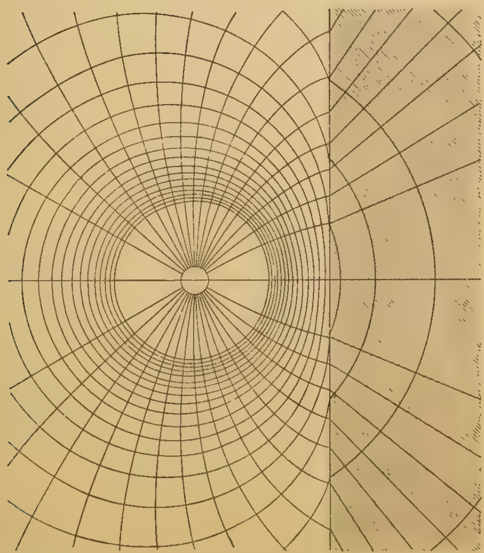

FIG. 72.

the condition will be satisfied. The surface density induced on the surface of separation is

$$\sigma' = -\frac{1}{4\pi}\{e - e' - (e + e')\}\frac{\cos\theta}{r^2} = \frac{e}{2\pi}\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\frac{a}{r^3},$$

so that the distribution is proportional to the distribution on a conducting plane, as in § 152. The lines of force in the medium 2 are straight, and the refraction on crossing the plane is shown in the diagram, Fig. 72, in which $\mu_2/\mu_1 = 4$. The greater μ_2 the less is the force in the medium 2, and if we make μ_2 infinite, the force in the medium 2 vanishes, and the surface density becomes

$$\sigma' = -\frac{e}{2\pi}\frac{a}{r^3},$$

as in the case of a conductor.

188. Slab in Uniform Field. Suppose a slab of inductivity μ_2 with parallel faces of infinite extent is placed in a uniform field parallel to the equipotential surfaces, and let the inductivity of the surrounding homogeneous medium be μ_1. Then the potential satisfies Laplace's equation in the slab as well as outside it. Accordingly the solution for either of the three parts of the field, in, above or below the slab, is a linear function of the single coordinate perpendicular to the equipotential planes, and the force has values which are constant, but different, in the three regions.

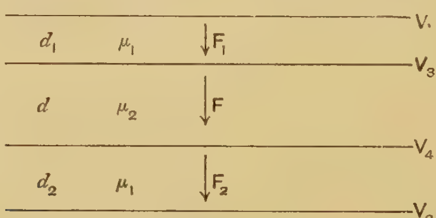

FIG. 73.

If V_1 and V_2 are any two equipotential planes outside the slab at distances d_1 and d_2 from its faces, V_3 and V_4 the potentials of the faces of the slab respectively facing them, and d the thickness of the slab, we have the conditions at the surfaces V_3, V_4

(1) $$\mu_1 F_1 = \mu_2 F, \quad \mu_2 F = \mu_1 F_2,$$

that is

(2) $$F_1 = F_2 = \frac{\mu_2}{\mu_1}F.$$

Now by § 145 we have

$$(3) \qquad F_1 = \frac{V_1 - V_3}{d_1}, \quad F = \frac{V_3 - V_4}{d}, \quad F_2 = \frac{V_4 - V_2}{d_2} = F_1,$$

so that we have the equations

$$(4) \qquad \begin{aligned} d_2(V_1 - V_3) &= d_1(V_4 - V_2), \\ \mu_1 d(V_1 - V_3) &= \mu_2 d_1(V_3 - V_4). \end{aligned}$$

Solving these for V_3 and V_4

$$(5) \qquad \begin{aligned} V_3 &= \frac{V_1(d\mu_1 + d_2\mu_2) + V_2 d_1\mu_2}{\mu_2(d_1 + d_2) + \mu_1 d}, \\ V_4 &= \frac{V_1 d_2\mu_2 + V_2(d_1\mu_2 + d\mu_1)}{\mu_2(d_1 + d_2) + \mu_1 d}, \end{aligned}$$

from which we get for the force outside the slab

$$(6) \qquad F_1 = \frac{V_1 - V_3}{d_1} = \frac{V_1 - V_2}{d_1 + d_2 + d\mu_1/\mu_2}.$$

In the electrical case if V_1, V_2, are the surfaces of conductors, the density on the upper plate V_1 is

$$(7) \qquad \sigma = \frac{\mathfrak{F}_1}{4\pi} = \frac{\mu_1 F_1}{4\pi},$$

and the capacity of the condenser of area S

$$(8) \qquad K = \frac{\sigma S}{V_1 - V_2} = \frac{\mu_1 S}{4\pi(d_1 + d_2 + d\mu_1/\mu_2)}.$$

By measuring the capacity with the slab and with it removed we may determine the dielectric constant of the slab in terms of that of the surrounding medium. If d_1 and d_2 are zero, the capacity is

$$(9) \qquad K = \frac{\mu_2 S}{4\pi d},$$

which is, as was stated in § 184, proportional to the dielectric constant. The apparent surface density on the upper face of the slab is

$$\sigma' = \frac{F - F_1}{4\pi} = \frac{\mu_1 - \mu_2}{4\pi\mu_2} F_1 = \frac{(\mu_1 - \mu_2)(V_1 - V_2)}{4\pi[\mu_2(d_1 + d_2) + \mu_1 d]},$$

so that the intensity of polarization is

$$I = -\sigma' = \frac{(\mu_2 - \mu_1)(V_1 - V_2)}{4\pi[\mu_2(d_1 + d_2) + \mu_1 d]},$$

which is in the direction of the force if $\mu_2 > \mu_1$.

189. Point-Charge in Sphere. Suppose we have a point-charge e at the center of a sphere of radius R of homogeneous substance of inductivity μ_1, surrounded by an infinite homogeneous medium of inductivity μ_2. Then Laplace's equation being satisfied in either medium, we may use the solution of § 142, and put V in either medium equal to a linear function of $1/r$

$$V_1 = \frac{A}{r} + B, \quad V_2 = \frac{A'}{r}.$$

The condition at the surface $r = R$ gives, since

$$\frac{\partial V_1}{\partial n_1} = \frac{A}{R^2}, \quad \frac{\partial V}{\partial n_2} = -\frac{A'}{R^2},$$

$$\mu_1 \frac{\partial V_1}{\partial n_1} + \mu_2 \frac{\partial V_2}{\partial n_2} = \frac{\mu_1 A - \mu_2 A'}{R^2} = 0,$$

so that

$$V_1 = \frac{e}{\mu_1 r} + c, \quad V_2 = \frac{e}{\mu_2 r},$$

since the integral of $\mathfrak{F}_n = -\mu \dfrac{\partial V}{\partial r}$ over any surface inclosing e must be $4\pi e$.

The apparent surface density and charge of the sphere are

$$\sigma' = -\frac{1}{4\pi}\left\{\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2}\right\} = -\frac{1}{4\pi}\frac{e}{R^2}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_2}\right\},$$

$$e' = 4\pi R^2 \sigma' = e\,(\mu_1 - \mu_2)/\mu_1\mu_2.$$

The real charge e at the center acts, by § 183 (20′), like an apparent charge e/μ_1, and the apparent charge of the sphere e' acts at outside points as if concentrated at the center. Accordingly the whole force in the medium 2 is

$$\frac{e/\mu_1 + e'}{r^2} = \frac{e}{\mu_2 r^2},$$

which is the same as found from $-\partial V_2/\partial r$.

190. Unit of Electricity or Magnetism. If the charge be situated in a medium of inductivity μ extending to infinity, the force of the field is, by the above, equal to $e/\mu r^2$ and the action of e on a charge of e_1 units is e_1 times as large, or $ee_1/\mu r^2$. Now the unit charge has been defined as the charge which repels an equal charge placed at the unit of distance from itself with a unit of force. We accordingly see that the magnitude of the unit will

depend on the medium, and if the experiment be made in a medium of inductivity μ the unit thus obtained will be larger in the ratio $\sqrt{\mu}$ than if it had been determined in a medium of unit inductivity. We also see that the dimensions of the unit involve μ, for we must have the dimensional equation

$$\frac{[e^2]}{[\mu L^2]} = \left[\frac{ML}{T^2}\right],$$

so that

$$[e] = [\mu^{\frac{1}{2}} M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}].$$

It is customary to choose the unit of inductivity so that the inductivity of empty space is unity, or as it is sometimes stated, the inductivity of the ether is unity. This is, as we have seen, purely arbitrary, as experiments enable us to determine only *ratios* of inductivities. The inductivity of air, both electric and magnetic, differs very little from that of a vacuum, so that for practical purposes we may consider the size of the units determined by experiments in air. We must notice that even if μ is put equal to unity its *dimensions* remain in the equation and the dimensions of μ we have no means of knowing. As the matter of dimensions is always more or less arbitrary, we may make any supposition that we please, until we are led to contradictory results. Two different suppositions are convenient. We may, when dealing with electrical quantities, assume that the dimensions of the electrical inductivity are zero. This gives the electrostatic system of units. We may on the other hand, when dealing with magnetic quantities, assume that the dimensions of the magnetic inductivity are zero. This gives the magnetic system. Both these systems are due to Gauss, and when we use both systems for their respective kinds of quantities, we shall say that the quantities are measured in *Gaussian* units. This has been the case in the preceding chapters. When we come to deal with both electrical and magnetic quantities at the same time, we must choose one or the other of these assumptions, as we shall find in the next chapter that both together are incompatible.

191. Susceptibility. The equation giving the apparent polarization of a medium of inductivity μ_2 surrounded by a medium of inductivity μ_1 is, (§ 186 (30))

$$I = \frac{1}{4\pi}\left(\frac{\mu_2 - \mu_1}{\mu_1}\right) F,$$

so that the polarization is proportional to the total force of the field, that is the sum of the external field and the field of force due to the polarization. The coefficient

$$\kappa_{21} = \frac{1}{4\pi} \frac{\mu_2 - \mu_1}{\mu_1},$$

is called the magnetic susceptibility, in the magnetic case. In the electric case the quantity κ has never come into practical use. The equation

$$I = \kappa F,$$

was the basis of Poisson's theory of magnetic induction, κ being supposed a quantity inherent in the body, and equal to zero for air. We see however that κ depends on the medium by which the body is surrounded, as well as on the body itself: κ_{21} may therefore be called the relative susceptibility of the body of inductivity μ_2 in a medium of susceptibility μ_1. If κ_{21} is positive, the polarization is in the direction of the polarizing force, and the body is said to be paramagnetic, or simply magnetic. If κ_{21} is negative, the polarization is in the opposite direction to the force, and the body is said to be diamagnetic. Accordingly any body immersed in a medium of greater inductivity than its own will appear diamagnetic. If we consider always the polarization of a body with respect to a vacuum, so that $\mu_1 = 1$, we may put

$$\kappa = \frac{1}{4\pi}(\mu - 1),$$

$$\mu = 1 + 4\pi\kappa.$$

Bodies are accordingly magnetic or diamagnetic as μ is greater or less than unity. It is evident that the assumption that κ is zero for a vacuum is arbitrary, in the same degree as, but independently of the assumption that the inductivity of vacuum is unity, for we might assume all apparent polarizations to be the differences of the polarizations of bodies from the polarizations of vacuum.

192. **Uniform Polarization by Induction.** When a body of different inductivity from the rest of the medium is inserted into a field of force, the configuration of the field is disturbed owing to induction, the polarization due to which produces new forces F_i which must be added to the forces of the undisturbed

field F_0. We shall now examine in what cases the introduction of a polarizable body into a *uniform* field will produce such a resultant field that the polarization of the body will be uniform. Let the potential of the undisturbed field be V_0 and the potential of the forces due to the induced polarization be V_i, so that the total potential of the field is

$$(1)\qquad V = V_0 + V_i.$$

If X_0, Y_0, Z_0 denote the constant components of the force of the undisturbed field, we have

$$(2)\qquad V_0 = C - X_0 x - Y_0 y - Z_0 z.$$

Let α, β, γ, be the constant direction cosines of the uniform polarization, so that

$$(3)\qquad A = I\alpha, \quad B = I\beta, \quad C = I\gamma.$$

Then since $I = \kappa F$ we must have for the total potential

$$(4)\qquad V = C' - Xx - Yy - Zz = C' - \frac{I}{\kappa}(\alpha x + \beta y + \gamma z).$$

But we have seen in § 123 (6), that if Ω be the potential of a single distribution of density unity occupying the space filled by the polarized body we have for the potential due to the polarization

$$(5)\qquad V_i = -I\frac{\partial \Omega}{\partial h} = -I\left\{\alpha\frac{\partial \Omega}{\partial x} + \beta\frac{\partial \Omega}{\partial y} + \gamma\frac{\partial \Omega}{\partial z}\right\}.$$

Consequently if we put for Ω

$$(6)\qquad \Omega = C'' - \tfrac{1}{2}\{Lx^2 + My^2 + Nz^2\},$$

so that

$$(7)\qquad V_i = I(L\alpha x + M\beta y + N\gamma z) = LAx + MBy + NCz,$$

$$(8)\qquad V = V_0 + V_i = C + (LA - X_0)x + (MB - Y_0)y + (NC - Z_0)z,$$

all our conditions will be fulfilled by taking

$$(9)\qquad LA - X_0 = -\frac{A}{\kappa}, \quad LB - Y_0 = -\frac{B}{\kappa}, \quad LC - Z_0 = -\frac{C}{\kappa}.$$

Now the only body for which Ω has the form of a quadratic function of the sort given is an ellipsoid. The values of the constants L, M, N in terms of the axes are given in § 113. We have accordingly found that an ellipsoid introduced into a uniform

field is uniformly polarized, and the field inside the ellipsoid is uniform. The field outside will not be uniform.

We obtain from equations (9) for the components of the polarization

$$A = \frac{\kappa X_0}{1 + \kappa L}, \quad B = \frac{\kappa Y_0}{1 + \kappa M}, \quad C = \frac{\kappa Z_0}{1 + \kappa N}, \tag{10}$$

and for the field in terms of the undisturbed field

$$X = \frac{X_0}{1 + \kappa L}, \quad Y = \frac{Y_0}{1 + \kappa M}, \quad Z = \frac{Z_0}{1 + \kappa N}. \tag{11}$$

The field in the polarized body is not in the direction of the undisturbed field, unless either $L = M = N$ (sphere), or the undisturbed field has the direction of one of the axes of the ellipsoid, when two of the components X_0, Y_0, Z_0 vanish. The force due to the polarization, F_i, has the components

$$\begin{aligned} X_i &= X - X_0 = -\frac{\kappa L}{1 + \kappa L} X_0, \\ Y_i &= Y - Y_0 = -\frac{\kappa M}{1 + \kappa M} Y_0, \\ Z_i &= Z - Z_0 = -\frac{\kappa N}{1 + \kappa N} Z_0, \end{aligned} \tag{12}$$

If κ is positive, this force is in the opposite direction to the undisturbed force. If the body be magnetic, and hard, so that when the field F_0 is removed, the polarization is retained, the force F_i alone acts, and tends to produce a magnetization in the reverse direction, or to demagnetize the body. The force F_i is accordingly called the *self-demagnetizing force*, and since we may write

$$X_i = -LA, \quad Y_i = -MB, \quad Z_i = -NC, \tag{13}$$

L, M, N are called the self-demagnetizing factors for the three axes.

193. Couple experienced by Ellipsoid. The couple experienced per unit of volume has, by § 120 (5), the components

$$BZ_0 - CY_0, \quad CX_0 - AZ_0, \quad AY_0 - BX_0.$$

Suppose that the force of the undisturbed field is in one of the principal diametral planes of the ellipsoid, then the couple tends

to turn it about the axis perpendicular to this plane. Suppose the force is in the XY-plane, and makes an angle θ with the X-axis. Then

$$X_0 = F_0 \cos\theta, \quad Y_0 = F_0 \sin\theta, \quad Z_0 = 0,$$

and the couple about the Z-axis acting on the whole ellipsoid of volume $4\pi abc/3$,

$$\frac{4\pi abc}{3}\left\{\frac{\kappa}{1+\kappa L} - \frac{\kappa}{1+\kappa M}\right\} X_0 Y_0 = \frac{4\pi abc\kappa^2 (M-L)}{3(1+\kappa L)(1+\kappa M)} F_0^2 \sin\theta\cos\theta.$$

The values of L, M, N are by § 113,

$$2\pi abc \int_0^\infty \frac{du}{(q^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

where L, M, N are obtained by putting a, b, c respectively for q. Accordingly L, M, N are in the reverse order of magnitude from a, b, c. Consequently if $a > b$ and $\theta < \pi/2$ the couple is positive, or from X to Y, that is, the ellipsoid tends to turn its longer axis parallel to the field, whether κ is positive or negative. This is in contradiction to a statement frequently made, that diamagnetic bodies tend to set their longest dimension across the field. They do not do so if the field is *uniform*. If an ellipsoid be suspended by a torsion fibre in a magnetic field, the field will cause it to vibrate more rapidly when its long axis is parallel to the field, and more slowly when it is across the field, than it would do in the absence of the field. It is however extremely difficult, if not impossible, to obtain a magnetic field nearly enough uniform to show these phenomena in diamagnetic bodies, on account of the extreme smallness of κ^2.

194. Polarization of Sphere. In the case of a sphere for inside points we have by § 80,

$$\text{(1)} \qquad \Omega = 2\pi\left(R^2 - \frac{r^2}{3}\right) = 2\pi R^2 - \frac{2\pi}{3}(x^2+y^2+z^2),$$

so that

$$\text{(2)} \qquad L = M = N = \frac{4\pi}{3},$$

which is the self-demagnetizing factor.

Accordingly the force is in the direction of the original field,

$$\text{(3)} \qquad F = \frac{F_0}{1+\frac{4\pi}{3}\kappa} = \frac{F_0}{1+\frac{\mu-\mu_1}{3\mu_1}} = \frac{3\mu_1}{\mu+2\mu_1} F_0,$$

and the polarization is

(4) $$I = \kappa F = \frac{F_0}{\frac{1}{\kappa} + \frac{4\pi}{3}} = \frac{F_0}{\frac{4\pi\mu_1}{\mu - \mu_1} + \frac{4\pi}{3}} = \frac{3(\mu - \mu_1)}{4\pi(\mu + 2\mu_1)} F_0.$$

The self-demagnetizing force is

(5) $$F_i = F - F_0 = \frac{\mu_1 - \mu}{\mu + 2\mu_1} F_0 = -\frac{4\pi}{3} I.$$

If μ be infinite this becomes equal to $-F_0$, so that the total force inside the sphere is zero. This is the case for a conducting sphere in an electric field, and is nearly the case for soft iron in a magnetic field.

Outside the sphere we have a different form for Ω,

(6) $$\Omega = \frac{4\pi}{3} \frac{R^3}{r},$$

so that

(7) $$V_i = -I \frac{\partial \Omega}{\partial h} = \frac{4\pi R^3 I}{3} \frac{\cos(hr)}{r^2}.$$

The field due to the polarization is accordingly, by § 123 (7), the same as the field of a doublet of moment $4\pi R^3 I/3 = -R^3 F_i$, and the total field outside the sphere is obtained by superposing this upon the uniform field F_0. If $\mu = \infty$ the moment of the sphere is $R^3 F_0$.

The lines of force due to a uniform field disturbed by a doublet

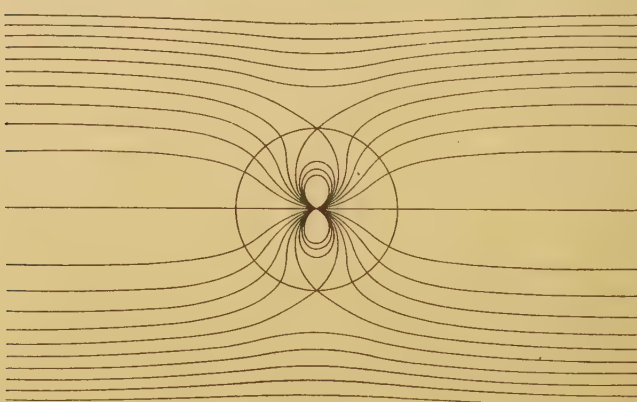

FIG. 74.

pointing in its own direction or the opposite are shown in Figs. 74 and 75 respectively.

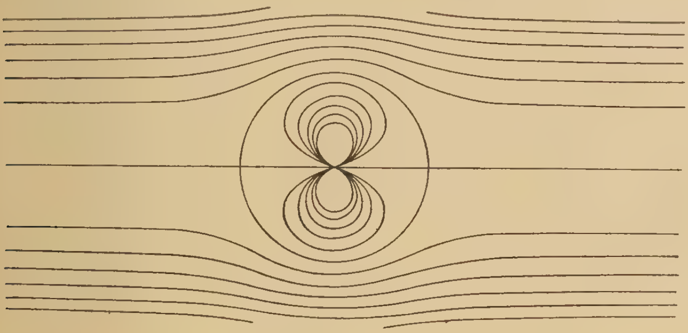

FIG. 75.

The field of the sphere in a uniform field is shown for $\mu/\mu_1 = 3$ and $\mu/\mu_1 = \infty$ respectively in Figs. 76 and 77. These figures were originally given in Lord Kelvin's Reprint of Papers on Electrostatics and Magnetism, (p. 492), where the equations of the curves are discussed. The figures have been re-drawn for this book, the lines being drawn for equal increments of the flux-function Ψ, or μ, § 103 (10).

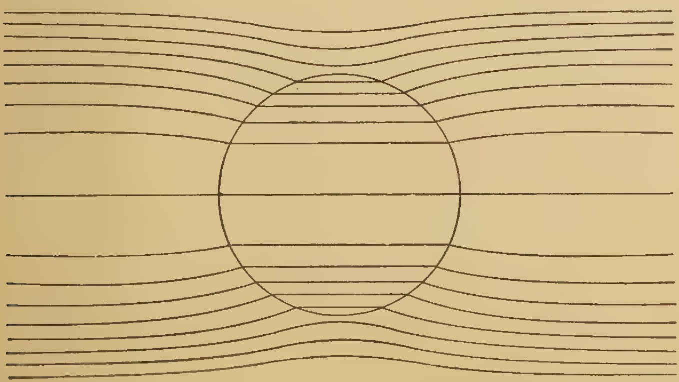

FIG. 76.

195. Infinite Elliptic Cylinder. If one of the axes of the ellipsoid is infinite, we have the case of an infinite elliptic cylinder. If $c = \infty$, N is zero, and L, M, reduce to trigonometric forms. The force parallel to the Z-axis is the same as that of the undisturbed

field. This is a consequence of the distributions on the ends being infinitely distant. We may then measure κ as the ratio of

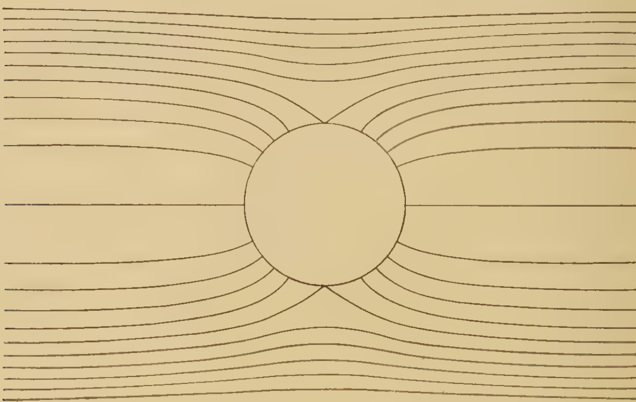

FIG. 77.

the longitudinal magnetization to force, a method often used in practice, and accurate only when the cylinder is extremely long.

196. Ellipsoid of Revolution. In the case of an ellipsoid of revolution the form of the integrals simplifies, and inserting in the formulæ of § 116 the eccentricity $e = \sqrt{a^2 - b^2}/a$ we obtain for a prolate ellipsoid

(1) $$L = 4\pi \frac{1-e^2}{e^2} \left\{ \frac{1}{2e} \log \frac{1+e}{1-e} - 1 \right\},$$

(2) $$M = N = 2\pi \left\{ \frac{1}{e^2} - \frac{1-e^2}{2e^2} \log \frac{1+e}{1-e} \right\},$$

and for an oblate ellipsoid

(3) $$L = 4\pi \left\{ \frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3} \sin^{-1} e \right\},$$

(4) $$M = N = 2\pi \left\{ \frac{\sqrt{1-e^2}}{e^3} \sin^{-1} e - \frac{1-e^2}{e^2} \right\}.$$

For $e = 0$ all these expressions become indeterminate, but on evaluating the indeterminate form they take the common value already found for the sphere. For $e = 1$ the expressions for the prolate ellipsoid become indeterminate, and on evaluation we find

$$L = 0, \quad M = N = 2\pi.$$

This gives the case of the infinitely long circulaı cylinder, for which, as we have previously found, the longitudinal demagnetizing factor vanishes, while for transverse magnetization it is equal to 2π.

When $e=1$ the expressions for the oblate ellipsoid give $L=4\pi$, $M=N=0$. This gives us the case of a disk magnetized normally, for which the demagnetizing factor is the largest possible, namely 4π, or parallel to the faces, when the demagnetizing factor is zero.

For a long prolate ellipsoid, for which e is nearly unity, we may conveniently use an approximate formula. Putting $m=a/b$ for the ratio of the length to the diameter, since

$$1-e^2=\frac{b^2}{a^2}=\frac{1}{m^2},$$

we have approximately

$$(5)\qquad L=\frac{4\pi}{m^2-1}\left\{\frac{m}{\sqrt{m^2-1}}\log\left(m+\sqrt{m^2-1}\right)-1\right\},$$

$$(6)\qquad M=N=\frac{4\pi}{m^2}(\log 2m-1).$$

A table of values of the demagnetizing factor is given by Ewing*, and a larger one by du Bois†.

197. Magnetization of Hollow Cylinder. We shall now consider a few cases of induction in which the induced magnetization is not uniform. In the first case let us consider the uniplanar problem of the transverse magnetization of an infinite homogeneous circular cylinder, placed in a field such that the lines of force are the intersections of cylindrical surfaces with planes perpendicular to the generators of the cylinder. If the cylinder is circular the method of development in series of circular harmonics, § 94, gives the general solution of the problem.

Let the cylinder be hollow, the inner radius being b and the outer a, the inductivity of the cylinder being μ_2, and of the space within and without μ_1. Let the undisturbed field, as before, be represented by F_0 with potential V_0, while the field due to the induced polarization is F_i with the potential V_i. We shall suppose that the bodies producing the field lie outside the

* Ewing, "Magnetic Induction in Iron and other Metals," p. 32.

† du Bois, "Magnetische Kreise, deren Theorie und Anwendung," p. 45.

cylinder, so that the potential V_0 and its derivatives are finite and continuous at the surfaces of the cylinder.

Let it be developed at the outer surface in the infinite series of harmonics

(1) $$V_0 = T_0(\phi) + T_1(\phi) + T_2(\phi) + \ldots\ldots$$

Then at points for which $\rho < a$ it is given by the series,

(2) $$V_0 = T_0 + \frac{\rho}{a} T_1 + \frac{\rho^2}{a^2} T_2 + \ldots\ldots$$

The potential V_i is represented by three different developments in the three different regions, (1) $\rho > a$, (2) $a > \rho > b$, and (3) $\rho < b$. We will distinguish these by an affix. Since V_i vanishes at infinity, we have outside the cylinder

(3) $$V_i^{(1)} = \sum_0^\infty A_n \rho^{-n} T_n.$$

In the substance of the cylinder we must take

(4) $$V_i^{(2)} = \sum_0^\infty (B_n \rho^n + C_n \rho^{-n}) T_n,$$

while in the cavity, since V_i is finite at the center,

(5) $$V_i^{(3)} = \sum_0^\infty D_n \rho^n T_n.$$

Since V_i is continuous, at the surface $\rho = a$ we have $V_i^{(1)} = V_i^{(2)}$, and as this must be identically true for all values of ϕ we must have for every term the coefficients of T_n equal.

(6) $$A_n a^{-n} = B_n a^n + C_n a^{-n}.$$

In like manner, at the surface $\rho = b$, we have for every term,

(7) $$D_n b^n = B_n b^n + C_n b^{-n}.$$

Beside the conditions of continuity, we have at each surface of the cylinder

(8) $$\mu_1 \frac{\partial V}{\partial n_1} + \mu_2 \frac{\partial V}{\partial n_2} = 0,$$

for the whole potential $V = V_0 + V_i$. The potential of the external field being continuous, with its derivatives as well, we have

(9) $$\frac{\partial V_0}{\partial n_1} + \frac{\partial V_0}{\partial n_2} = 0,$$

which being multiplied by μ_2 and subtracted from (8) gives

(10) $$\mu_1 \frac{\partial V_i}{\partial n_1} + \mu_2 \frac{\partial V_i}{\partial n_2} = (\mu_2 - \mu_1) \frac{\partial V_0}{\partial n_1}.$$

At the surface $\rho = a$ this gives, differentiating (2), (3), (4), by ρ,

(11) $$\mu_1 \sum_0^\infty (-A_n n a^{-(n+1)} T_n)$$
$$- \mu_2 \sum_0^\infty (B_n n a^{n-1} - C_n n a^{-(n+1)}) T_n = (\mu_2 - \mu_1) \sum_0^\infty n a^{-1} T_n,$$

and consequently for every n

(12) $$-\mu_1 A_n a^{-n} - \mu_2 (B_n a^n - C_n a^{-n}) = (\mu_2 - \mu_1),$$

and at $\rho = b$,

(13) $$-\mu_1 \sum_0^\infty D_n n b^{n-1} T_n$$
$$+ \mu_2 \sum_0^\infty (B_n n b^{n-1} - C_n n b^{-(n+1)}) = -(\mu_2 - \mu_1) \sum_0^\infty \frac{n b^{n-1}}{a^n} T_n,$$

and consequently

(14) $$-\mu_1 D_n b^n + \mu_2 (B_n b^n - C_n b^{-n}) = -(\mu_2 - \mu_1) \left(\frac{b}{a}\right)^n.$$

The four linear equations (6), (7), (12), and (14) determine the four constants A_n, B_n, C_n, D_n.

Solving, we obtain for their values, putting

$$\frac{\mu_2 + \mu_1}{\mu_2 - \mu_1} = M, \quad N_n = \frac{1}{M^2 - \left(\frac{b}{a}\right)^{2n}},$$

(15)
$$A_n = a^n M \left\{\left(\frac{b}{a}\right)^{2n} - 1\right\} N_n,$$
$$B_n = a^{-n} \left\{\left(\frac{b}{a}\right)^{2n} - M\right\} N_n,$$
$$C_n = b^{2n} a^{-n} (M - 1) N_n,$$
$$D_n = a^{-n} \left\{\left(\frac{b}{a}\right)^{2n} - 1\right\} N_n.$$

Since the absolute value of M is greater than 1, and since $b/a < 1$, N_n is always positive, and accordingly D_n is always negative. Accordingly the effect of the induced polarization

within the hollow is to oppose the external field. We have for the whole potential in the cavity, by (2), (5), and (15)

$$(16) \qquad V^{(3)} = V_0^{(3)} + V_i^{(3)} = \sum_0^\infty \left(\frac{\rho}{a}\right)^n \left(1 - \frac{1 - \left(\frac{b}{a}\right)^{2n}}{M^2 - \left(\frac{b}{a}\right)^{2n}}\right) T_n.$$

The absolute value of the coefficient of any T_n is less, the smaller the ratio b/a, that is, the thicker the walls of the cylinder. By making the walls thick enough, we can make the coefficient approach as nearly as we please the value $1 - 1/M^2$. Now this is smaller, the greater the ratio μ_2/μ_1. For μ_2/μ_1 infinite the internal field would be reduced to zero, as in the case of the sphere.

This principle was used by Lord Kelvin in his marine galvanometer, in which a thick cylinder of iron shields the galvanometer from the influence of external magnetic fields. Such an arrangement is now nearly always necessary to protect magnetic instruments from the field of electric railroads (in America).

In case the external field is uniform, the internal field is also. We then have, if F_0 is the strength of the external field,

$$V_0 = F_0 \rho \cos\phi = \frac{\rho}{a} T_1,$$

$$(17) \qquad V^{(3)} = \left(1 - \frac{1 - \left(\frac{b}{a}\right)^2}{M^2 - \left(\frac{b}{a}\right)^2}\right) F_0 \rho \cos\phi.$$

In this case if $\mu_2/\mu_1 = 1000$, and if the thickness of the cylinder is only one-tenth of its outer diameter, the field within is reduced to two per cent. of the value outside. The effectiveness of the shielding is thus plainly shown.

The total field is shown for this case in Fig. 78, for which

$$b/a = 2/3, \quad \mu_2/\mu_1 = 10.$$

This represents approximately the distribution of the lines of force in a ring-armature of a dynamo.

198. Magnetization of Hollow Sphere. The case of the sphere may be treated in precisely the same manner as the case of

the cylinder, substituting spherical for circular harmonics. Let us

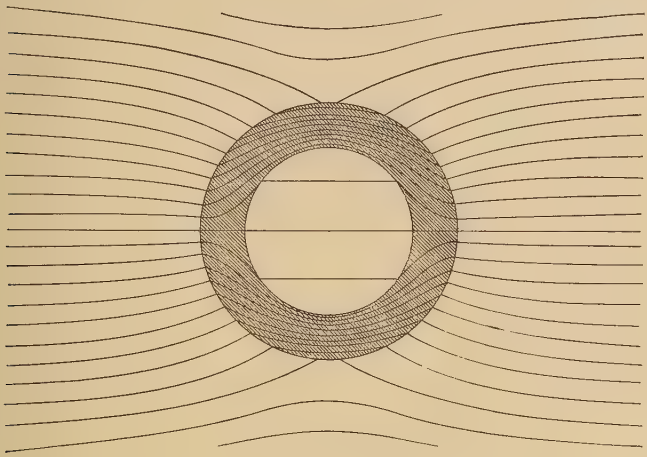

FIG. 78.

again suppose that the bodies producing the undisturbed field are outside, so that at the outer surface of the sphere

(1) $$V_0 = Y_0(\theta, \phi) + Y_1(\theta, \phi) + Y_2(\theta, \phi) + \dots.$$

Accordingly for $r < a$,

(2) $$V_0 = Y_0 + \frac{r}{a} Y_1 + \frac{r^2}{a^2} Y_2 + \dots.$$

The potential of the induced polarization is given by

(3) $$V_i^{(1)} = \sum_0^\infty A_n r^{-(n+1)} Y_n, \quad r > a,$$

(4) $$V_i^{(2)} = \sum_0^\infty (B_n r^n + C_n r^{-(n+1)}) Y_n, \quad a > r > b,$$

(5) $$V_i^{(3)} = \sum_0^\infty D_n r^n Y_n, \quad r < b.$$

The conditions of continuity of V_i give as before

(6) $$A_n a^{-(n+1)} = B_n a^n + C_n a^{-(n+1)},$$

(7) $$D_n b^n \qquad = B_n b^n + C_n b^{-(n+1)}.$$

The condition (10) of the preceding section gives the corresponding equations,

$$(11)\quad \mu_1 \sum_0^\infty \{-A_n(n+1)a^{-(n+2)}\} Y_n$$
$$- \mu_2 \sum_0^\infty \{B_n n a^{n-1} - C_n(n+1)a^{-(n+2)}\} Y_n = (\mu_2 - \mu_1) \sum_0 n a^{-1} Y_n,$$

$$(12)\quad -\mu_1 A_n (n+1) a^{-(n+1)}$$
$$- \mu_2 \{B_n n a^n - C_n (n+1) a^{-(n+1)}\} = (\mu_2 - \mu_1) n,$$

$$(13)\quad -\mu_1 \sum_0^\infty D_n n b^{n-1} Y_n$$
$$+ \mu_2 \sum_0^\infty \{B_n n b^{n-1} - C_n (n+1) b^{-(n+2)}\} Y_n = -(\mu_2 - \mu_1) \sum_0^\infty n \frac{b^{n-1}}{a^n} Y_n,$$

$$(14)\quad -\mu_1 D_n n b^n + \mu_2 \{B_n n b^n - C_n (n+1) b^{-(n+1)}\}$$
$$= -(\mu_2 - \mu_1) n \left(\frac{b}{a}\right)^n.$$

Solving the equations (6), (7), (12), (14), we get, putting

$$M_n = \frac{(n+1)\mu_2 + n\mu_1}{(n+1)(\mu_2 - \mu_1)}, \quad M_n' = \frac{n\mu_2 + (n+1)\mu_1}{n(\mu_2 - \mu_1)},$$

$$N_n = \frac{1}{MM' - \left(\frac{b}{a}\right)^{2n+1}}$$

$$(15)\quad \begin{aligned} A_n &= a^{n+1} M_n \left\{\left(\frac{b}{a}\right)^{2n+1} - 1\right\} N_n, \\ B_n &= a^{-n} \left\{\left(\frac{b}{a}\right)^{2n+1} - M_n\right\} N_n, \\ C_n &= \frac{2n+1}{n+1} \frac{\mu_1}{\mu_2 - \mu_1} a^{-n} b^{2n+1} N_n, \\ D_n &= -a^{-n} \left\{1 - \left(\frac{b}{a}\right)^{2n+1}\right\} N_n. \end{aligned}$$

The results are similar to those in the last section. Since M_n and M_n' have the same sign, and are greater in absolute value than unity, N_n is positive, and D_n is negative. The field in the cavity is given by

$$(16)\quad V^{(3)} = V_0^{(3)} + V_i^{(3)} = \sum_0^\infty \left(\frac{r}{a}\right)^n \left(1 - \frac{1 - \left(\frac{b}{a}\right)^{2n+1}}{M_n M_n' - \left(\frac{b}{a}\right)^{2n+1}}\right) Y_n.$$

For a uniform external field

$$V_0 = F_0 r \cos\theta = \frac{r}{a} Y_1,$$

the internal field is uniform, and given by

$$V_3 = \left(1 - \frac{1 - \left(\frac{b}{a}\right)^3}{M_1 M_1' - \left(\frac{b}{a}\right)^3}\right) F_0 r \cos\theta.$$

If $b/a = 9/10$, $\mu_2/\mu_1 = 1000$, the internal field is 1/67 of the external.

The sphere thus shields more effectively than the cylinder, as might be expected. A table of the relative strengths of the internal field for various ratios of b/a and for $\mu_2/\mu_1 = 100$ and $= 1000$, is given by J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, p. 264.

If $b = 0$ the results of this section agree with those of § 194. For instance in the sphere

$$V_2 = \left(1 - \frac{M_n}{M_n M_n'}\right) F_0 r \cos\theta = \frac{3\mu_1}{\mu_2 + 2\mu_1} F_0 r \cos\theta,$$

agreeing with § 194 (3).

199. Forces acting on the Polarized Body. In virtue of the polarization of a body whose inductivity differs from that of the surrounding medium the body experiences certain forces. These forces may be calculated by considering the work done when the induced body is moved from one part of the field to another, during which motion its polarization will in general change.

Before considering the general problem, let us, to fix the ideas, examine the case of an electrical condenser. We have seen that the capacity is proportional to the inductivity of the dielectric. Accordingly for a given charge, the difference of potential of the plates is inversely as the inductivity, consequently the force of the field varies in the same ratio. The energy being proportional to the product of the charge by the difference of potential is accordingly inversely proportional to the inductivity. Now since, the charge being given, the energy tends to decrease, if the dielectric is movable, and its inductivity variable, it will tend to

move so that portions of greater inductivity shall be drawn into the field. If on the other hand the potential of the condenser plates is maintained constant, the charge is directly proportional to the inductivity, so that the energy is also directly proportional.

We have seen in § 140 that in this case the energy tends to increase, so that again the forces tend to bring substance of greater inductivity into the field. These properties of the energy should not be confused with the maximum property mentioned in §§ 119, 180, for there the variation was in V, tending to make it differ from the values necessitated by the differential equation

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right)=-4\pi\rho;$$

μ being unvaried. Here the variation is in μ, and may be made to depend on geometrical parameters fixing the position of the polarizable bodies, precisely as in § 140 we had changes in geometrical parameters, and in this case the variations of V must be such as are *consistent* with the above differential equation.

We may look at the matter from a slightly different point of view. Since we found in § 140 that the capacity tends to increase when the forces of the system produce motion, the system will move so as to increase μ. The capacity will be increased when a body of greater inductivity moves into stronger parts of the field, consequently magnetic bodies are drawn into the strong parts of the field, while diamagnetic bodies are repelled from the stronger portions to the weaker portions. This property was correctly stated by Faraday, and was demonstrated by Lord Kelvin. It is this tendency of diamagnetic bodies to move to the weaker parts of the field that often makes them set themselves across the field, instead of along it as they should do in a *uniform* field.

We may calculate the mechanical forces experienced by unit of volume of a substance by the proposition that the work done by the forces in a displacement is equal to the loss of energy of the system. Let us call the force per unit volume Ξ, H, Z. Then if a body is displaced so that a point x, y, z comes into the position $x+\delta x$, $y+\delta y$, $z+\delta z$, and the corresponding change in W be δW, we have

(1) $$\delta W=-\iiint_{\infty}(\Xi\delta x+\mathrm{H}\delta y+\mathrm{Z}\delta z)\,d\tau.$$

During the displacement the distribution of the field varies, but if we use the form of W given in § 119, namely,

$$W = 2W_d - W_f,$$

we have the important simplification that $\delta_V W = 0$*, for after the motion as well as before the potential satisfies the conditions § 180 (8) and (9). Accordingly in considering the variation δW we have to consider only the variation of ρ and μ produced by the motion, the change of W in other ways being taken account of in the condition $\delta_V W = 0$. The change in ρ at any point is caused by matter differently charged coming to the point, and we find as in § 38, putting $dm = \rho d\tau$,

$$(2) \qquad \delta\rho = -\left\{\frac{\partial(\rho\delta x)}{\partial x} + \frac{\partial(\rho\delta y)}{\partial y} + \frac{\partial(\rho\delta z)}{\partial z}\right\}.$$

In like manner μ has changed to the value it formerly had at the point $x - \delta x$, $y - \delta y$, $z - \delta z$, which has moved to x, y, z, so that

$$(3) \qquad \delta\mu = -\left\{\frac{\partial\mu}{\partial x}\delta x + \frac{\partial\mu}{\partial y}\delta y + \frac{\partial\mu}{\partial z}\delta z\right\}.$$

Accordingly (considering surface distributions as a limiting case of volume distributions) since we have

$$0 = \delta_V W = \iiint_\infty \delta V \,.\, \rho d\tau$$

$$- \frac{1}{4\pi}\iiint_\infty \mu \left\{\frac{\partial V}{\partial x}\frac{\partial\delta V}{\partial x} + \frac{\partial V}{\partial y}\frac{\partial\delta V}{\partial y} + \frac{\partial V}{\partial z}\frac{\partial\delta V}{\partial z}\right\} d\tau,$$

we obtain the change in W as

$$(4) \qquad \delta W = -\iiint_\infty V\left\{\frac{\partial(\rho\delta x)}{\partial x} + \frac{\partial(\rho\delta y)}{\partial y} + \frac{\partial(\rho\delta z)}{\partial z}\right\} d\tau$$

$$+ \frac{1}{8\pi}\iiint_\infty \left\{\frac{\partial\mu}{\partial x}\delta x + \frac{\partial\mu}{\partial y}\delta y + \frac{\partial\mu}{\partial z}\delta z\right\}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau,$$

and integrating the first integral by parts, the surface integral vanishing at infinity,

$$(5) \qquad \delta W = \iiint_\infty \rho\left\{\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\right\} d\tau$$

$$+ \frac{1}{8\pi}\iiint_\infty \left\{\frac{\partial\mu}{\partial x}\delta x + \frac{\partial\mu}{\partial y}\delta y + \frac{\partial\mu}{\partial z}\delta z\right\}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} d\tau.$$

* (To the first order.)

Since this is to be equal to

$$-\iiint_\infty (\Xi\delta x + \mathrm{H}\delta y + \mathrm{Z}\delta z)\, d\tau$$

for arbitrary values of δx, δy, δz we must have everywhere

$$\begin{aligned}
\Xi &= -\rho\frac{\partial V}{\partial x} - \frac{1}{8\pi}\frac{\partial\mu}{\partial x}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\},\\
\mathrm{H} &= -\rho\frac{\partial V}{\partial y} - \frac{1}{8\pi}\frac{\partial\mu}{\partial y}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\},\\
\mathrm{Z} &= -\rho\frac{\partial V}{\partial z} - \frac{1}{8\pi}\frac{\partial\mu}{\partial z}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\}.
\end{aligned} \tag{6}$$

The first term is the force exerted by the field on the real charge at each point. It is of course in the direction of the field. The second term is due to the polarization, and is proportional to the *square* of the field strength, and in the direction of fastest decrease of μ. That is, any point of a polarized body tends to move toward the side on which the inductivity is less, or to bring more inductive matter into the field, as stated above. The form of deduction here given is due to Helmholtz*.

200. Stresses in the Medium. The modern theory of electricity and magnetism, due to Faraday and Maxwell, assumes that bodies do not act directly on other bodies at a distance, but by means of actions transmitted through the intervening medium from particle to particle. The influence of the medium has been made apparent in this chapter, as we in fact started from the expression of the energy as being distributed in all space. It remains to find a system of stresses that shall account for the electrical or magnetic forces which have been here investigated.

If forces Ξ, H, Z per unit volume act on all portions of a body, for example gravity, these forces will throw the body into a state of strain, and in order to produce equilibrium the applied forces Ξ, H, Z must be balanced by a set of elastic stresses developed in the body. These are forces acting from point to point in the body, and may be specified as follows. Suppose at any point P the body divided into two portions, 1 and 2, by a plane whose normal is n. If we consider a small area dS of this plane containing the point P the portions of the body on the two sides of the plane

* *Wiss. Abh.* Bd. I., p. 811. See note in Appendix.

exert forces on each other, whose combined action may for either part be represented by a single resultant force applied to dS. Let the force acting through the area dS on the portion of the body 1 be denoted by $F_n dS$, F_n is not in general normal to dS, but has a tangential component. This tends to cause the two portions 1 and 2 to slide over each other, or to be *sheared*. The normal component of F_n, if directed toward the body 2, tends to make the two portions of the body approach each other, and is called a *traction* or *tension*, as in the case of a stretched rope. If the force F_n on 1 is directed toward 1, the force is called a *pressure*, as in the case of liquid pressure. A traction will be considered positive, that is the force acting *on* a portion of the body has a positive component along the normal drawn *outward* from that portion. We shall denote the components of F_n by X_n, Y_n, Z_n, the suffix n denoting the direction of the normal to the element of surface to which they are applied. If we consider three sides of an infinitesimal cube at any point, we may specify the stress at that point by giving the components of the stresses on each side, those on the side perpendicular to the X-axis being X_x, Y_x, Z_x, those on the side perpendicular to the Y-axis being X_y, Y_y, Z_y, and those on the face perpendicular to the Z-axis being X_z, Y_z, Z_z. If we consider the equilibrium of an infinitesimal tetrahedron formed by cutting off one corner of this cube by a plane whose normal is n (Fig. 80), the areas of its four faces being dS_x, dS_y, dS_z, dS_n, (the suffixes denoting their normals) and its volume being $d\tau$, we have for the equations of equilibrium, resolving along the three axes,

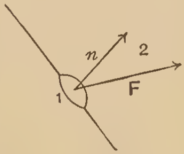

FIG. 79.

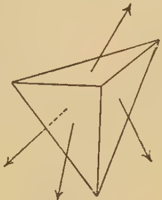
FIG. 80.

$$
(1)\qquad
\begin{aligned}
\Xi d\tau + X_x dS_x + X_y dS_y + X_z dS_z - X_n dS_n &= 0,\\
\mathrm{H} d\tau + Y_x dS_x + Y_y dS_y + Y_z dS_z - Y_n dS_n &= 0,\\
\mathrm{Z} d\tau + Z_x dS_x + Z_y dS_y + Z_z dS_z - Z_n dS_n &= 0.
\end{aligned}
$$

Now the faces dS_x, dS_y, dS_z are the projections of the face dS_n on the coordinate planes, and accordingly

$$
\begin{aligned}
dS_x &= dS_n \cos(nx),\\
dS_y &= dS_n \cos(ny),\\
dS_z &= dS_n \cos(nz).
\end{aligned}
$$

If we now let the dimensions of the tetrahedron diminish indefinitely, the volume $d\tau$ is of a higher order than the surface of any face and can accordingly be neglected, accordingly the equations of equilibrium become, dS_n dividing out,

$$(2)\qquad \begin{aligned} X_n &= X_x \cos(nx) + X_y \cos(ny) + X_z \cos(nz), \\ Y_n &= Y_x \cos(nx) + Y_y \cos(ny) + Y_z \cos(nz), \\ Z_n &= Z_x \cos(nx) + Z_y \cos(ny) + Z_z \cos(nz), \end{aligned}$$

which proves the statement that the stress at any point, involving the action on a plane element in *any* direction at the point, may be expressed in terms of the nine components at the point,

$$X_x,\ Y_x,\ Z_x,\ X_y,\ Y_y,\ Z_y,\ X_z,\ Y_z,\ Z_z.$$

Let us now consider the condition of any finite portion of matter τ. Let the body-forces Ξ, H, Z, per unit of volume be applied to each element. If now the forces X_n, Y_n, Z_n applied to each unit of surface are to produce the same effect as the given system of body forces, then the system of body forces with their signs reversed, together with the surface forces, would produce equilibrium. For equilibrium we must have, resolving in the X-direction,

$$(3)\qquad \iint X_n dS - \iiint \Xi\, d\tau = 0.$$

Let us now express X_n in terms of the nine components by the equations (2),

$$(4)\qquad \iint \{X_x \cos(n_e x) + X_y \cos(n_e y) + X_z \cos(n_e z)\}\, dS - \iiint \Xi\, d\tau = 0.$$

Transforming the surface integral into a volume integral we obtain

$$(5)\qquad \iiint \left\{ \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} - \Xi \right\} d\tau = 0,$$

and if every portion of the body is to remain in equilibrium under the stresses, in order that the integral shall vanish for every field of integration we must have everywhere

$$(6)\qquad \Xi = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}.$$

In like manner we find

$$(6') \qquad \begin{aligned} \mathrm{H} &= \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}, \\ \mathrm{Z} &= \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}, \end{aligned}$$

as the equations of equilibrium.

In order to explain electrical and magnetic forces by means of stresses we must therefore be able to transform the expressions already found for Ξ, H, Z, into forms involving partial derivatives as above.

Introducing into the expression for Ξ the value of ρ from

$$-\rho = \frac{1}{4\pi}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right)\right\},$$

and transforming the derivatives we obtain

$$(7) \qquad \begin{aligned} \Xi &= \frac{1}{4\pi}\frac{\partial V}{\partial x}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right)\right\} \\ &\quad - \frac{1}{8\pi}\frac{\partial \mu}{\partial x}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} \\ &= \frac{1}{8\pi}\frac{\partial}{\partial x}\left[\mu\left\{\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial y}\right)^2 - \left(\frac{\partial V}{\partial z}\right)^2\right\}\right] \\ &\quad + \frac{1}{4\pi}\frac{\partial}{\partial y}\left\{\mu\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}\right\} + \frac{1}{4\pi}\frac{\partial}{\partial z}\left\{\mu\frac{\partial V}{\partial x}\frac{\partial V}{\partial z}\right\}. \end{aligned}$$

The expression now has the required form of a sum of three derivatives. If we perform similar transformations on H and Z we shall find that the equations of equilibrium are satisfied by putting

$$(8) \qquad \begin{aligned} X_x &= \frac{\mu}{8\pi}\left\{\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial y}\right)^2 - \left(\frac{\partial V}{\partial z}\right)^2\right\} = \frac{1}{8\pi}\{2\mathfrak{X}X - \mathfrak{F}F\}, \\ Y_y &= \frac{\mu}{8\pi}\left\{-\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 - \left(\frac{\partial V}{\partial z}\right)^2\right\} = \frac{1}{8\pi}\{2\mathfrak{Y}Y - \mathfrak{F}F\}, \\ Z_z &= \frac{\mu}{8\pi}\left\{-\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\} = \frac{1}{8\pi}\{2\mathfrak{Z}Z - \mathfrak{F}F\}; \\ Y_z = Z_y &= \frac{\mu}{4\pi}\frac{\partial V}{\partial y}\frac{\partial V}{\partial z} = \frac{1}{4\pi}\mathfrak{Y}Z = \frac{1}{4\pi}\mathfrak{Z}Y, \\ Z_x = X_z &= \frac{\mu}{4\pi}\frac{\partial V}{\partial z}\frac{\partial V}{\partial x} = \frac{1}{4\pi}\mathfrak{Z}X = \frac{1}{4\pi}\mathfrak{X}Z, \\ X_y = Y_x &= \frac{\mu}{4\pi}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y} = \frac{1}{4\pi}\mathfrak{X}Y = \frac{1}{4\pi}\mathfrak{Y}X. \end{aligned}$$

Since $Y_z = Z_y$, etc., it is easy to see that the couple tending to turn any element of volume about either of the axes vanishes, as is the case with ordinary elastic stresses. If the body is not isotropic this condition does not hold.

We shall now apply the expressions found to determine the nature of the stress in two particular cases. First, let the element dS be perpendicular to a line of force. Then we have

$$\cos(nx) = \frac{X}{F}, \quad \cos(ny) = \frac{Y}{F}, \quad \cos(nz) = \frac{Z}{F},$$

and using these values in the equation (2)

$$X_n = X_x \frac{X}{F} + X_y \frac{Y}{F} + X_z \frac{Z}{F},$$

$$X_n = \frac{1}{8\pi}\{2\mathfrak{X}X - \mathfrak{F}F\}\frac{X}{F} + \frac{1}{4\pi}\frac{\mathfrak{X}Y^2}{F} + \frac{1}{4\pi}\frac{\mathfrak{X}Z^2}{F} = \frac{1}{8\pi}\mathfrak{F}X,$$

(9) $$Y_n = \frac{1}{4\pi}\frac{\mathfrak{Y}X^2}{F} + \frac{1}{8\pi}\{2\mathfrak{Y}Y - \mathfrak{F}F\}\frac{Y}{F} + \frac{1}{4\pi}\frac{\mathfrak{Y}Z^2}{F} = \frac{1}{8\pi}\mathfrak{F}Y,$$

$$Z_n = \frac{1}{4\pi}\frac{\mathfrak{Z}X^2}{F} + \frac{1}{4\pi}\frac{\mathfrak{Z}Y^2}{F} + \frac{1}{8\pi}\{2\mathfrak{Z}Z - \mathfrak{F}F\}\frac{Z}{F} = \frac{1}{8\pi}\mathfrak{F}Z.$$

These components of F_n are equal to $\mathfrak{F}F/8\pi$ multiplied by the direction cosines of F, which is in the direction of the normal n. That is the force F_n is perpendicular to its plane. A plane possessing this property is called a principal plane of the stress. The stress being positive represents a tension. Accordingly the medium is in a state of tension along the lines of force, of an amount per unit of surface equal to $\mathfrak{F}F/8\pi$, which, it may be noticed, is the amount of energy of the medium per unit volume.

Consider secondly an element tangent to a line of force. Then we have

$$\frac{X}{F}\cos(nx) + \frac{Y}{F}\cos(ny) + \frac{Z}{F}\cos(nz) = 0.$$

Multiplying this equation by $F\mathfrak{X}/4\pi$ and subtracting it from the expression for X_n gives

$$X_n = \frac{1}{8\pi}\{2\mathfrak{X}X - \mathfrak{F}F\}\cos(nx) + \frac{1}{4\pi}\mathfrak{X}Y\cos(ny) + \frac{1}{4\pi}\mathfrak{X}Z\cos(nz)$$

(10)

$$- \frac{1}{4\pi}\{\mathfrak{X}X\cos(nx) + \mathfrak{X}Y\cos(ny) + \mathfrak{X}Z\cos(nz)\} = -\frac{\mathfrak{F}F}{8\pi}\cos(nx).$$

In like manner

$$Y_n = -\frac{\mathfrak{F}F}{8\pi}\cos(ny), \quad Z_n = -\frac{\mathfrak{F}F}{8\pi}\cos(nz).$$

Here again the components of F_n are equal to $-\mathfrak{F}F/8\pi$ multiplied by the direction cosines of n, or the force is normal to its plane. Consequently any plane tangent to a line of force is a principal plane of the stress, and the stress is symmetrical about the line of force. The negative sign shows that the stress is a pressure. The state of stress consisting of tension along the lines of force combined with an equal pressure at right angles to them was described by Faraday*, who expressed the matter in words that state in effect that the lines of force tend to contract and to repel each other.

This may be illustrated by supposing the medium to be divided into filaments along the lines of force, and these again to be subdivided into short filaments. Then each short filament is a polarized body which acts like a doublet, and since unlike poles of successive elements are in juxtaposition, the filaments all attract each other endwise. For filaments lying side by side, however, since like poles are together, there is a sidewise repulsion.

Fig. 81.

201. Permanent Magnets and Electrets. Intrinsic Polarization. The fundamental laws of magnetic and electric induction may be summed up in the statement that in soft iron and in similarly acting bodies the force is lamellar, and μ times the force is solenoidal. Or in brief

(I) $$\text{curl } F = 0,$$

(I′) $$\text{div}(\mu F) = 0.$$

Iron for which this statement is true is said to be perfectly soft. When the external field affecting such iron is removed, the polarization disappears. As a matter of fact, this is an ideal condition not exactly realized by any sort of real iron, for when the external field is removed, a part of the polarization persists. This is called residual magnetization. The harder the iron or steel, the greater is the fraction of the induced polarization which

* Faraday, *Exp. Res.* (1297).

persists. A substance in which, when the external field is removed, the whole induced polarization remains, is called perfectly hard, and a body consisting of such substance is called a permanent magnet. The inductivity of such a body is to be considered the same as of air. Such bodies do not exist any more than perfectly soft ones. We may however treat actual bodies as if they were formed by the superposition of perfectly hard and perfectly soft matter. The portion of the polarization which permanently remains is called the *intrinsic* polarization*. In order to carry out the analogy, Heaviside has proposed to call a dielectric permanently polarized body an *electret*, and its polarization *electrization*. Certain natural crystals when heated assume this condition.

The permanent or intrinsic polarization now forms a *real* magnetic or electric charge, and if the intrinsic polarization be denoted by I_0 with components A_0, B_0, C_0, we have for the real density

(2) $$\rho = -\left\{\frac{\partial A_0}{\partial x} + \frac{\partial B_0}{\partial y} + \frac{\partial C_0}{\partial z}\right\},$$

with a similar expression for σ.

Comparing this with the expression for ρ in § 182 (15), we find

(3) $$\frac{1}{4\pi}\left\{\frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z}\right\} = -\left\{\frac{\partial A_0}{\partial x} + \frac{\partial B_0}{\partial y} + \frac{\partial C_0}{\partial z}\right\},$$

or the divergence of the induction is equal to 4π times the *convergence* of the intrinsic polarization. Comparing the expressions for the apparent density, that is the sum of the real and induced, in terms of the force F, § 186 (33), and in terms of the total polarization I, § 120 (6), we find

(4) $$\rho' = \frac{1}{4\pi}\left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} = -\left\{\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right\}.$$

Accordingly

(5) $$\frac{\partial}{\partial x}(X + 4\pi A) + \frac{\partial}{\partial y}(Y + 4\pi B) + \frac{\partial}{\partial z}(Z + 4\pi C) = 0,$$

or more briefly,

(5) $$\operatorname{div}(\overline{F} + 4\pi\overline{I}) = 0.$$

The solenoidal vector-sum, $\overline{F} + 4\pi\overline{I}$, has been called in § 121, the induction. We shall call it the Maxwellian induction, and denote it by $\mathfrak{F}_M$, since it corresponds to the definition of the

* Thomson. Reprint of Papers on Electrostatics and Magnetism, p. 578.

induction given by Maxwell. It is solenoidal in intrinsically magnetized bodies as well as elsewhere. The induction, $\mathfrak{F}$, which is divergent in intrinsically magnetized bodies, and which is defined as μF, we shall call the Hertzian induction, and denote by $\mathfrak{F}_H$. In magnetically soft bodies these two inductions are identical, but in intrinsically polarized bodies they differ.

If we write equation (3) as

(6) $$\operatorname{div}(\bar{\mathfrak{F}}_H + 4\pi \bar{I}_0) = 0,$$

and from it subtract

(5) $$\operatorname{div}(\bar{F} + 4\pi \bar{I}) = 0,$$

we have

(7) $$\operatorname{div}(\bar{\mathfrak{F}}_H - \bar{F}) = 4\pi \operatorname{div}(\bar{I} - \bar{I}_0).$$

Now if we call I_i the induced polarization, we have as always

(8) $$I_i = \kappa F = \frac{(\mu - 1)}{4\pi} F, \quad \bar{I} = \bar{I}_0 + \bar{I}_i.$$

Inserting these in (7)

$$\operatorname{div}(\bar{\mathfrak{F}}_H - \bar{F}) = 4\pi \operatorname{div} \bar{I}_i = \operatorname{div}\{(\mu - 1)\bar{F}\},$$

and transposing $\operatorname{div} \bar{F}$,

(9) $$\operatorname{div} \mathfrak{F}_H = \operatorname{div}(\mu F),$$

agreeing with the definition of $\mathfrak{F}_H$.

202. Heaviside's treatment of Intrinsic Polarization. The treatment given by Heaviside differs in several respects from that just given. According to that author the induction is always solenoidal, so that true magnetic charges do not exist. The only reason given for this assumption seems to the present writer insufficient, being, as stated by Heaviside, "to exclude unipolar magnets." It appears that the exclusion of unipolar magnets merely means that for any magnet the *integral charge* is zero,

$$\iint \sigma dS + \iiint \rho d\tau = 0,$$

which simply means that the distribution is what we have called polarization, and lays no restriction on the divergence of the polarization or induction. It might be supposed that Heaviside's induction was what is here called the Maxwellian induction, were it not for the fact that he says that "we use always" $\mathfrak{F} = \mu F$. In

order to make these two statements, which we hold to be mutually exclusive where there is intrinsic polarization, appear consistent, Heaviside proceeds in the following, as it appears to us, artificial manner. In our notation, Heaviside* considers the field F as made up of a part $h = 4\pi I_0/\mu$, defined as the *intrinsic force*, together with a lamellar force which we shall denote by F_h (F in Heaviside's notation),

$$F = h + F_h, \text{ curl } F_h = 0.$$

In order to make the induction solenoidal, he then puts

$$I_i = \kappa F_h,$$

instead of $I_i = \kappa F$. Then the induction is defined, not as

$$\mathfrak{F}_M = F + 4\pi I,$$

but as

$$\mathfrak{F} = F_h + 4\pi I.$$

Inserting for I the sum of the intrinsic and induced polarizations, this becomes

$$\begin{aligned}\mathfrak{F} &= F_h + 4\pi (I_0 + I_i) = F_h + \mu h + (\mu - 1) F_h \\ &= \mu F_h + \mu h = \mu F.\end{aligned}$$

This gives, in conjunction with the equation supposed to be fundamental, namely

$$\text{div } \mathfrak{F} = 0,$$

the equation

$$\frac{1}{4\pi} \text{div } (\mu F_h) = -\frac{1}{4\pi} \text{div } \mu h = -\text{div } I_0,$$

which we may compare with our equation (6). Accordingly Heaviside's μF_h has the property of our Hertzian induction. The difference in Heaviside's treatment may be summed up as: 1. A different definition of the total field. 2. Induced polarization produced by only a part of the field. 3. The Hertzian induction considered solenoidal, even in case there is intrinsic polarization.

We have stated the difficulties of Heaviside's treatment as they appear to us, without wishing to dispute the dicta of so weighty an authority. The theory as we have given it seems to be that of Helmholtz and Hertz, both of whom explicitly state that real magnetism exists in permanent magnets. Neither they, however, nor any other author, so far as known to the present writer, have

* Papers, Vol. I., pp. 453—4.

worked the matter out in detail as has Heaviside, nor have any problems been solved in which a difference becomes of importance. In either treatment, the flux of induction issuing from a magnet is the same, which is the quantity with which we are concerned in practice, the ambiguity existing only in the substance of the intrinsic magnet. The difference between intrinsic and other magnets is that in the former two *independent* vectors are necessary to characterize the state of the body, while in the latter one suffices. These may be taken as

$$\mathfrak{F}_H \text{ and } \mathfrak{F}_M, \text{ as } \mathfrak{F}_H, I, \text{ or as } \mathfrak{F}_M, I.$$

203. Variability of μ. Hysteresis. Throughout this chapter it has been assumed that the value of μ at any point was constant for that point. This assumption is not borne out by the facts, but was necessary in order to make the subject amenable to mathematical treatment. It is found that μ is a function of the strength of the field, and that for magnetic bodies, in which this phenomenon has been most carefully investigated, as the force increases, μ diminishes, finally tending towards the limit unity, so that the ratio of the induction to the force approaches unity. At the same time the difference between the induction and the force tends towards a constant maximum value, which is equal to 4π times the greatest intensity of magnetization that the substance can assume. This is known as the intensity of saturation. For wrought-iron this intensity of saturation has been found to be about 1700 C.G.S. units. The variability of μ does not affect the validity of Ohm's Law, which determines the distribution of the tubes of induction, although it seriously complicates the mathematical theory. In fact no cases of magnetization have been worked out taking account of the dependence of μ upon F. But this is not the only defect of our theory. It has been found that for a given value of F there is not a single determinate value of μ, but that the value depends not only on the *actual* value of F, but upon the values which have acted at the point in question at previous times. If we plot a curve having as abscissas the values of F at a given point at various times and as ordinates the values of $\mathfrak{F}$ at the corresponding times, we may express this phenomenon by saying that the value of μ at any point of the diagram depends on the path by which the substance has been brought to the point, that is, on the whole history of the field at the point. This

phenomenon, discovered by Warburg*, and thoroughly investigated by Ewing†, was named by the latter *Hysteresis*, to denote the after-effects of the fields to which the substance has been submitted. Warburg and Ewing found that if the field was increased to a certain value, then decreased, and then varied successively between the same limiting values, the path of the representative point on the F-$\mathfrak{F}$ diagram was a closed curve, which was re-traversed after the first periodic cycle. This is called the hysteresis-loop, and its area has an important physical significance. Such a loop is shown in Fig. 82. If instead of continuing to repeat the same cycle we vary F between different limits the point may take any position between the two limiting curves of the loop, as shown in Fig. 83, both these figures being copied from Ewing.

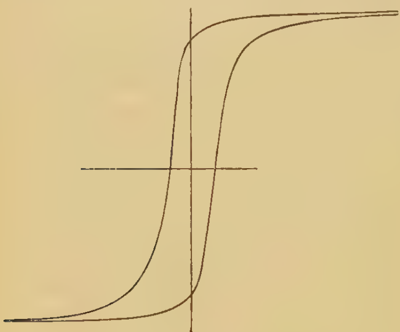

FIG. 82.

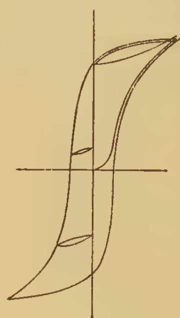

FIG. 83.

If the cycle be so chosen that at some point, F, while decreasing, passes through the value zero, the value of I calculated as the corresponding value of $\mathfrak{F}/4\pi$ is the residual magnetization. If the force F is still further decreased, its value when $I=0$, $\mathfrak{F}=F$, is called, after Hopkinson, the *coercive force*, since it measures the negative force necessary to destroy the residual magnetization.

Besides these phenomena of hysteresis, there is another more complicated effect, which causes the magnetization to arrive at its final value only gradually, taking a certain time to reach its permanent value. This is denoted by the name of viscous hysteresis, magnetic lag, or after-effect (Nachwirkung), to distinguish it from the proper or *static* hysteresis just described.

* Warburg, *Wied. Ann.* 13, p. 141, 1881.
† Ewing, *Phil. Trans.* CLXXVI., p. 523, 1885.

204. Dissipation of energy in Static Hysteresis. Since we have seen that μ is not uniquely determined by the value of F, so it must be for the energy of the field. Accordingly the forces acting on polarized bodies cannot be derived from a single-valued potential, but must be non-conservative. In taking a body through a cycle of magnetization, accordingly, a certain portion of the work done upon it fails to be stored up as energy, and is therefore dissipated into heat. We may easily find an expression for the value of this dissipated energy. The potential energy of a polarized body in a field whose potential is V is, by § 126 (2), equal to

$$W = \iiint \left(A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} + C\frac{\partial V}{\partial z} \right) d\tau,$$

or in terms of the field

$$W = -\iiint (AX + BY + CZ)\, d\tau.$$

If we consider an element of volume $d\tau$, and suppose it moved to a point where the field is

$$X + dX, \quad Y + dY, \quad Z + dZ,$$

the work dW done upon the particle during the motion is accordingly equal to the increase in the value of the energy,

(1) $$dW = -d\tau\,(A\,dX + B\,dY + C\,dZ).$$

In the second position the values of A, B, C have changed to the values

$$A + dA, \quad B + dB, \quad C + dC,$$

but the change made by using these values in the expression for the work would be of the second order and may be neglected. If instead of moving the particle we change the strength of the field the work done will be the same. Inserting the values of A, B, C in terms of the induction and force we obtain

(2) $$dW = -\frac{d\tau}{4\pi}\{(\mathfrak{X} - X)\,dX + (\mathfrak{Y} - Y)\,dY + (\mathfrak{Z} - Z)\,dZ\}.$$

If now we vary X, Y, Z through a cycle of values, coming back to the value from which we started, the integral

(3) $$\int X\,dX,$$

vanishes, since the value of X^2 at both limits is the same. The integral

$$-\int \mathfrak{X}\, dX$$

may be integrated by parts, giving

$$-\mathfrak{X}X\Big|_1^2 + \int X\, d\mathfrak{X}.$$

Of this the integrated part vanishes, since, as found by Warburg and Ewing, after the cycle has been once traversed $\mathfrak{X}$ returns to the same value on traversing the complete cycle. We thus find that in taking the particle through the whole cycle of magnetic operations, and leaving it in its original state, we have done a quantity of work, which is equal, not to zero, but to

$$\frac{d\tau}{4\pi}\int X\, d\mathfrak{X} + Y d\mathfrak{Y} + Z d\mathfrak{Z},$$

the integral being taken around a closed loop. Each term of the integral must of course be obtained from a separate loop. The whole energy dissipated in the body is

$$\frac{1}{4\pi}\iiint\left\{\int X\, d\mathfrak{X} + Y d\mathfrak{Y} + Z d\mathfrak{Z}\right\} d\tau.$$

Of course the general theory is so complicated that it is not even to be assumed that when we have carried the magnetization through a closed cycle in one point of the body we have done so at all points. In practice we can calculate the dissipation only in the case of a uniformly polarized body, where A, B, C are the same at all points of the body and in the direction of the force. The cycle is then the same for all points, and the energy dissipated is equal to

$$\text{vol. of body} \times \frac{1}{4\pi}\int F\, d\mathfrak{F}.$$

The integral

$$\int F\, d\mathfrak{F}$$

is evidently the area of the hysteresis-loop. This area is independent of the time of description of the cycle. In the case of viscous hysteresis there is an additional dissipation which depends in a complicated manner on the rate of description of the cycle.

205. Hysteresis couple. In the examples of §§ 192—198, it is evident that a sphere or cylinder turned about an axis of symmetry in the field would experience no resisting couple, for no work would be done against the forces of the field. In like manner an ellipsoid would require on the whole no work to rotate it about an axis, for the forces hindering the motion in one part of the revolution would have corresponding forces helping the motion in another part of the revolution. If hysteresis exists, however, the case is quite different. Then the ellipsoid in a position in which its long axis makes a *diminishing* acute angle with the direction of the field experiences a mechanical couple tending to accelerate its motion. The magnetic force parallel to the long axis is then increasing, so that when the force has reached the same value in the symmetrical position in which the axis makes the same angle with the direction of the field, but on the other side, F being then on the decreasing branch of the hysteresis-loop, the value of the magnetization is greater, so that the mechanical force, which now retards the motion, is greater. Accordingly the motion is on the whole retarded*, and it is easy to see that the mean retarding couple is proportional to the mean difference of the ordinates on the upper and lower branches of the loop, that is to the area of the loop. Upon this principle is based Ewing's Hysteresis indicator†, in which a long sample of iron is rapidly revolved between the poles of a magnet, and the mean couple between them measured by the pull on the magnet. The couple is, as seen above, independent of the time of revolution.

* An effect of this sort was observed in diamagnetic and very slightly magnetic bodies by Mr. A. P. Wills, in the physical laboratory of Clark University, in the fall of 1895, and was discovered independently by Mr. Wm. Duane, in the physical laboratory of the University of Berlin. *Wied. Ann.* Bd. 58, p. 517, 1896.

† Ewing, *Journ. Inst. Elec. Eng.* 24, p. 398, 1895.

CHAPTER X.

CONDUCTION IN DIELECTRICS.

206. **Variable Flow. Relaxation-Time.** We have hitherto supposed dielectrics to be perfect insulators. This can hardly be said to be the case, even for the best insulators. On the other hand, although, as we have seen, the greater the inductivity of a dielectric, the more nearly does it act, as far as concerns electrostatic distributions, like a conductor, it is by no means likely that the inductivity of conductors is infinite. Still less is it likely that it is zero. We shall now consider the consequences of considering a dielectric to possess, in addition to its electrical inductivity μ, an electric conductivity λ. We shall now deal with currents which are not in the steady state, and shall require to assume that at any instant Ohm's Law determines the distribution of the currents, namely

$$q = \lambda F.$$

This assumption is justified by experiment. Instead of the solenoidal condition for the current, however, we must obtain a new equation. This is obtained by the consideration that, if we consider a portion of substance τ bounded by a closed surface S, the total charge within that surface increases in any interval of time by the amount of total current flowing into τ through the surface, that is, if n is the internal normal

$$(1) \qquad \frac{d}{dt}\iiint \rho\, d\tau = \iint \{u \cos(nx) + v \cos(ny) + w \cos(nz)\}\, dS$$

$$= -\iiint \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right\} d\tau.$$

Since this equation must hold for any portion of space, we must have everywhere

$$(2) \qquad \frac{d\rho}{dt} = -\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right\}.$$

But in a dielectric,

$$\rho = \frac{1}{4\pi}\left\{\frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z}\right\}. \tag{3}$$

Differentiating (3) by t, and eliminating $\frac{d\rho}{dt}$ from (2), we obtain for a conducting dielectric

$$\frac{\partial}{\partial x}\left\{u + \frac{1}{4\pi}\frac{d\mathfrak{X}}{dt}\right\} + \frac{\partial}{\partial y}\left\{v + \frac{1}{4\pi}\frac{d\mathfrak{Y}}{dt}\right\} + \frac{\partial}{\partial z}\left\{w + \frac{1}{4\pi}\frac{d\mathfrak{Z}}{dt}\right\} = 0. \tag{4}$$

If we put u, v, w, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ in terms of the field, assuming that the substance is homogeneous as regards both μ and λ, this becomes

$$\frac{\partial}{\partial x}\left\{\lambda X + \frac{1}{4\pi}\mu\frac{dX}{dt}\right\} + \frac{\partial}{\partial y}\left\{\lambda Y + \frac{1}{4\pi}\mu\frac{dX}{dt}\right\} + \frac{\partial}{\partial z}\left\{\lambda Z + \frac{1}{4\pi}\mu\frac{dZ}{dt}\right\} = 0, \tag{5}$$

or in terms of the density

$$\frac{4\pi\lambda}{\mu}\rho + \frac{d\rho}{dt} = 0. \tag{6}$$

Integrating this differential equation, we have

$$\rho = \rho_0 e^{-\frac{4\pi\lambda}{\mu}t}. \tag{7}$$

Accordingly whatever charge the body has originally decreases in geometrical ratio as the time increases in arithmetical progression. The constant $T = \mu/4\pi\lambda$, which is the time it takes for the density at any point to fall to $1/e$ of its original value, has been called by Cohn* the relaxation-time, a term used by Maxwell in connection with the Kinetic Theory of Gases. For ordinary metallic conductors this time is so short as to have hitherto defied observation. The importance of its discovery was recognized by the committee setting subjects for an international prize competition in 1893, who proposed this as one of the questions for investigation†. It appeared that no experimenter ventured to attack the problem, it being evidently considered too difficult. The finite relaxation-time was determined for so good a conductor as water in some remarkable experiments by Cohn and Arons‡, who are entitled to the credit of discovering the finiteness of T for conductors.

* Cohn, *Wied. Ann.* 40, p. 625, 1890.

† Elihu Thomson Prize, *Electrician*, 1892.

‡ Cohn u. Arons. "Leitungsvermögen und Dielektricitätsconstante." *Wied. Ann.* 28, p. 454, 1886.

207. Method of Cohn and Arons. Consider a condenser A, which may or may not be connected in parallel with the condenser B and the resistance wire R. Let the capacity of A be K, the inductivity of the dielectric μ. Let the conductivity of the dielectric in A be λ and in B zero. Then the charge of one of the plates 1 of A is in terms of the induction, § 182 (16),

$$(8) \qquad e_1 = \frac{1}{4\pi}\iint_{S_1}(\mathfrak{X}\cos(nx) + \mathfrak{Y}\cos(ny) + \mathfrak{Z}\cos(nz))\,dS.$$

On the other hand the quantity flowing through the dielectric in the condenser in unit time is

$$(9) \qquad -\frac{de_1}{dt} = \iint_{S_1}\{u\cos(nx) + v\cos(ny) + w\cos(nz)\}\,dS,$$

so that, assuming λ and μ constant,

$$\frac{de_1}{dt} + \frac{4\pi\lambda}{\mu}e_1 = 0,$$

$$(10) \qquad e_1 = e_0 e^{-\frac{4\pi\lambda}{\mu}t}.$$

If we assume that an electromotive force is applied to the plates in order to establish a steady difference of potential V_0 until a steady state of flow is attained, we have everywhere in the dielectric $\rho = 0$. If the electromotive force is suddenly removed, we have from that time on

$$\rho = 0, \quad e_1 = e_0 e^{-\frac{4\pi\lambda}{\mu}t},$$

and accordingly the difference of potential of the condenser plates is

$$(11) \qquad V = V_0 e^{-\frac{4\pi\lambda}{\mu}t}.$$

If the difference of potential V can be measured by an electrometer at any time t, we have

$$(12) \qquad T = \frac{\mu}{4\pi\lambda} = \frac{t}{\log V_0 - \log V}.$$

If in the second place the condenser A is connected in parallel with the condenser B and wire of resistance R, we have for the charge e_1' of the plate 1 of B, $e_1' = K'V$ where K' is the capacity of B.

If after the steady state is established, we remove the electromotive force and leave the system to itself, we have flowing through the wire R per unit of time the quantity

$$\frac{V}{R}.$$

Accordingly we have for the decrease of the charges

$$(13)\qquad -\frac{d(e_1+e_1')}{dt}=\iint_{S_1} q\cos(qn)\,dS+\frac{V}{R}=-\iint\lambda\frac{\partial V}{\partial n}\,dS+\frac{V}{R},$$

which when combined with the equation

$$(8)\qquad e_1=\frac{1}{4\pi}\iint\mathfrak{F}\cos(\mathfrak{F}n)\,dS=-\frac{1}{4\pi}\iint\mu\frac{\partial V}{\partial n}\,dS,$$

gives the differential equation

$$(14)\qquad \frac{d(e_1+e_1')}{dt}+\frac{4\pi\lambda}{\mu}e_1=-\frac{V}{R}.$$

Substituting for the charges e_1, e_1', their values in terms of the difference of potential V, we have

$$(15)\qquad \frac{d(KV+K'V)}{dt}+\frac{4\pi\lambda}{\mu}KV+\frac{V}{R}=0,$$

which being integrated gives

$$(16)\qquad V=V_0e^{-t\left(\frac{4\pi\lambda}{\mu}K+\frac{1}{R}\right)/(K+K')}.$$

Putting $R=\infty$, $K'=0$ we obtain the solution (11) just found. Considering the condenser B alone discharging through the wire, we obtain, putting $K=0$,

$$(17)\qquad V=V_0e^{-\frac{t}{K'R}}.$$

A conducting condenser accordingly behaves, when left to itself, exactly like a perfectly insulating condenser discharging through a wire. The relaxation-time of such a condenser is KR, but for a conducting condenser, although we may use the same formula, the relaxation time is independent of the form or dimensions of the condenser, since, as we have seen in § 184, if K_0 be the capacity of the condenser with air as a dielectric, we have

$$K=\mu K_0,\quad R=\frac{1}{4\pi\lambda K_0},\quad KR=\frac{\mu}{4\pi\lambda}=T.$$

The relaxation-time is accordingly a *characteristic constant* of the medium, and may be determined independently of other

media, whereas we may determine only the *ratio* of the *inductivity* of a medium to that of a standard medium.

If we make two experiments with the above combination of condensers, one with A alone, which gives T, and a second with A and B which gives

$$\left(\frac{K}{T}+\frac{1}{R}\right)\Big/(K+K'),$$

if we know K' and R we may from these two results determine K, and if the condenser is made in any shape suitable for calculating K_0 from geometrical data, we can then determine μ. In this manner Cohn found for water $\mu = 73{\cdot}6$, the largest value of the electric inductivity yet found for any substance. In the case of metals, all that we know is that T is extremely small. This is of course due to the large value of λ, so that whether μ is large or small we have as yet no means of knowing.

208. Condenser with two Dielectrics. Absorption. In the preceding section we have seen that a charge residing in any part of a conducting dielectric will gradually disappear, and that no electricity will accumulate at any part of such a dielectric. We have considered only the discharge or leakage of a condenser, starting from a state of steady flow. We shall now consider the state which precedes the attainment of the steady state when an electromotive force is suddenly applied to produce a difference of potential between the plates of the condenser. We shall also suppose that the condenser contains two dielectrics of different properties, and for simplicity we shall consider only a plane condenser. Let the potentials of the two plates be V_1 and V_2, and let that of the plane separating the two dielectrics be V_3.

Let the thickness, inductivity and conductivity of the upper dielectric be d_1, μ_1, λ_1, and of the lower d_2, μ_2, λ_2. The force in the upper dielectric will be the same at all points, F_1, which however depends on the time. In the lower dielectric let the force be F_2, also a function of the time.

Let the currents in the two dielectrics be q_1 and q_2 respectively, and let F_1, F_2, q_1, q_2, be considered positive when measured from V_1 to V_2. Let the condenser plates, of area S, be connected by a wire of no resistance, into which we can suddenly introduce an electromotive force E, which can be suddenly removed. The

circuit may also be broken. If I is the current in the wire from the plate 2 to plate 1, we have, since whatever charge arrives by the wire is uniformly distributed over the plates

(1) $$S\frac{d\sigma_1}{dt}=I-q_1S=I-\lambda_1F_1S,$$

(2) $$S\frac{d\sigma_2}{dt}=-I+q_2S=-I+\lambda_2F_2S,$$

(3) $$S\frac{d\sigma_3}{dt}=(q_1-q_2)S=\lambda_1F_1-\lambda_2F_2.$$

The densities are determined by the equations

(4) $$\sigma_1=\frac{1}{4\pi}\mathfrak{F}_1=\frac{\mu_1}{4\pi}F_1,$$

(5) $$\sigma_2=-\frac{1}{4\pi}\mathfrak{F}_2=-\frac{\mu_2}{4\pi}F_2,$$

(6) $$\sigma_3=\frac{1}{4\pi}(\mathfrak{F}_2-\mathfrak{F}_1)=\frac{1}{4\pi}(\mu_2F_2-\mu_1F_1).$$

Beside these we have always, taking the line integral of the force plate to plate, the equation

(7) $$d_1F_1+d_2F_2=V_1-V_2.$$

From (1) and (4), (2) and (5)

(8) $$\frac{I}{S}-\lambda_1F_1=\frac{\mu_1}{4\pi}\frac{dF_1}{dt},$$

(9) $$\frac{I}{S}-\lambda_2F_2=\frac{\mu_2}{4\pi}\frac{dF_2}{dt},$$

and integrating from $t=0$ to $t=\tau$,

$$\frac{I}{S}\int_0^\tau Idt-\int_0^\tau\lambda_1F_1dt=\frac{\mu_1}{4\pi}\{F_1(\tau)-F_1(0)\},$$

$$\frac{I}{S}\int_0^\tau Idt-\int_0^\tau\lambda_2F_2dt=\frac{\mu_2}{4\pi}\{F_2(\tau)-F_2(0)\}.$$

If $\overline{F}_1$ be the greatest value of F_1 in the interval $t=0$, $t=\tau$, we have

$$\int_0^\tau F_1dt<\overline{F}_1\tau,$$

and since F_1 is finite, as we decrease τ indefinitely, we have in the limit, since $F_1(0)$, $F_2(0)$ are zero,

$$(10) \qquad e = \int I dt = \frac{\mu_1}{4\pi} F_1 S = \frac{\mu_2}{4\pi} F_2 S.$$

That is, the forces jump suddenly from zero to F_1 and F_2, while the total quantity of electricity $e = \int I dt$ passes from one plate to the other. This is called the instantaneous charge.

From the equations (10), (7), we find

$$(11) \qquad \begin{aligned} F_1 &= (V_1 - V_2) \frac{\mu_2}{d_1\mu_2 + d_2\mu_1}, \\ F_2 &= (V_1 - V_2) \frac{\mu_1}{d_1\mu_2 + d_2\mu_1}, \end{aligned}$$

$$(12) \qquad e = \frac{V_1 - V_2}{4\pi} \frac{\mu_1\mu_2 S}{d_1\mu_2 + d_2\mu_1},$$

the same as if there were no conductivity, as in § 188. The ratio

$$(13) \qquad \frac{e}{V_1 - V_2} = \frac{S}{4\pi\,(d_1/\mu_1 + d_2/\mu_2)},$$

or the instantaneous capacity, is the same as the true capacity. If we now keep the electromotive force E in the wire, electricity continues to flow into the condenser, its plates always maintaining the same difference of potential $V_1 - V_2 = E$. The capacity appears to increase without limit. In order to examine what goes on, we must integrate the differential equations. Eliminating I from (8) and (9),

$$(14) \qquad \frac{\mu_1}{4\pi} \frac{dF_1}{dt} + \lambda_1 F_1 = \frac{\mu_2}{4\pi} \frac{dF_2}{dt} + \lambda_2 F_2.$$

By means of the equation (7) we may introduce F_2 in terms of F_1 and E, and differentiating the equation (7),

$$d_1 \frac{dF_1}{dt} + d_2 \frac{dF_2}{dt} = 0,$$

from which we may obtain dF_2/dt in terms of dF_1/dt, giving finally

$$(15) \qquad \frac{dF_1}{dt} + \frac{4\pi\,(\lambda_1 d_2 + \lambda_2 d_1)}{\mu_1 d_2 + \mu_2 d_1} F_1 = \frac{4\pi\lambda_2}{\mu_1 d_2 + \mu_2 d_1} E,$$

as the differential equation for F_1. This is to be integrated with

the arbitrary constant so determined that for $t=0$, F_1 has the value

$$F_1^{(0)} = E\frac{\mu_2}{d_1\mu_2 + d_2\mu_1}. \tag{16}$$

The integral is accordingly

$$F_1 = E\left\{\frac{\lambda_2}{\lambda_1 d_2 + \lambda_2 d_1} + \left(\frac{\mu_2}{\mu_1 d_2 + \mu_2 d_1} - \frac{\lambda_2}{\lambda_1 d_2 + \lambda_2 d_1}\right) e^{-\frac{4\pi(\lambda_1 d_2 + \lambda_2 d_1)}{\mu_1 d_2 + \mu_2 d_1} t}\right\}, \tag{17}$$

and from this and (7)

$$F_2 = E\left\{\frac{\lambda_1}{\lambda_1 d_2 + \lambda_2 d_1} + \left(\frac{\mu_1}{\mu_1 d_2 + \mu_2 d_1} - \frac{\lambda_1}{\lambda_1 d_2 + \lambda_2 d_1}\right) e^{-\frac{4\pi(\lambda_1 d_2 + \lambda_2 d_1)}{\mu_1 d_2 + \mu_2 d_1} t}\right\}. \tag{18}$$

From (4), (5), (6), putting for brevity

$$\frac{\mu_1 d_2 + \mu_2 d_1}{4\pi(\lambda_1 d_2 + \lambda_2 d_1)} = T, \quad \frac{\lambda_2}{\lambda_1 d_2 + \lambda_2 d_1} = a_1, \quad \frac{\lambda_1}{\lambda_1 d_2 + \lambda_2 d_1} = a_2,$$

$$\frac{\mu_2}{\mu_1 d_2 + \mu_2 d_1} = b_1, \quad \frac{\mu_1}{\mu_1 d_2 + \mu_2 d_1} = b_2,$$

we have for the densities,

$$\sigma_1 = \frac{E\mu_1}{4\pi}\{a_1 + (b_1 - a_1)e^{-\frac{t}{T}}\},$$

$$\sigma_2 = -\frac{E\mu_2}{4\pi}\{a_2 + (b_2 - a_2)e^{-\frac{t}{T}}\}, \tag{19}$$

$$\sigma_3 = -(\sigma_1 + \sigma_2) = \frac{E}{4\pi}\frac{(\lambda_1\mu_2 - \lambda_2\mu_1)}{\lambda_1 d_2 + \lambda_2 d_1}(1 - e^{-\frac{t}{T}}).$$

The plane 3 accordingly acquires a charge, which is not the case if the dielectric is homogeneous, or if the relaxation-times of the two dielectrics are equal. We shall distinguish the values of F, σ, etc., attained after the time t_1 by an affix, $F_1^{(1)}$. Suppose that the circuit be now broken. We accordingly have $I=0$ and therefore

$$\frac{\mu_1}{4\pi}\frac{dF_1}{dt} + \lambda_1 F = 0,$$

$$\frac{\mu_2}{4\pi}\frac{dF_2}{dt} + \lambda_2 F = 0, \tag{20}$$

and the charges of the three planes begin to die away at different rates. At any subsequent time later by an interval t_2,

(21)
$$F_1^{(2)} = F_1^{(1)}\, e^{-\frac{4\pi\lambda_1}{\mu_1}t_2} = F_1^{(1)}\, e^{-\frac{t_2}{T_1}},$$
$$F_2^{(2)} = F_2^{(1)}\, e^{-\frac{4\pi\lambda_2}{\mu_2}t_2} = F_2^{(1)}\, e^{-\frac{t_2}{T_2}},$$

(22)
$$\sigma_1^{(2)} = \sigma_1^{(1)}\, e^{-\frac{t_2}{T_1}},$$
$$\sigma_2^{(2)} = \sigma_2^{(1)}\, e^{-\frac{t_2}{T}}$$
$$\sigma_3^{(2)} = -\left(\sigma_1^{(1)}\, e^{-\frac{t_2}{T_1}} + \sigma_2^{(1)}\, e^{-\frac{t_2}{T_2}}\right).$$

If the condenser be now left to itself, the charges will finally entirely disappear. If however the plates are short circuited, we have the same conditions as in the first stage, with $E=0$. Accordingly the forces change suddenly from $F_1^{(2)}$, $F_2^{(2)}$ to $F_1^{(3)}$, $F_2^{(3)}$, and there passes through the wire the instantaneous discharge

(23)
$$e' = \int I dt = \frac{\mu_1 (F_1^{(3)} - F_1^{(2)}) S}{4\pi} = \frac{\mu_2 (F_2^{(3)} - F_2^{(2)}) S}{4\pi}.$$

We now have, by (7)
$$d_1 F_1^{(3)} + d_2 F_2^{(3)} = 0,$$
and since
$$\mu_1 F_1^{(3)} - \mu_2 F_2^{(3)} = \mu_1 F_1^{(2)} - \mu_2 F_2^{(2)} = -4\pi\sigma_3^{(2)},$$
we obtain
$$F_1^{(3)} = -\frac{4\pi d_2 \sigma_3^{(2)}}{\mu_1 d_2 + \mu_2 d_1}, \quad F_2^{(3)} = \frac{4\pi d_1 \sigma_3^{(2)}}{\mu_1 d_2 + \mu_2 d_1},$$
$$e' = -\left(\frac{\mu_1 d_2 \sigma_3^{(2)}}{\mu_1 d_2 + \mu_2 d_1} + \sigma_1^{(2)}\right) S.$$

(If t_2, the time of leakage through the condenser, be zero, and if t_1 the time of charge, be either zero or infinity, we find that conduction is without effect, and the instantaneous discharge, $-e'$, is equal to the instantaneous charge.)

There now remain the charges
$$\sigma_1^{(3)} = \frac{\mu_1 F_1^{(3)}}{4\pi} = -\frac{d_2 \mu_1 \sigma_3^{(2)}}{\mu_1 d_2 + \mu_2 d_1},$$
$$\sigma_2^{(3)} = -\frac{\mu_2 F_2^{(3)}}{4\pi} = -\frac{d_1 \mu_2 \sigma_3^{(2)}}{\mu_1 d_2 + \mu_2 d_1},$$
$$\sigma_3^{(3)} = \sigma_3^{(2)}.$$

If the circuit be again broken, the circumstances are the same as in stage 2, so that if the condenser is subsequently again short-circuited, we obtain a new instantaneous discharge, called the residual discharge, and this may be repeated as often as we please. It will be seen that the residual discharge arises from the charge σ_3 that has accumulated by conduction on the plane 3, and that there will accordingly be no residual discharge in a condenser in which the relaxation-time is the same in every part. This is a type of what would occur in any non-homogeneous dielectric, and it is in this manner that Maxwell gave a possible explanation of the phenomena of electric absorption, and of residual charge (Rückstand). Maxwell's explanation has found confirmation in experimental results of Rowland and Nichols, Hertz, Arons, and Muraoka*, all of whom found that when the dielectric was perfectly homogeneous there was no residual charge.

209. Total and Displacement current. In the fundamental equation § 206 (4), we see that the vector

$$\bar{q} + \frac{1}{4\pi}\frac{d\overline{\mathfrak{F}}}{dt},$$

whose components are

$$u + \frac{1}{4\pi}\frac{d\mathfrak{X}}{dt}, \quad v + \frac{1}{4\pi}\frac{d\mathfrak{Y}}{dt}, \quad w + \frac{1}{4\pi}\frac{d\mathfrak{Z}}{dt},$$

is solenoidal. If we consider the condition at the surface of an ordinary conductor, in which we consider $\mathfrak{F} = 0$, surrounded by an insulator (in which $q = 0$), we have

$$\frac{d\sigma}{dt} = -\{u \cos(n_i x) + v \cos(n_i y) + w \cos(n_i z)\},$$

$$\sigma = \frac{1}{4\pi}\{\mathfrak{X} \cos(n_e x) + \mathfrak{Y} \cos(n_e y) + \mathfrak{Z} \cos(n_e z)\},$$

so that here also the solenoidal condition is fulfilled. The vector $\bar{q} + \frac{1}{4\pi}\frac{d\overline{\mathfrak{F}}}{dt}$ is called by Maxwell the *total current*. It is a fundamental principle of Maxwell's theory that the magnetic effects of

* Rowland and Nichols, *Phil. Mag.* (5) 11, p. 414, 1881; Hertz, *Wied. Ann.* 20, p. 279, 1883; Arons, *Wied. Ann.* 35, p. 291, 1888; Muraoka, *Wied. Ann.* 40, p. 328, 1890.

the current are due to the total current, and not to the conduction current alone. In insulators the part $d\mathfrak{F}/dt\,.\,4\pi$, which alone exists in insulators is called the *displacement current*, since Maxwell calls $\mathfrak{F}/4\pi$ the electric displacement. The corresponding magnetic quantity, which, since there is no magnetic conduction, constitutes the magnetic current, has important physical properties, which will be considered in Chapter XIII.

PART III.

THE ELECTROMAGNETIC FIELD.

CHAPTER XI.

ELECTROMAGNETISM.

210. Magnetic Force due to Linear Current. The discovery was made by Oersted*, in 1820, that if a linear circuit be traversed by an electric current, the space in its neighborhood constitutes a field of magnetic force. The nature of the forces of the field was completely investigated by Ampère†, who found that they were of the same nature as if they proceeded from permanent magnets. They accordingly have a potential, which, with its first derivatives, is continuous and vanishes at infinity, and which satisfies Laplace's equation at all points outside of the conducting wire, supposing that a single homogeneous medium is present. We have however seen that a single-valued, or uniform function having all these properties vanishes everywhere. Accordingly the magnetic potential due to a current is not uniform.

* Oersted, *Experimenta circa effectum Conflictus Electrici in Acum Magneticam*, Copenhagen, 1820.

† Ampère, "Mémoire sur la théorie mathematique des phénomènes électro-dynamiques, uniquement deduite de l'expérience. *Gilbert's Ann.* 67, 1821; *Mém. de l'Acad.* t. 6, Ann. 1823.

We shall in future denote the magnetic force by H, its components by L, M, N, the magnetic induction by $\mathfrak{B}$, its components by $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, and the magnetic potential by Ω, reserving the notation F, X, Y, Z; $\mathfrak{F}$, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, V, for the corresponding electric quantities. For the electric inductivity we shall use the letter ϵ, leaving μ for the magnetic inductivity. These distinctions have not before been necessary, since we have not at the same time considered both electrical and magnetic quantities, as we must do from now on. If we form the line integral of magnetic force from a point A to a point B, we have

$$(1) \qquad \int_A^B Ldx + Mdy + Ndz = \Omega_A - \Omega_B,$$

which must be independent of the path AB, for otherwise, by changing the path infinitely little, we should, starting with the given value Ω_A, cause Ω_B to change by an infinitely small amount, and could thus cause Ω_B to take at the same point a series of continuously varying values. The integral is accordingly the same for all paths that can be changed into one another by continuous deformation. If, however, the current separates two paths ACB, ADB, the integral is not the same for both. In other words, while the integral around any closed path not linked with the circuit is zero, the integral around a path linked with the circuit is not. But the integral around any two closed paths each linked once with the circuit is the same, for they may be continuously deformed into each other. Or in other words, we may connect two such paths 1 and 2, Fig. 84, by a path PQ. The integral around the circuit $ABPQDCQPA$, which is not linked with the current, is zero, but this is equal to the sum of the integrals $PABP$ around 1 in the positive direction, together with the integral $QDCQ$ around 2 in the negative direction, while the integrals over the coincident paths PQ, QP in opposite directions destroy each other. Accordingly

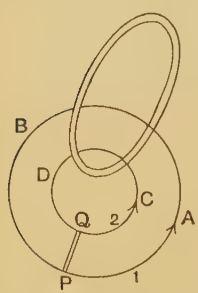

FIG. 84.

$$\int PABP = \int QCDQ.$$

We shall say that two geometrical circuits are linked positively,

when, given a direction of circulation about each circuit, the direction of circulation in one circuit agrees with the forward motion of a right-handed screw, whose rotation corresponds to the direction of circulation in the other circuit. Fig. 85 represents two circuits linked positively above and negatively below. By an extension of the above reasoning we see that the integral around any circuit linked n times in the positive manner with the current is nJ, where J is the integral around any circuit linked once. Accordingly the potential at any point is an infinitely valued function, whose values differ from each other by integral multiples of J. We may however make the potential a uniform function, if we prevent passage from one point to another by paths not continuously deformable into each other, that is, if we reduce the doubly connected space about the current to a singly-connected one by means of a diaphragm covering the current circuit. Then no two paths can be separated by the current. If we consider the potential at two points infinitely near each other but lying on opposite sides of the diaphragm, Fig. 86, to get from one to the other we must perform a closed circuit about the current, so that their potential differs by the amount J, accordingly in crossing the diaphragm, the potential is discontinuous, the amount of the discontinuity being

$$\Omega_A - \Omega_B = J,$$

where A is on the positive side of the diaphragm. There is, however, no discontinuity nor lack of uniformity in the derivatives of Ω.

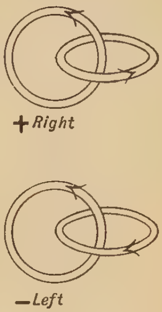

FIG. 85.

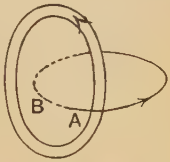

FIG. 86.

If we now consider all space, except a small sphere of radius R with center at the point P, and apply to it Green's theorem

$$(2) \qquad \iint\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS = \iiint(V\Delta U - U\Delta V)\,d\tau,$$

where for V we put the magnetic potential Ω, and for U the function $1/r$, where r is the distance from P, the volume integrals vanish, and the surface integrals are to be taken over the infinite

sphere, where they vanish, over the small sphere about P, where we have as in § 83, on making R decrease indefinitely,

$$\iint\left(\frac{1}{r}\frac{\partial\Omega}{\partial n} - \Omega\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\right)dS = 4\pi\Omega_P,$$

and over the two sides of the diaphragm, where we have

$$\iint\left(\frac{1}{r}\frac{\partial\Omega}{\partial n_1} - \Omega_1\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1}\right)dS + \iint\left(\frac{1}{r}\frac{\partial\Omega}{\partial n_2} - \Omega_2\frac{\partial\left(\frac{1}{r}\right)}{\partial n_2}\right)dS.$$

Since, however, $\partial\Omega/\partial n$ is continuous, the first terms in the two integrals cancel each other, the normals n_1 and n_2 being in opposite directions, and since

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1} = -\frac{\partial\left(\frac{1}{r}\right)}{\partial n_2},$$

this becomes

$$\iint(\Omega_2 - \Omega_1)\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1}dS,$$

and finally

$$(3)\qquad 4\pi\Omega_P + \iint(\Omega_2 - \Omega_1)\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1}dS = 0.$$

Since $\Omega_1 - \Omega_2 = J$ we have

$$(4)\qquad \Omega_P = \frac{J}{4\pi}\iint\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1}dS,$$

so that the action of the current is the same as that of a magnetic double-layer or shell of strength

$$(5)\qquad \Phi = \frac{J}{4\pi} = \frac{\Omega_1 - \Omega_2}{4\pi}.$$

This result was given by Ampère, by different reasoning. Experiment shows that the magnetic forces are proportional to the strength of the current, so that if A be a factor of proportionality,

$$(6)\qquad \Omega_P = AI\iint\frac{\partial\left(\frac{1}{r}\right)}{\partial n_1}dS = AI\omega,$$

the surface integral, being by Gauss's theorem, § 39, equal to ω the solid angle subtended by the current circuit at the point P. The positive side of the shell and the one toward which the normal is to be drawn is the side toward which a right-handed screw advancing in the direction of the normal to the diaphragm, would move when rotating with the current. The line of force is positively linked with the current. Since the potential everywhere, except in the substance of the conductor, satisfies Laplace's equation, the force is everywhere solenoidal, and the tubes of force are endless, and are all linked once with the current.

211. Electromagnetic Units. The determination of the factor A, which is a natural constant, is a matter of experiment. It is extremely small, that is, an enormous number of electrostatic units of electricity must pass in unit time in order that the current may produce magnetic forces of appreciable amount. If, however, we choose a *new unit* for I, defined by the assumption $A = 1$, so that

$$\Omega = I\omega,$$

we get a new system of measuring currents known as the electromagnetic system. The unit magnetic potential is defined as the potential at unit distance from the unit magnetic pole in vacuo, accordingly the electromagnetic unit of current is referred at once to a magnetic pole, instead of to an electrified point. From this definition of the new unit of current we may at once obtain a whole system of electrical units. We define the new unit of quantity of electricity as the quantity passing in unit of time when a steady current of one electromagnetic unit flows. From this definition of unit charge we obtain, as before, new units of field, of electric potential, of resistance, capacity, and the rest. Conversely if, measuring the current in electrostatic measure, we put $A = 1$ we shall get a new unit of magnetic potential, from which we may obtain a complete set of units for magnetic quantities, all referred to the unit of electric charge, instead of to the unit magnetic pole. We may thus measure electric quantities in the electromagnetic system, or magnetic quantities in the electrostatic system, or as before, each kind of quantity in its own appropriate system, thus obtaining the Gaussian system.

212. Dimensions of the Units. If we denote the numeric of a quantity when measured in the electrostatic system by the suffix e and when measured in the electromagnetic or magnetic system by the suffix m, we have for the magnetic potential

(1) $$\Omega_m = AI_e\omega = I_m\omega,$$

(2) $$\Omega_e = I_e\omega = \frac{1}{A} I_m\omega.$$

Consequently the number A denotes the ratio of the numeric of a certain current when measured electromagnetically, to the numeric of the same quantity measured electrostatically, or $1/A$ is the number of electrostatic units of current in one electromagnetic unit. If m denote a magnetic charge, we have the dimensional equation, by § 190,

(3) $$[\Omega] = \left[\frac{m}{\mu L}\right],$$

the quantities being measured in either system. Also since the dimensions of solid angle are zero, the dimensions of Ω are the same as of I, and

(4) $$[I] = [\Omega] = \left[\frac{m}{\mu L}\right].$$

Since the unit of electric charge in either system is obtained from the unit of current multiplied by the unit of time,

(5) $$[e] = [IT] = \left[\frac{mT}{\mu L}\right],$$

and we accordingly have for the ratio of the two units of electricity or of current, inserting the suffix m in (5)*

(6) $$\left[\frac{1}{A}\right] = \left[\frac{e_e}{e_m}\right] = \left[\frac{I_e}{I_m}\right] = \left[\frac{e_e \mu_m L}{m_m T}\right].$$

Now the fundamental assumption in defining the magnetic system was that the dimensions of μ were zero. Also the assumption defining the electrostatic system was that the dimensions of ϵ were zero. Accordingly the dimensions of e_e, and of m_m, both belonging to the Gaussian system, and defined by precisely the same considerations, namely

(7) $$\left[\frac{e^2}{\epsilon L^2}\right] = \left[\frac{m^2}{\mu L^2}\right] = \left[\frac{ML}{T^2}\right],$$

* Evidently any dimensional equation holds when either suffix e or m is inserted on *both* sides.

are *the same*. Hence the dimensions of the quantity $1/A$ are accordingly the same as those of a velocity. All that has been said of course applies to any absolute system of units, and has no restriction to the C.G.S. system. If the units of length, mass, and time are given, we can by definition immediately obtain the unit of electricity in either the electrostatic or electromagnetic system, and by experiment determine the number of electrostatic units contained in one electromagnetic. If the unit of mass is now changed, and we define our electrical units as before, the size of both units of electricity has changed, but in the same ratio, so that the number of one kind contained in one of the other is the same as before. If, on the other hand, we change the unit either of length or time, the two electrical units change, but in different ratios, so that the numeric expressing the number of one kind in one of the other is changed from its former value. It has, however, changed in precisely the same way that the numeric expressing any given velocity has changed, so that we may say that the number $1/A$ represents a certain definite velocity, which is totally independent of the units chosen. When the units of mass, length, and time have been settled upon, the numeric of this velocity may be given. This velocity will be denoted by $\mathbf{v}$. It is to be noticed that the determination of the quantity $\mathbf{v}$ depends upon the determination of a certain numeric, the units being settled upon, and that there is nothing of the nature of an actual velocity involved. We shall, therefore, not as yet be understood to speak of $\mathbf{v}$ as a velocity, but merely as a quantity whose numerical expression changes like that of a velocity, with any change of units. The quantity $\mathbf{v}$ is the most important electrical natural constant. Numerous determinations of its value have been made, the first by Wilhelm Weber* and Rudolf Kohlrausch, in 1856. The number now generally accepted is

$$\mathbf{v} = 3 \times 10^{10} \text{ cm./sec.}$$

Electrical and magnetic potential are defined in terms of work, so that

$$(8) \qquad [eV] = [m\Omega] = [ML^2T^{-2}],$$

which agrees with the other possible definition

$$(3) \qquad [V] = \left[\frac{e}{\epsilon L}\right], \quad [\Omega] = \left[\frac{m}{\mu L}\right].$$

* Weber, *Elektrodynamische Maassbestimmungen* IV. 1856; *Werke*, Bd. III. p. 609.

From (8) and (6) we obtain

$$(9) \qquad [e_e V_e] = [e_m V_m],$$

$$(9') \qquad \left[\frac{V_e}{V_m}\right] = \left[\frac{e_m}{e_e}\right] = [A] = \left[\frac{1}{\mathbf{v}}\right],$$

and there are $\mathbf{v}$ electromagnetic units of potential in one electrostatic unit. Capacity is defined as ratio of charge to potential, so that

$$(10) \qquad [K] = \left[\frac{e}{V}\right],$$

from which

$$(10') \qquad \left[\frac{K_e}{K_m}\right] = \left[\frac{e_e}{e_m}\frac{V_m}{V_e}\right] = [\mathbf{v}^2],$$

or there are $\mathbf{v}^2$ electrostatic units of capacity in one electromagnetic unit. Resistance is defined as ratio of potential to current, so that

$$(11) \qquad [R] = \left[\frac{V}{I}\right],$$

$$(11') \qquad \left[\frac{R_e}{R_m}\right] = \left[\frac{V_e}{V_m}\frac{I_m}{I_e}\right] = \left[\frac{1}{\mathbf{v}^2}\right],$$

or there are $\mathbf{v}^2$ electromagnetic units in one electrostatic unit.

213. Practical System. The absolute system of units was due to Gauss, and was introduced to practice by Weber. The system was first made practicable for general use by the exertions of the British Association, which issued copies of the unit of resistance, and decided on various multiples of the C.G.S. electromagnetic units for practical units. Its action has been seconded by international congresses, at Paris in 1881, 1884 and 1889, and at Chicago in 1893, which determined on the following multiples of the electromagnetic units:

1 Volt	$= 10^8$	C.G.S.	electromagnetic units	of	Potential.
1 Ohm	$= 10^9$	„	„	„	Resistance.
1 Ampère	$= 10^{-1}$	„	„	„	Current.
1 Coulomb	$= 10^{-1}$	„	„	„	Electric Charge.
1 Farad	$= 10^{-9}$	„	„	„	Capacity.
1 Joule	$= 10^7$	„	„	„	Work.
1 Watt	$= 10^7$	„	„	„	Activity.

The prefixes *mega* and *micro* are used before the preceding names of the units to denote respectively multiplication and division by

a million. These units form a consistent system, so that electrical relations involving quantities measured in these units require no numerical factors. For instance, a current of one ampère is produced when an electromotive force of one volt is impressed in a circuit whose resistance is one ohm, and the activity of one watt thereby exerted dissipates energy at the rate of one joule per second.

214. Electrostatic compared with Practical Units. From the above definitions with the value of **v** given and equation (9′) we find

1 C.G.S. electrostatic unit of Potential = 300 Volts.

From (10′)

1 Farad contains $9 \cdot 10^{11}$ C.G.S. electrostatic units of Capacity.

The electrostatic unit of capacity is the unit of length, accordingly

1 Microfarad = 900,000 cm. of Capacity.

A sphere of nine kilometers radius in free space would have a capacity of one microfarad. From (6)

1 Coulomb = $3 \cdot 10^9$ C.G.S. electrostatic units of Electric Charge.

From (11′)

900,000 Megohms = 1 C.G.S. electrostatic unit of Resistance.

From equation (7) we may find the dimensions of e and m, when those of ϵ and μ are settled upon. Any convention that may be made gives us a possible system of units. It must be noticed, however, that there is always a relation between the dimensions of ϵ and μ. From equations (4) and (5)

$$\left[\frac{e}{T}\right] = [I] = \left[\frac{m}{\mu L}\right].$$

Squaring this and dividing by

$$\left[\frac{e^2}{\epsilon L^2}\right] = \left[\frac{m^2}{\mu L^2}\right] \tag{7}$$

we obtain

$$\left[\frac{\epsilon L^2}{T^2}\right] = \left[\frac{1}{\mu}\right],$$

$$[\epsilon\mu] = \left[\frac{1}{\mathbf{v}^2}\right].$$

Consequently the dimensions of the product of the electric and magnetic inductivities must in any system be those of the square of the reciprocal of a velocity. The *absolute* dimensions of either factor are arbitrary. Attempts have been made to settle the absolute dimensions of ϵ or μ, but they are evidently based upon misconceptions of the theory of dimensions. The two common assumptions are, that

$$\epsilon = 1, \quad \mu = \frac{1}{\mathbf{v}^2}.$$

This gives the electrostatic system. Secondly we may assume

$$\mu = 1, \quad \epsilon = \frac{1}{\mathbf{v}^2}.$$

This gives the electromagnetic system. We shall, when dealing principally with the magnetic properties of currents, use the electromagnetic system, but when dealing equally with electrical and magnetic phenomena, to avoid ambiguity, we shall, following Helmholtz and Hertz, use the Gaussian system, measuring all electrical quantities in the electrostatic system, all magnetic quantities in the magnetic system, and introducing the factor A, with the numerical value $1/\mathbf{v}$. A complete table of dimensions of the various units is given at the end of Chapter XIII.

215. Potential due to Circular Current. The potential at P due to a current being $\Omega = I\omega$, where ω is the solid angle subtended at P by the current circuit, if P is situated at a distance x from the center of a circular current of radius R, on the line through its center O perpendicular to its plane, we have for the area of the segment of the sphere of unit radius about P cut off by the right cone whose vertex is P, and base the current,

$$\text{(1)} \qquad \omega = 2\pi \int_0^\alpha \sin\theta d\theta = 2\pi(1 - \cos\alpha)$$

$$= 2\pi \left\{1 - \frac{x}{\sqrt{R^2 + x^2}}\right\} = 2\pi \left\{1 - \frac{1}{\sqrt{1 + \frac{R^2}{x^2}}}\right\}.$$

This may also be obtained, according to § 123, by differentiating the expression for the potential of a disc at a point on the axis.

The force in the direction of the axis is

$$L = -\frac{\partial \Omega}{\partial x} = \frac{2\pi R^2 I}{(x^2 + R^2)^{\frac{3}{2}}}.$$

At the center of the circle

$$L = \frac{2\pi I}{R}.$$

From this expression comes the definition often given of the unit of current as that current which, flowing in a circle of unit radius, produces the field 2π at its center, or less correctly, the current, which, flowing in an arc equal to the radius in a unit circle, produces unit field at the center.

The expression for the force is an example of the proposition that similar geometrical circuits traversed by equal currents, produce at corresponding points forces inversely proportional to their linear dimensions. For at corresponding points the solid angle, and therefore the potential is the same. In the circuit of n times the dimensions, the potential changes by equal amounts for displacements of n times the length, hence for equal displacements the change is $1/n$ as great, and the force is n times smaller.

When the point is not on the axis of the circle, the cone, having an oblique section circular, is elliptic, and we must calculate the area of the spherical ellipse cut out by it from the unit sphere. This involves an elliptic integral.

We may however develop the result in an infinite series of zonal spherical harmonics, as in the case of the potential of a disc, in § 102. Developing the above expression for ω at points on the axis by the binomial theorem, we have

$$\omega = 2\pi \left[1 - \left\{1 + \left(\frac{R}{x}\right)^2\right\}^{-\frac{1}{2}}\right]$$

$$= 2\pi \left\{\frac{1}{2}\left(\frac{R}{x}\right)^2 - \frac{1.3}{2.4}\left(\frac{R}{x}\right)^4 + \frac{1.3.5}{2.4.6}\left(\frac{R}{x}\right)^6 - \dots\right\} \quad x > R,$$

$$\omega = 2\pi \left[1 - \frac{x}{R}\left\{1 + \left(\frac{x}{R}\right)^2\right\}^{-\frac{1}{2}}\right]$$

$$= 2\pi \left\{1 - \frac{x}{R} + \frac{1}{2}\left(\frac{x}{R}\right)^3 - \frac{1.3}{2.4}\left(\frac{x}{R}\right)^5 + \dots\right\} \quad x < R.$$

Accordingly for points not on the axis, at a distance r from the center of the circle,

$$\omega = 2\pi \left\{ \frac{1}{2}\left(\frac{R}{r}\right)^2 P_1 - \frac{1.3}{2.4}\left(\frac{R}{r}\right)^4 P_3 + \frac{1.3.5}{2.4.6}\left(\frac{R}{r}\right)^6 P_5 - \dots \right\} r > R,$$

$$\omega = 2\pi \left\{ 1 - \frac{r}{R} P_1 + \frac{1}{2}\left(\frac{r}{R}\right)^3 P_3 - \frac{1.3}{2.4}\left(\frac{r}{R}\right)^5 P_5 + \dots \right\} r < R.$$

In order to find the direction of the force we must differentiate this in the directions parallel and perpendicular to the axis, and take the resultant. A figure of the lines of force is given by Maxwell, Plate 18.

216. Infinite Straight Current. Law of Biot and Savart. If we have a current flowing through a straight linear conductor of infinite length, we may consider the circuit completed by conductors lying at an infinite distance all in the same plane. The solid angle subtended by the circuit at a point P will be that sector of the unit sphere with center at P included between the plane through the straight conductor and P, and a plane through P parallel to a given plane, which is assumed to be the plane of the circuit. This angle being ϕ, we have the ratio of the solid angle ω to the surface of the unit sphere equal to the ratio of the plane angle to the circumference of the unit circle,

$$\frac{\omega}{4\pi} = \frac{\phi}{2\pi}, \quad \omega = 2\phi, \quad \Omega = 2I\phi.$$

But ϕ is equal to the angle made by a plane through P and the conductor with a fixed plane through the conductor. Consequently the equipotential surfaces are planes through the conductor, and the lines of force are circles whose planes are perpendicular to the conductor.

The line integral of force about a circle of radius r is the value of the force H, which is tangential to the circle, times the length of the circumference, and this must be equal to $4\pi I$,

$$4\pi I = 2\pi r H.$$

Accordingly the value of the force is

$$H = \frac{2I}{r}.$$

This is the law of Biot and Savart*.

* Biot et Savart, *Ann. Chim. Phys.* 15, p. 222, 1820.

217. Force due to any Linear Current. If the potential at a point P is Ω and at a neighboring point Q is $\Omega + \delta\Omega$, where the distance $PQ = \delta h$, and if H is the magnetic force at P, we have

(1) $$H \cos(Hh) = -\frac{\partial \Omega}{\partial h},$$

(2) $$\delta\Omega = -H\delta h \cos(Hh).$$

This change in the potential is the same as the change that would be made in the potential at P by moving the whole circuit parallel to itself the same distance δh in the opposite direction. The change $\delta\Omega$ is proportional to the change $\delta\omega$ made in the solid angle subtended at P due to the motion of the circuit, which is easily seen to be exactly the solid angle subtended at P by the narrow ribbon of cylindrical surface whose edges are the initial and final positions of the circuit, and whose generating lines are equal and parallel to δh. But any arc ds has described in the motion an area dS of a parallelogram equal to

(3) $$dS = ds\delta h \sin(ds, \delta h),$$

and if n be the normal to this element of area, we have for the element $d\delta\omega$ of the solid angle subtended by it at P,

(4) $$d\delta\omega = \frac{dS \cos(nr)}{r^2} = ds\delta h \sin(ds, \delta h) \cos(nr),$$

where r is the distance of the element from P. Consequently integrating around the ribbon

(5) $$\delta\omega = \delta h \int \frac{\sin(ds, \delta h) \cos(nr)}{r^2} ds,$$

(6) $$\delta\Omega = I\delta\omega.$$

If we consider that each element of the current of length ds contributes to the field the potential $d\Omega$ and the force dH, we have, by (2),

(7) $$-dH\delta h \cos(dH, \delta h) = d\delta\Omega = I \frac{r\, ds\, \delta h \sin(ds, \delta h) \cos(nr)}{r^3}.$$

The numerator is the volume of the parallelepiped whose sides are r, ds, δh. It therefore vanishes if the direction of δh coincides with that of r.

There is accordingly no component of the force in the direction of r, or the force is perpendicular to r. In like manner if δh has

the direction of ds, the force vanishes, so that the force is perpendicular to ds. If δh is perpendicular to the plane of r and ds, we have

$$\cos(dH,\ \delta h) = 1, \quad \sin(ds,\ \delta h) = 1,$$

(8) $$dH = \frac{rds\cos(nr)}{r^3} = \frac{rds\sin(r,\ ds)}{r^3}.$$

Accordingly if we call $d\sigma$ the component of ds perpendicular to r, the magnetic force due to the whole conducting circuit will be obtained if we suppose each element ds to contribute to the field the amount

(9) $$dH = \frac{d\sigma}{r^2},$$

which has the direction perpendicular to the element ds and the radius r.

The total field is the vector sum of all these infinitesimal parts. The proper sign to be chosen may be found by considering the way in which the lines of force are linked with the current, and we find that the direction of the force is given by the rotation of a right-handed screw advancing with the current in the direction of $d\sigma$. The complete specification may be most concisely stated by saying that the force due to the element ds is $1/r^3$ times the vector product of Ids and r, the vector r being drawn from the element ds to the point P,

(10) $$\overline{dH} = \frac{I}{r^3}\mathbf{V}\overline{ds}\,.\,\bar{r}.$$

The resolution of the field into elementary fields is artificial, for the field is of course due to the whole closed circuit. Moreover the resolution may be performed in an infinite number of ways, for it is the integral of the above differential taken around the whole circuit which gives the field.

We may consequently add to the differential above the differential of any function of the coordinates of the element ds, for in integration around the circuit this function returns to its original value so that the integral vanishes.

If the coordinates of a point in the current circuit are x_1, y_1, z_1, those of P, x, y, z, since the direction cosines of $\bar{r}$ and $\overline{ds_1}$ are

respectively

$$\frac{x-x_1}{r},\quad \frac{y-y_1}{r},\quad \frac{z-z_1}{r},$$

$$\frac{dx_1}{ds_1},\quad \frac{dy_1}{ds_1},\quad \frac{dz_1}{ds_1},$$

we have for the components of the vector-product representing the field due to an element ds_1,

$$dL = \frac{I}{r^3}\{dy_1(z-z_1) - dz_1(y-y_1)\},$$

(11)
$$dM = \frac{I}{r^3}\{dz_1(x-x_1) - dx_1(z-z_1)\},$$

$$dN = \frac{I}{r^3}\{dx_1(y-y_1) - dy_1(x-x_1)\}.$$

We may obtain the same result by the use of Stokes's Theorem, § 31. Since

$$\Omega = I\omega = I\iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\,dS,$$

the component of the field in the direction h is

(12)
$$H\cos(Hh) = -\frac{\partial\Omega}{\partial h} = -I\iint \frac{\partial^2\left(\frac{1}{r}\right)}{\partial h\,\partial n}\,dS.$$

Let the constant direction cosines of h be α, β, γ, and those of n be λ, μ, ν, variable over the surface of the diaphragm. Then

(13)
$$\frac{\partial}{\partial h} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial n} = \lambda\frac{\partial}{\partial x_1} + \mu\frac{\partial}{\partial y_1} + \nu\frac{\partial}{\partial z_1}.$$

Now since

$$r^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2,$$

we have

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = -\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1},\quad \frac{\partial\left(\frac{1}{r}\right)}{\partial y} = -\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1},\quad \frac{\partial\left(\frac{1}{r}\right)}{\partial z} = -\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1},$$

so that

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial h} = -\left(\alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1} + \beta\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1} + \gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\right),$$

(14) $$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial h\,\partial n} = -\left(\lambda\frac{\partial}{\partial x_1} + \mu\frac{\partial}{\partial y_1} + \nu\frac{\partial}{\partial z_1}\right)\left(\alpha\frac{\partial}{\partial x_1} + \beta\frac{\partial}{\partial y_1} + \gamma\frac{\partial}{\partial z_1}\right)\left(\frac{1}{r}\right)$$

$$= -\left\{\lambda\alpha\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x_1^2} + \mu\beta\frac{\partial^2\left(\frac{1}{r}\right)}{\partial y_1^2} + \nu\gamma\frac{\partial^2\left(\frac{1}{r}\right)}{\partial z_1^2}\right.$$

$$\left. + (\mu\gamma+\nu\beta)\frac{\partial^2\left(\frac{1}{r}\right)}{\partial y_1\partial z_1} + (\nu\alpha+\lambda\gamma)\frac{\partial^2\left(\frac{1}{r}\right)}{\partial z_1\partial x_1} + (\lambda\beta+\mu\alpha)\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x_1\partial y_1}\right\}.$$

Now since $1/r$ satisfies the equation

$$\Delta\left(\frac{1}{r}\right) = 0,$$

we may write

$$-\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x_1^2} = \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y_1^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z_1^2},$$

with similar substitutions for $\partial^2(1/r)/\partial y_1^2$ and $\partial^2(1/r)/\partial z_1^2$. Making these substitutions, and arranging the terms differently, we obtain

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial h\,\partial n} = \lambda\left[\frac{\partial}{\partial y_1}\left(\alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1} - \beta\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\right) - \frac{\partial}{\partial z_1}\left(\gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1} - \alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\right)\right]$$

(15) $$+\mu\left[\frac{\partial}{\partial z_1}\left(\beta\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1} - \gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}\right) - \frac{\partial}{\partial x_1}\left(\alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1} - \beta\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\right)\right]$$

$$+\nu\left[\frac{\partial}{\partial x_1}\left(\gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1} - \alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\right) - \frac{\partial}{\partial y_1}\left(\beta\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1} - \gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}\right)\right].$$

Consequently if we put

$$U_h = I\left(\beta\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1} - \gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}\right),$$

(16)
$$V_h = I\left(\gamma\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1} - \alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\right),$$

$$W_h = I\left(\alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1} - \beta\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\right),$$

making Q_h the resultant of U_h, V_h, W_h equal to I times the vector product of the unit vector h and the vector parameter of $1/r$, the force in the direction h is

(17)
$$H\cos(Hh) = -\iint\left\{\lambda\left(\frac{\partial W_h}{\partial y_1} - \frac{\partial V_h}{\partial z_1}\right) + \mu\left(\frac{\partial U_h}{\partial z_1} - \frac{\partial W_h}{\partial x_1}\right) + \nu\left(\frac{\partial V_h}{\partial x_1} - \frac{\partial U_h}{\partial y_1}\right)\right\} dS$$
$$= -\iint(\text{curl } Q_h)\cos(\text{curl } Q_h, n)\, dS.$$

But by Stokes's theorem this is equal to the line integral

(18)
$$-\int(U_h dx_1 + V_h dy_1 + W_h dz_1) = -\int Q_h \cos(Q_h ds)\, ds_1,$$

around the current circuit.

Accordingly attributing to each element ds the amount of field dH,

(19)
$$dH\cos(dH, h) = \alpha dL + \beta dM + \gamma dN$$
$$= -(U_h dx_1 + V_h dy_1 + W_h dz_1),$$

and since this must hold for every value of α, β, γ, equating their coefficients we obtain

$$dL = I\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}dy_1 - \frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}dz_1\right\},$$

$$dM = I\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}dz_1 - \frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}dx_1\right\},$$

$$dN = I\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}dx_1 - \frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}dy_1\right\},$$

which give the values obtained in (11).

218. Forces on Conductor carrying Current. The magnetic energy of a pole m at P in the field due to a current is

$$(1)\qquad W = m\Omega = mI\iint \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\,dS$$

$$= I\iint \frac{\partial\left(\frac{m}{r}\right)}{\partial n}\,dS = I\iint \frac{\partial\Omega_m}{\partial n}\,dS,$$

where Ω_m is the potential due to the pole. For any number of of poles, in like manner Ω_m being the potential due to them all,

$$(2)\qquad W = I\iint \frac{\partial\Omega_m}{\partial n}\,dS = -I\iint H_m\cos(H_m n)\,dS,$$

which is the flux of force through the current circuit in the negative direction, due to all magnets. The potential energy tends to decrease, consequently a current in a magnetic field tends to move so as to make the surface integral a maximum, that is, to embrace the largest possible number of tubes of force linked with it in the positive direction. This statement of the mechanical action of magnetic forces on a current is due to Faraday.

219. Mechanical Force acting on Element of Circuit. We may consider the forces acting on the whole circuit as the resultant of the forces acting on each element ds_1, with the same degree of arbitrariness as in the case of the field due to the current. By the principle of reaction the force on ds_1 due to the presence of a unit pole P must be equal and opposite to the force dL, dM, dN on the unit pole, due to the current element ds_1. Consequently if $d\Xi$, $d\mathrm{H}$, dZ, are the components of the mechanical force acting on ds_1

$$d\Xi = \frac{I}{r^3}\{dy_1(z_1 - z) - dz_1(y_1 - y)\},$$

$$(3)\qquad d\mathrm{H} = \frac{I}{r^3}\{dz_1(x_1 - x) - dx_1(z_1 - z)\},$$

$$d\mathrm{Z} = \frac{I}{r^3}\{dx_1(y_1 - y) - dy_1(x_1 - x)\}.$$

But

$$\frac{1}{r^2}\frac{x_1 - x}{r} = L_m,\quad \frac{1}{r^2}\frac{y_1 - y}{r} = M_m,\quad \frac{1}{r^2}\frac{z_1 - z}{r} = N_m,$$

are the components of the field at ds_1 due to the unit pole at P.

Consequently

$$(4)\qquad \begin{aligned} d\Xi &= I\,(N_m dy_1 - M_m dz_1), \\ d\mathrm{H} &= I\,(L_m dz_1 - N_m dx_1), \\ dZ &= I\,(M_m dx_1 - L_m dy_1), \end{aligned}$$

and the whole force due to the presence of any number of magnetic bodies producing a field L, M, N is the resultant of all the individual actions

$$(5)\qquad \begin{aligned} d\Xi &= I\,(N dy_1 - M dz_1), \\ d\mathrm{H} &= I\,(L dz_1 - N dx_1), \\ dZ &= I\,(M dx_1 - L dy_1). \end{aligned}$$

That is: the mechanical force on the element is the vector product of the current element Ids_1 and of the magnetic field where it is situated*.

Suppose that the magnetic field is due to a second element ds_2 of strength I_2 at a distance r from ds_1. Then since by (11) § 217, putting ds_2 for ds_1, x_1, y_1, z_1 for x, y, z,

$$dL = \frac{I_2}{r^3}\{dy_2(z_1 - z_2) - dz_2(y_1 - y_2)\},$$

$$dM = \frac{I_2}{r^3}\{dz_2(x_1 - x_2) - dx_2(z_1 - z_2)\},$$

$$dN = \frac{I_2}{r^3}\{dx_2(y_1 - y_2) - dy_2(x_1 - x_2)\},$$

we have for the mechanical force acting on ds_1, by (5),

$$(6)\quad d^2\Xi = \frac{I_1 I_2}{r^3}\left[dy_1\{dx_2(y_1 - y_2) - dy_2(x_1 - x_2)\} - dz_1\{dz_2(x_1 - x_2) - dx_2(z_1 - z_2)\}\right].$$

Adding and subtracting the term $dx_1 dx_2 (x_1 - x_2)/r^3$ this may be written

$$(7)\quad \begin{aligned} d^2\Xi &= \frac{I_1 I_2}{r^2}\left\{-\frac{(x_1 - x_2)}{r}(dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2) + dx_2\left(\frac{x_1 - x_2}{r}dx_1 + \frac{y_1 - y_2}{r}dy_1 + \frac{z_1 - z_2}{r}dz_1\right)\right\} \\ &= \frac{I_1 I_2 ds_1 ds_2}{r^2}\{\cos(rx)\cos(ds_1 ds_2) - \cos(ds_2 x)\cos(r ds_1)\}, \end{aligned}$$

r being drawn from ds_1 to ds_2.

* It would be hard to devise a simpler rule for remembering the direction of the force than the one given on p. 12.

In like manner

$$d^2\mathrm{H} = \frac{I_1 I_2 ds_1 ds_2}{r^2}\{\cos(ry)\cos(ds_1 ds_2) - \cos(ds_2 y)\cos(rds_1)\},$$

$$d^2\mathrm{Z} = \frac{I_1 I_2 ds_1 ds_2}{r^2}\{\cos(rz)\cos(ds_1 ds_2) - \cos(ds_2 z)\cos(rds_1)\}.$$

The resultant d^2R has a component

$$\frac{I_1 I_2 ds_1 ds_2 \cos(ds_1 ds_2)}{r^2},$$

in the direction of r, and one of magnitude

$$-\frac{I_1 I_2 ds_1 ds_2 \cos(r, ds_1)}{r^2},$$

in the direction ds_2. This resolution into infinitesimal forces is unfortunate on account of the lack of symmetry with regard to the two elements.

220. Mutual Energy of two Currents. The whole force acting on the circuit 1 is found by integrating the expressions (7) already found for $d^2\Xi$, $d^2\mathrm{H}$, $d^2\mathrm{Z}$, around both circuits 1 and 2,

$$(8)\quad \Xi = I_1 I_2 \int_1\int_2 \frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\{dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2\}$$
$$-\int_1\int_2 dx_2\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}dx_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}dy_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}dz_1\right\},$$

with similar expressions for H and Z. If we now suppose the circuit 1 displaced or deformed in any manner, so that a point x_1, y_1, z_1, is displaced by the amount δx_1, δy_1, δz_1 the circuit 2 being fixed, the forces Ξ, H, Z do the work

$$(9)\quad \Xi\delta x_1 + \mathrm{H}\delta y_1 + Z\delta z_1 =$$

$$I_1 I_2\int_1\int_2\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\delta x_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}\delta y_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\delta z_1\right\}\{dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2\}$$
$$-\int_1\int_2(dx_2\delta x_1 + dy_2\delta y_1 + dz_2\delta z_1)\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}dx_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial y_2}dy_2 + \frac{\partial\left(\frac{1}{r}\right)}{\partial z_2}dz_2\right\}.$$

The second factor in the second integral may be written

$$\frac{d\left(\frac{1}{r}\right)}{ds_1}ds_1,$$

and we may then perform the integration around the circuit 1, integrating by parts, obtaining

$$\begin{aligned}\int_1 (dx_2\delta x_1 + dy_2\delta y_1 + dz_2\delta z_1)\frac{d\left(\frac{1}{r}\right)}{ds_1}ds_1 \\ = \frac{1}{r}(dx_2\delta x_1 + dy_2\delta y_1 + dz_2\delta z_1)/ \\ - \int_1 \frac{1}{r}\left(dx_2\frac{d\delta x_1}{ds_1} + dy_2\frac{d\delta y_1}{ds_1} + dz_2\frac{d\delta z_1}{ds_1}\right)ds_1.\end{aligned}$$

The integrated part vanishes, for the factors δx_1, δy_1, δz_1 are the same for the beginning and end of the circuit. Accordingly the expression for the work becomes

$$\begin{aligned}I_1 I_2\int_1\int_2\left\{\frac{\partial\left(\frac{1}{r}\right)}{\partial x_1}\delta x_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial y_1}\delta y_1 + \frac{\partial\left(\frac{1}{r}\right)}{\partial z_1}\delta z_1\right\}\{dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2\} \\ (10) \qquad + \int_1\int_2 \frac{1}{r}(dx_2\delta dx_1 + dy_2\delta dy_1 + dz_2\delta dz_1) \\ = \delta_1\left[I_1 I_2\int_1\int_2\frac{1}{r}(dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2)\right],\end{aligned}$$

δ_1 denoting the change made by changing $x_1\, y_1\, z_1$, keeping $x_2\, y_2\, z_2$ constant. We have accordingly obtained the work as the change due to the motion in the value of a line integral around both circuits. Consequently the mechanical forces are derivable from a force-function, and the integral represents the negative mutual potential energy due to the magnetic forces acting between the two currents.

$$\begin{aligned}(11) \qquad -W = I_1 I_2\int_1\int_2\frac{1}{r}(dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2) \\ = I_1 I_2\int_1\int_2\frac{\cos(ds_1 ds_2)}{r}ds_1 ds_2.\end{aligned}$$

This form of the integral was given by Franz Emil Neumann* in 1845 and is generally known in Germany by the name of the Electrodynamic Potential of the currents 1 and 2.

* Neumann, "Allgemeine Gesetze der inducirten Ströme." *Abh. Berl. Akad.* 1845.

221. Various Resolutions into Elementary Forces. The value of the integral will not be changed if we add to the integrand any expression,

$$\frac{\partial^2 F}{\partial s_1 \partial s_2},$$

where F is any function of r, for it will disappear when integrated around either circuit. If we put $F = r$,

$$\frac{\partial r}{\partial s_1} = \frac{\partial r}{\partial x_1}\frac{dx_1}{ds_1} + \frac{\partial r}{\partial y_1}\frac{dy_1}{ds_1} + \frac{\partial r}{\partial z_1}\frac{dz_1}{ds_1}.$$

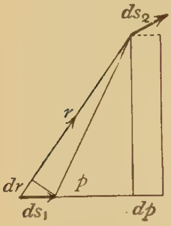

FIG. 87.

If, Fig. 87, we drop a perpendicular from ds_2 on the tangent at ds_1, and call the length of the tangent thus cut off p, we see by infinitesimal geometry that

$$-ds_1 \cos(r, ds_1) = dr, \qquad ds_2 \cos(ds_1, ds_2) = dp.$$

Accordingly

$$\frac{\partial r}{\partial s_1} = -\cos(r, ds_1),$$

(12) $$\frac{\partial p}{\partial s_2} = \cos(ds_1, ds_2),$$

$$\frac{\partial r}{\partial s_2} = \cos(r, ds_2).$$

But $$p = r\cos(r, ds_1) = -r\frac{\partial r}{\partial s_1}.$$

Consequently

(13) $$\cos(ds_1, ds_2) = \frac{\partial p}{\partial s_2} = -\frac{\partial}{\partial s_2}\left(r\frac{\partial r}{\partial s_1}\right) = -\frac{\partial r}{\partial s_2}\frac{\partial r}{\partial s_1} - r\frac{\partial^2 r}{\partial s_1 \partial s_2},$$

and

(14) $$\frac{\partial^2 r}{\partial s_1 \partial s_2} = -\frac{1}{r}\left\{\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2} + \cos(ds_1, ds_2)\right\}$$
$$= \frac{\cos(r, ds_1)\cos(r, ds_2) - \cos(ds_1, ds_2)}{r},$$

and multiplying this by an arbitrary constant $(1-k)/2$, and adding to the integrand in (11)*

(15) $$-W = I_1 I_2 \int_1 \int_2 \frac{1}{r}\left\{\frac{(1+k)}{2}\cos(ds_1, ds_2) + \frac{(1-k)}{2}\cos(r, ds_1)\cos(r, ds_2)\right\} ds_1 ds_2.$$

The value $k = 1$ gives Neumann's form of the integral, from which may be obtained the resolution into elementary forces already

* Helmholtz, *Wiss. Abh.* Bd. I. p. 567.

given. For $k=-1$ we get a resolution into forces proposed by Weber and C. Neumann, and for $k=0$ one implicitly suggested by Maxwell. Let us examine the case $k=-1$.

$$(16)\qquad -W = I_1 I_2 \int_1\int_2 \frac{\cos(r, ds_1)\cos(r, ds_2)}{r} ds_1 ds_2$$
$$= -I_1 I_2 \int_1\int_2 \frac{1}{r}\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2} ds_1 ds_2.$$

From this we obtain

$$(17)\quad -\delta W = I_1 I_2 \int_1\int_2 \left\{\frac{1}{r^2}\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2}\delta r - \frac{1}{r}\frac{\partial r}{\partial s_2}\frac{\partial \delta r}{\partial s_1} - \frac{1}{r}\frac{\partial r}{\partial s_1}\frac{\partial \delta r}{\partial s_2}\right\} ds_1 ds_2.$$

Integrating by parts, the second term around the circuit 1, and the third around the circuit 2, the integrated parts vanishing in both cases,

$$(18)\qquad -\delta W = I_1 I_2 \int_1\int_2 \left\{\frac{1}{r^2}\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2} + \frac{\partial}{\partial s_1}\left(\frac{1}{r}\frac{\partial r}{\partial s_2}\right) + \frac{\partial}{\partial s_2}\left(\frac{1}{r}\frac{\partial r}{\partial s_1}\right)\right\}\delta r\, ds_1 ds_2$$
$$= I_1 I_2 \int_1\int_2 \left\{-\frac{1}{r^2}\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2} + \frac{2}{r}\frac{\partial^2 r}{\partial s_1 \partial s_2}\right\}\delta r\, ds_1 ds_2.$$

Since the integrand contains the factor δr, work is done only when the distances apart of some of the pairs of elements are changed, and we may resolve the action into attractions between ds_1 and ds_2 of the magnitude

$$(19)\qquad I_1 I_2 \frac{1}{r^2}\left\{\frac{\partial r}{\partial s_1}\frac{\partial r}{\partial s_2} - 2r\frac{\partial^2 r}{\partial s_1 \partial s_2}\right\} ds_1 ds_2$$
$$= I_1 I_2 \frac{1}{r^2}\{2\cos(ds_1, ds_2) - 3\cos(r, ds_1)\cos(r, ds_2)\}.$$

This form for the elementary forces was given by Ampère*. According to this form, we see that parallel elements perpendicular to the line joining them attract each other with a force

$$\frac{2 I_1 I_2 ds_1 ds_2}{r^2}.$$

Parallel elements having the direction of the line joining them repel each other with a force

$$\frac{I_1 I_2 ds_1 ds_2}{r^2},$$

* Ampère. "Mémoire sur la théorie mathématique des phenomènes électrodynamiques." *Mém de l'Acad.* T. VI., 1823.

while mutually perpendicular elements exert no action on each other if either is perpendicular to the line joining them.

222. Currents distributed in three Dimensions. We have seen in § 210 that a current I is equivalent to a magnetic shell of strength

$$\Phi = \frac{J}{4\pi},$$

where J is the line integral of the magnetic force around a circuit positively linked once with the circuit, and in the electromagnetic system $I = \Phi$. Accordingly

$$J = 4\pi I.$$

If now we consider steady currents distributed in any manner in a conducting body with current density q, the integral of magnetic force around any closed curve depends only on the tubes of flow with which it is linked, being equal to 4π times the total current through the curve. Consequently

(1)
$$\int L dx + M dy + N dz$$
$$= 4\pi \iint \{u \cos(nx) + v \cos(ny) + w \cos(nz)\}\, dS,$$

the surface integral being taken over any surface bounded by the curve. But by Stokes's theorem

$$\int L dx + M dy + N dz$$
$$= \iint \left\{ \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \right) \cos(nx) + \left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \right) \cos(ny) \right.$$
$$\left. + \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \cos(nz) \right\} dS.$$

The surface integrals can be equal for all surfaces bounded by any curve whatsoever only if we have everywhere

(2)
$$4\pi u = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z},$$
$$4\pi v = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x},$$
$$4\pi w = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}.$$

These are the fundamental equations of electromagnetism. In the Gaussian system, we must introduce the factor A on the left. By these equations the solenoidal vector q is expressed as the curl of the magnetic force H. The magnetic forces cannot be derived from a potential except where there is no current, but must be found by integration of the partial differential equations (2). In order to show how this may always be accomplished, we shall prove a general theorem.

223. Vector Potentials. Helmholtz's Theorem. Any uniform, continuous, vector point-function vanishing at infinity may be expressed as the sum of a lamellar and a solenoidal part, and the solenoidal part may be expressed as the curl of a vector point-function. A vector point-function is completely determined if its divergence and curl are everywhere given.

Let R be the given vector, with components X, Y, Z. Let us suppose it possible to express it as the sum of the vector parameter of a scalar function ϕ and the curl of a vector-function Q, whose components are U, V, W. Then

(1)
$$X = \frac{\partial\phi}{\partial x} + \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z},$$
$$Y = \frac{\partial\phi}{\partial y} + \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x},$$
$$Z = \frac{\partial\phi}{\partial z} + \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}.$$

Finding first the divergence of R,

$$\operatorname{div} R = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = \Delta\phi,$$

for the curl of any vector is solenoidal, § 35.

But by § 85 (18) we know that if ϕ and its first derivatives are everywhere finite and continuous, we have

(2)
$$\phi = -\frac{1}{4\pi}\iiint_\infty \frac{\Delta\phi}{r}\,d\tau.$$

Since R is continuous by hypothesis, div R is finite, so that

(3)
$$\phi = -\frac{1}{4\pi}\iiint_\infty \left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\}\frac{d\tau}{r}.$$

Consequently the lamellar part of R is determined by its divergence. Secondly finding the curl of R, say ω, with components ξ, η, ζ,

$$
(4) \quad \begin{aligned} \xi = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z} + \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y} + \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}\right) \\ &= -\Delta U + \frac{\partial}{\partial x}\left\{\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right\}. \end{aligned}
$$

Since Q is as yet undetermined except by the partial differential equations (1) we may impose on it the condition of being solenoidal,

$$
(5) \quad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0.
$$

Hence

$$
\xi = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = -\Delta U,
$$

and in like manner

$$
(6) \quad \begin{aligned} \eta &= \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} = -\Delta V, \\ \zeta &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = -\Delta W. \end{aligned}
$$

But since R is continuous, curl R must be finite, and therefore as before

$$
(7) \quad \begin{aligned} U &= \frac{1}{4\pi}\iiint_\infty \frac{\xi}{r}\,d\tau, \\ V &= \frac{1}{4\pi}\iiint_\infty \frac{\eta}{r}\,d\tau, \\ W &= \frac{1}{4\pi}\iiint_\infty \frac{\zeta}{r}\,d\tau. \end{aligned}
$$

The vector Q, whose components are $1/4\pi$ times the potentials of the scalar functions ξ, η, ζ, the components of ω, is derived from $\omega/4\pi$ by the operation Pot, considering ω as a vector, so that we may write

$$
(8) \quad \bar{Q} = \frac{1}{4\pi}\,\text{Pot}\,\bar{\omega},
$$

and call $4\pi Q$ the *Vector Potential of* ω. Since the solenoidal part of R is the curl of Q, we shall also say that Q is the vector potential *belonging to* R. We accordingly see that the solenoidal part of R is determined by curl R, and accordingly the vector

is uniquely determined by its divergence and curl. This theorem was given by Helmholtz in his celebrated paper on Vortex Motion*.

224. Symbolic Formulae. These relations may be concisely expressed by means of Hamilton's and Gibbs's symbols ∇ and Pot (§ 78). In words we may say that any solenoidal vector is the curl of the vector potential belonging to it, which is the vector potential of $1/4\pi$ times its curl.

By virtue of the definition of Hamilton's operator we have the vector equation

$$(9) \qquad \overline{R} = \overline{\nabla\phi} + \overline{\nabla\overline{Q}},$$

so that we may call the sum of the scalar ϕ and the vector Q the *quaternion potential* belonging to R, from which R is derived by the single vector operation ∇. Inserting the values of ϕ and Q,

$$(10) \qquad \overline{R} = \nabla\left\{\frac{1}{4\pi}(-\text{Pot div } R + \overline{\text{Pot curl } R})\right\},$$

so that the operator $(\overline{\text{Pot}}\text{ curl} - \text{Pot div})/4\pi$ is the inverse of ∇, when applied to a vector-function.

For a lamellar vector we have

$$(11) \qquad \text{curl } \overline{R} = 0, \quad \overline{R} = -\frac{1}{4\pi}\nabla \text{ Pot div } \overline{R},$$

and for a solenoidal vector

$$(12) \quad \text{div } R = 0, \quad R = \frac{1}{4\pi}\nabla \text{ Pot curl } R = \frac{1}{4\pi}\text{ curl Pot curl } R.$$

Taking the curl of ω, we find in like manner

$$\text{curl } \overline{\omega} = \text{curl}^2\, \overline{R} = -\overline{\Delta\overline{R}},$$

(R being solenoidal) so that

$$(13) \qquad \overline{R} = \frac{1}{4\pi}\overline{\text{Pot}}\text{ curl}^2\, \overline{R}.$$

In fact since the operations of definite integration and partial differentiation are commutative, the operations Pot and curl must be.

* Helmholtz. "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle's Journal*, Bd. 55, 1858, p. 25. *Wiss. Abh.* Bd. I. p. 101.

225. Magnetic Force from Current. Applying Helmholtz's theorem to the solenoidal vector H, the magnetic force, and calling the components of the vector potential F, G, H,* and using the fundamental equations (7), together with § 222 (2), we obtain

$$(14)\qquad L=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z},\quad M=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x},\quad N=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y};$$

$$(15)\qquad F=\iiint_\infty \frac{u}{r}\,d\tau,\quad G=\iiint_\infty \frac{v}{r}\,d\tau,\quad H=\iiint_\infty \frac{w}{r}\,d\tau,$$

or the vector potential belonging to the magnetic force is the vector potential of the current density.

226. Energy of Magnetic Field of Currents. The magnetic energy of the field is by § 118 (10),

$$(1)\qquad W_m=\frac{1}{8\pi}\iiint_\infty (L^2+M^2+N^2)\,d\tau,$$

and introducing the vector potential this becomes

$$(2)\quad W_m=\frac{1}{8\pi}\iiint_\infty\left\{L\left(\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}\right)+M\left(\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}\right)+N\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)\right\}d\tau.$$

Integrating by parts for any volume τ bounded by a closed surface S,

* It is to be noticed that the letter H is here unfortunately used for both the resultant magnetic force and one component of the vector-potential. This is because we have followed Maxwell in using the letters F, G, H. The ambiguity need cause no confusion.

(3) $$\iiint\left\{L\left(\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}\right)+M\left(\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}\right)+N\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)\right\}d\tau$$
$$=\iint\{(MH-NG)\cos(nx)+(NF-LH)\cos(ny)$$
$$+(LG-MF)\cos(nz)\}\,dS$$
$$+\iiint\left\{F\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right)+G\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)+H\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)\right\}d\tau.$$

This important theorem in integration may be abbreviated as

(4) $$\iiint\{\overparen{H\,\mathrm{curl}\,Q}-\overparen{Q\,\mathrm{curl}\,H}\}\,d\tau=\iint\mathbf{V}.HQ\cos(n,\mathbf{V}.HQ)\,dS.$$

The integral representing the energy is extended over infinite space, and the surface integral vanishes at infinity. Inserting the value of curl H in terms of the current density, § 222 (2), we obtain

(5) $$W_m=\frac{1}{2}\iiint(Fu+Gv+Hw)\,d\tau,$$

and since no portion of space contributes to the integral unless it is traversed by currents, we may take the integral simply through conductors carrying currents. The components of the vector potential are however themselves triple integrals over the same portions of space, so that if we distinguish a second point of integration by an accent, we have the double volume integral

(6) $$\frac{1}{2}\iiiint\!\!\iint\frac{uu'+vv'+ww'}{r}\,d\tau d\tau'=\frac{1}{2}\iiiint\!\!\iint\frac{qq'\cos(qq')}{r}\,d\tau d\tau',$$
$$r^2=(x-x')^2+(y-y')^2+(z-z')^2,$$

where each point of integration traverses the whole volume occupied by currents.

This form of the energy corresponds to the form in terms of density given in § 117 (5), the integrals being there taken through all distributions of matter.

If we perform the volume integration by dividing the space up into current tubes, of infinitesimal cross-section S, ds being the length of the generating curve, and $I=qS$ the total current in the tube, we have for the element of volume $d\tau=Sds$, so that the integral becomes

(7) $$\frac{1}{2}\iiiint\!\!\iint\frac{II'\cos(ds\,ds')}{r}\,ds\,ds',$$

both ds and ds' traversing all current tubes. The sextuple integral is here interpreted as a line integral around every current tube and then an integration for the double infinity of tubes for each variable s and s'. If the currents consist of two linear circuits, or closed tubes of infinitesimal cross-section and strengths I_1 and I_2, the sextuple integral reduces to a double-line-integral, and since *both* variables s and s' are to traverse *both* circuits, we may divide the integral up into four parts according as s or s' coincide with s_1 or s_2,

$$W_m = \frac{I_1^2}{2}\int\limits_{s=s_1}\int\limits_{s'=s_1}\frac{\cos(ds\,ds')}{r}\,ds\,ds' + \frac{I_1 I_2}{2}\int\limits_{s=s_1}\int\limits_{s'=s_2}\frac{\cos(ds\,ds')}{r}\,ds\,ds'$$

(8)

$$+\frac{I_1 I_2}{2}\int\limits_{s=s_2}\int\limits_{s'=s_1}\frac{\cos(ds\,ds')}{r}\,ds\,ds' + \frac{I_2^2}{2}\int\limits_{s=s_2}\int\limits_{s'=s_2}\frac{\cos(ds\,ds')}{r}\,ds\,ds'.$$

The second and third integrals are equal, for it is evidently a matter of indifference which point of integration is associated with either circuit, so that we may write for the sum of these two terms

$$I_1 I_2\int_1\int_2\frac{\cos(ds_1 ds_2)}{r}\,ds_1\,ds_2,$$

where each point of integration goes once around *one* of the circuits.

This term is equal to the *negative* of the mutual potential energy of the electromagnetic forces acting between the two currents, as found in § 220 (11).

In like manner the first and last terms, where each point of integration goes once around the same circuit, are the negatives respectively of the potential energy of either current in its own field, from which the electromagnetic forces acting between its different parts may be calculated. If we call the integrals

$$L_1 = \int_1\int_1\frac{\cos(ds\,ds')}{r}\,ds\,ds', \quad L_2 = \int_2\int_2\frac{\cos(ds\,ds')}{r}\,ds\,ds',$$

$$M_{12} = \int_1\int_2\frac{\cos(ds_1 ds_2)}{r}\,ds_1\,ds_2,$$

we may say then that the magnetic energy of the field due to both currents

(9) $$W_m = \tfrac{1}{2}L_1 I_1^2 + M_{12} I_1 I_2 + \tfrac{1}{2}L_2 I_2^2,$$

is the negative of the total potential energy. But the potential energy tends to decrease, and if the current strengths are constant, while the circuits are moved or deformed, their position and form being specified by a certain number of geometrical parameters q_s, the forces P_s according to these parameters are given by

$$\Sigma_s P_s \delta q_s = -\delta W = \delta W_m = \tfrac{1}{2} I_1^2 \delta L_1 + I_1 I_2 \delta M_{12} + \tfrac{1}{2} I_2^2 \delta L_2; \tag{10}$$

$$P_s = \tfrac{1}{2} I_1^2 \frac{\partial L_1}{\partial q_s} + I_1 I_2 \frac{\partial M_{12}}{\partial q_s} + \tfrac{1}{2} I_2^2 \frac{\partial L_2}{\partial q_s}. \tag{11}$$

The magnetic energy of the field then tends to increase, and we find the system behaving in the same manner as a cyclic system during an isocyclic motion, § 70. The energy which must be furnished to the system during a motion caused by the electromagnetic forces must be double the amount of work done by the electromagnetic forces, which is equal to the loss of potential energy, and must be furnished by the impressed electromotive forces that maintain the currents. We have already seen that in the case of concealed motions we cannot always tell whether energy is potential or kinetic, and that in cyclic systems the *kinetic* energy has the properties of a force function for either isocyclic or adiabatic motions. We are therefore led naturally to consider a system of currents as a cyclic system, and, instead of considering W as potential energy, to consider $W_m = -W$ as kinetic energy. We shall henceforth call it the *electrokinetic energy*, and denote it by T.

These considerations, assimilating an electrical system to a mechanical system, are due principally to Maxwell, and by means of them we shall in the next chapter be able to deduce the laws of induction of currents.

If in the integral (5) we integrate over current-tubes in the manner just explained, for $u d\tau$ we must put

$$q \cos (qx) S ds = I dx \text{ etc.},$$

so that we obtain for each current

$$T = \frac{I}{2} \int (F dx + G dy + H dz), \tag{12}$$

where the integral is around its own circuit, but F, G, H are the definite integrals over all currents, as previously used. Applying Stokes's theorem to the above line-integral, we obtain

$$(13)\quad T=\frac{I}{2}\iint\left\{\left(\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}\right)\cos(nx)+\left(\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}\right)\cos(ny)\right.$$
$$\left.+\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)\cos(nz)\right\}dS$$

over any surface bounded by the current.

But by the equations § 225 (14),

$$(14)\qquad T=\frac{I}{2}\iint\{L\cos(nx)+M\cos(ny)+N\cos(nz)\}\,dS,$$

or the electrokinetic energy of a system of currents is equal to one-half the sum of the strengths of each current multiplied by the total flux of magnetic force through its own circuit in the positive direction. The part of the flux due to the current itself constitutes the term $\frac{1}{2}LI^2$, while for any two currents 1 and 2, the portions consisting of one-half the strength of either times the flux through its circuit due to the other current, being equal to the two middle terms of (8), are equal. We may consequently express the mutual kinetic energy of two currents as the strength of either multiplied by the flux through its circuit of the magnetic force due to the other.

227. Mechanical Forces. We may deduce the mechanical forces acting on conductors carrying currents from the expressions found in § 219 (7). Calling the forces per unit of volume Ξ, H, Z, and writing for Idx the value in terms of the current density $ud\tau$, we have

$$(1)\qquad \iiint\Xi\,d\tau=\iiiint\!\!\iint\frac{\partial\left(\frac{1}{r}\right)}{\partial x}(uu'+vv'+ww')\,d\tau\,d\tau'$$
$$-\iiiint\!\!\iint u'\left\{u\frac{\partial\left(\frac{1}{r}\right)}{\partial x}+v\frac{\partial\left(\frac{1}{r}\right)}{\partial y}+w\frac{\partial\left(\frac{1}{r}\right)}{\partial z}\right\}d\tau\,d\tau'.$$

The first terms in the first and second integrals destroy each other. The second terms may be written respectively, since the accented quantities are independent of the unaccented,

$$\iiint v\frac{\partial}{\partial x}\left\{\iiint\frac{v'}{r}d\tau'\right\}d\tau=\iiint v\frac{\partial G}{\partial x}\,d\tau,$$

and

$$-\iiint v\frac{\partial}{\partial y}\left\{\iiint\frac{u'}{r}d\tau'\right\}d\tau=-\iiint v\frac{\partial F}{\partial y}\,d\tau.$$

Consequently we get

$$(2) \qquad \iiint \Xi \, d\tau = \iiint \left\{ v \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) - w \left(\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} \right) \right\} d\tau$$

$$= \iiint (vN - wM) \, d\tau,$$

and we obtain for the mechanical forces on the conductor per unit volume,

$$(3) \qquad \begin{aligned} \Xi &= vN - wM, \\ \mathrm{H} &= wL - uN, \\ \mathrm{Z} &= uM - vL. \end{aligned}$$

The mechanical force per unit volume is the vector product of the current density and the magnetic field.

228. Effect of Heterogeneous Medium. Let us consider what changes are necessitated in our equations by the presence of magnetizable bodies, so that the magnetic inductivity μ is not constant throughout space. In the reasoning of § 210 we supposed the magnetic force to be both lamellar and solenoidal in all space not traversed by currents. As soon as we have variations in the inductivity, the force is in general no longer solenoidal, but the induction is. We cannot, however, apply the reasoning unchanged to the induction, for this, in general, is not lamellar. The reasoning connecting the current strength with the work of carrying a pole around a closed circuit is however unchanged, and if the circuit lie in any other medium than air, the work is the same as if the circuit lay in air, namely zero if the circuit is not linked with the current, $4\pi nI$ if linked n times positively. For consider a circuit composed of two infinitely near circuits each embracing the current once, corresponding points of the two lying infinitely near each other on opposite sides of a surface separating air from another medium. Then if we carry a pole around the circuit in air in one direction, and back around the circuit in the other medium in the opposite direction, since the double circuit is not linked with the current no work has been done. For otherwise, in going around the double circuit in one direction or the other, we might store up energy, as much as we pleased, by repeating the operation. But this would be in opposition to the principle of conservation of energy, which says that the energy is definitely determined when the positions and strengths of poles and currents are given. Consequently our

electromagnetic equations § 222 (2) remain unaltered. When we consider the energy and the mechanical forces, however, we have changes. The potential due to a current is no longer proportional to the solid angle subtended by it, and accordingly we can no longer deduce the forces as simple line-integrals. We must now write for the energy of the field, by § 180,

$$(1) \qquad T = \frac{1}{8\pi} \iiint_{\infty} (L\mathfrak{L} + M\mathfrak{M} + N\mathfrak{N})\, d\tau,$$

so that if a given current is placed in an infinite homogeneous medium, since the distribution of the force is independent of the medium, as long as it is homogeneous, the induction, and therefore the energy, are directly proportional to the inductivity. Contrast this behaviour of a current with that of a permanent magnet, which in different homogeneous media always emits the same total flux of induction, while the force and therefore the energy are inversely proportional to the inductivity. The flux of *force* emitted by the conductor carrying current is constant.

Since the magnetic force is no longer solenoidal, it can no longer be represented as the curl of a vector potential. The induction, on the contrary, can be so represented, and the vector potential belongs to the magnetic induction.

$$(2) \qquad \begin{aligned} \mathfrak{L} &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \\ \mathfrak{M} &= \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \\ \mathfrak{N} &= \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}. \end{aligned}$$

On account of this change it is no longer possible to integrate the equations § 222 (2) in the same simple manner as in § 225, for while the current is the curl of the magnetic force, it is the induction that is the curl of the vector potential. Taking the curl of the induction,

$$(3) \qquad \frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z} = \mu \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \right) + N \frac{\partial \mu}{\partial y} - M \frac{\partial \mu}{\partial z} = -\Delta F,$$

or using § 222 (2),

$$4\pi\mu u + N \frac{\partial \mu}{\partial y} - M \frac{\partial \mu}{\partial z} = -\Delta F,$$

so that the vector potential is not related simply to the current, but the magnetic forces still occur in the differential equation. We have the same difficulty as occurs when we undertake to find the potential of the field when the inductivity varies. As we there made use of an apparent density, so here we may define an apparent current as

$$u' = \mu u + \frac{1}{4\pi}\left\{\frac{\partial \mu}{\partial y} N - \frac{\partial \mu}{\partial z} M\right\} \text{ etc.,}$$

so that the vector potentials are

$$F = \iiint \frac{u'}{r}\, d\tau, \quad G = \iiint \frac{v'}{r}\, d\tau, \quad H = \iiint \frac{w'}{r}\, d\tau.$$

If each magnetizable body is homogeneous,

$$\frac{\partial \mu}{\partial x} = \frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial z} = 0,$$

and the apparent currents are the true currents multiplied by the inductivity, except at the surface separating two media, where the derivatives of μ, and consequently the values of ΔF, ΔG, ΔH, are infinite. We have the above form for F, G, H only when ΔF, etc. are finite, and when they are infinite at a surface we must proceed as in the case of a surface distribution of matter, that is we must consider an apparent current-sheet between the two media. Considering two surfaces infinitely near each other and situated on opposite sides of a surface of discontinuity of μ at a distance dn from each other, and integrating the equation (3) over the volume of the thin sheet between them we obtain*

$$(4) \qquad \iiint\left\{\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right\} dn\, dS$$

$$= \iiint\left\{\frac{\partial \mathfrak{N}}{\partial n}\cos(ny) - \frac{\partial \mathfrak{M}}{\partial n}\cos(nz)\right\} dn\, dS$$

$$= \iint\{(\mathfrak{N}' - \mathfrak{N})\cos(ny) - (\mathfrak{M}' - \mathfrak{M})\cos(nz)\}\, dS$$

$$= -\iiint \Delta F\, d\tau = -\iint\left(\frac{\partial F}{\partial n_1} + \frac{\partial F}{\partial n_2}\right) dS,$$

* The second integrand in (4) is equal to the first since n is the direction of *most rapid* (infinitely rapid) *change* in the functions $\mathfrak{N}$, $\mathfrak{M}$, in the infinitely thin sheet.

$\mathfrak{L}'$, $\mathfrak{M}'$, $\mathfrak{N}'$, being the components of the induction on the side toward which n is drawn, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, on the side from which it is drawn. We must now, as shown in § 85 (18), add to the volume integral already found for F the surface integral

$$-\frac{1}{4\pi}\iint\left\{\frac{\partial F}{\partial n_1}+\frac{\partial F}{\partial n_2}\right\}\frac{dS}{r},$$

which is the effect of an apparent current whose x-component per unit of surface is $1/4\pi$ times

$$\begin{aligned}(\mathfrak{N}'-\mathfrak{N})\cos(ny)-(\mathfrak{M}'-\mathfrak{M})\cos(nz)\\ =\mathfrak{B}'\{\cos(\mathfrak{B}'z)\cos(ny)-\cos(\mathfrak{B}'y)\cos(nz)\}\\ -\mathfrak{B}\{\cos(\mathfrak{B}z)\cos(ny)-\cos(\mathfrak{B}y)\cos(nz)\}.\end{aligned}$$

Now the normal component of the induction is continuous, its tangential component being discontinuous, while the tangential component of the force is continuous. The normal plane tangent to the line of force is the same in both media, and the amount of the discontinuity in the tangential component of the induction is

$$\mathfrak{B}'\sin(\mathfrak{B}'n)-\mathfrak{B}\sin(\mathfrak{B}n).$$

Referring now to the definition of a vector product, we see that the first parenthesis above is the x-component of the vector product of the induction and a unit vector in the direction of the normal, which vector product has the magnitude $\mathfrak{B}'\sin(\mathfrak{B}'n)$. The apparent current is accordingly in the surface, perpendicular to the normal plane containing the line of force where it crosses the surface, and its magnitude per unit of surface is $1/4\pi$ times the discontinuity in the tangential induction. If the lines of force are normal to the surface, the apparent surface current vanishes*. If, however, there is a surface carrying a *true* current-sheet, by the same reasoning, applied to equations § 222 (2), we find a discontinuity in the component of the force tangent to the surface and perpendicular to the current of amount 4π times the current density.

229. Mutual Energy of Magnets and Currents. If we have permanent magnets and currents situated in a homogeneous medium of unit inductivity, we may represent their mutual energy in two ways. We may in the first place consider the magnets to

* This apparent current-sheet was overlooked by Maxwell, and it was not until the appearance of the Third Edition of his Treatise that the correction was made by J. J. Thomson.

be traversed by apparent currents and current-sheets, as in the preceding section in the case of temporary magnets. We there introduced the discontinuity in the induction, but we might have introduced the intensity of magnetization. In the case of the permanent magnet this will be more convenient—in either case the form of the vector potential will be the same.

We have for the potential due to a magnet in a homogeneous medium of unit inductivity, § 122 (3),

$$(1)\qquad \Omega = \iiint \left\{ A' \frac{\partial\left(\frac{1}{r}\right)}{\partial a} + B' \frac{\partial\left(\frac{1}{r}\right)}{\partial b} + C' \frac{\partial\left(\frac{1}{r}\right)}{\partial c} \right\} d\tau',$$

where A', B', C' are the values at a, b, c,

$$d\tau' = da\,db\,dc,$$

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

The field at any point x, y, z has the component

$$(2)\qquad L = -\frac{\partial\Omega}{\partial x} = -\iiint \left\{ A' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x\partial a} + B' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x\partial b} + C' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x\partial c} \right\} d\tau'.$$

Now the derivatives of $1/r$ with respect to a, b, c are the negatives of its derivatives with respect to x, y, z, so that we may write

$$(3)\qquad L = \iiint \left\{ A' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + B' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x\partial y} + C' \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x\partial z} \right\} d\tau'.$$

But since $1/r$ is harmonic we may put

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} = -\left\{ \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2} \right\},$$

and thus the integral becomes

$$L = \iiint \left\{ \frac{\partial}{\partial y}\left(B' \frac{\partial\left(\frac{1}{r}\right)}{\partial x} - A' \frac{\partial\left(\frac{1}{r}\right)}{\partial y} \right) - \frac{\partial}{\partial z}\left(A' \frac{\partial\left(\frac{1}{r}\right)}{\partial z} - C' \frac{\partial\left(\frac{1}{r}\right)}{\partial x} \right) \right\} d\tau'$$

(4)

$$= \frac{\partial}{\partial y} \iiint \left\{ A' \frac{\partial\left(\frac{1}{r}\right)}{\partial b} - B' \frac{\partial\left(\frac{1}{r}\right)}{\partial a} \right\} d\tau - \frac{\partial}{\partial z} \iiint \left\{ C' \frac{\partial\left(\frac{1}{r}\right)}{\partial a} - A' \frac{\partial\left(\frac{1}{r}\right)}{\partial c} \right\} d\tau.$$

But since $\mu = 1$ we have

$$L = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z},$$

and accordingly the components of the vector potential may be taken as

$$F = \iiint \left\{ B' \frac{\partial \left(\frac{1}{r}\right)}{\partial c} - C' \frac{\partial \left(\frac{1}{r}\right)}{\partial b} \right\} d\tau',$$

(5) $$G = \iiint \left\{ C' \frac{\partial \left(\frac{1}{r}\right)}{\partial a} - A' \frac{\partial \left(\frac{1}{r}\right)}{\partial c} \right\} d\tau',$$

$$H = \iiint \left\{ A' \frac{\partial \left(\frac{1}{r}\right)}{\partial b} - B' \frac{\partial \left(\frac{1}{r}\right)}{\partial a} \right\} d\tau'.$$

Every element of volume produces at x, y, z a portion of vector potential equal to $1/r^2$ times the vector product of its magnetization by its vector distance from the point x, y, z. The mutual energy of currents and magnets is then obtained by the equation § 226 (5), omitting the factor $\frac{1}{2}$. This method of treatment is that of Maxwell*.

From the above form for the vector potentials we may easily express the solenoidal vector F, G, H as itself the curl of another vector potential. For again replacing derivatives of $1/r$ by a, b, c by derivatives by x, y, z,

(6) $$F = -\iiint \left(B' \frac{\partial \left(\frac{1}{r}\right)}{\partial z} - C' \frac{\partial \left(\frac{1}{r}\right)}{\partial y} \right) d\tau'$$
$$= \frac{\partial}{\partial y} \iiint \frac{C'}{r} d\tau' - \frac{\partial}{\partial z} \iiint \frac{B'}{r} d\tau',$$

so that if we introduce the vector potential of magnetization, with components

$$P = \iiint \frac{A'}{r} d\tau',$$

(7) $$Q = \iiint \frac{B'}{r} d\tau',$$

$$R = \iiint \frac{C'}{r} d\tau',$$

* *Treatise*, Vol. II., Art. 405.

the vector potential belonging to the magnetic force is its curl.

$$F = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},$$

(8) $$G = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},$$

$$H = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

This leads us to a second manner of obtaining the mutual energy, due to Helmholtz. The expression for the energy of a permanent magnet in a magnetic field L, M, N is, § 126 (2),

(9) $$W = -\iiint (AL + BM + CN)\, d\tau,$$

where the volume of integration is that occupied by magnets, or it may be extended to infinity, since elsewhere

$$A = B = C = 0.$$

We may transform the integral into one taken throughout the space occupied by currents. If we introduce the vector potential of intensity of magnetization, we have from (7), if the magnetization is everywhere finite,

$$A = -\frac{1}{4\pi}\Delta P,$$

(10) $$B = -\frac{1}{4\pi}\Delta Q,$$

$$C = -\frac{1}{4\pi}\Delta R.$$

Introducing these values of A, B, C into the integral (9),

(11) $$W = \frac{1}{4\pi}\iiint_{\infty} (L\Delta P + M\Delta Q + N\Delta R)\, d\tau,$$

and transforming each term by Green's theorem in its second form, the surface integrals vanishing at infinity,

(12) $$W = \frac{1}{4\pi}\iiint_{\infty} (P\Delta L + Q\Delta M + R\Delta N)\, d\tau.$$

Let us now substitute for L, M, N their values in terms of the vector potential belonging to them, noticing that, since the differ-

ential operator Δ is commutative with any partial differentiation, we may write

$$\Delta L = \Delta\left(\frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}\right) = \frac{\partial \Delta H}{\partial y} - \frac{\partial \Delta G}{\partial z}. \tag{13}$$

But if the vector potentials F, G, H are those of the currents u, v, w,

$$\Delta F = -4\pi u,$$
$$\Delta G = -4\pi v,$$
$$\Delta H = -4\pi w,$$

so that finally

$$W = -\iiint_\infty \left\{P\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + Q\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + R\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} d\tau, \tag{14}$$

which by the theorem of § 226 (3), (4) is equal to

$$W = -\iiint_\infty \left\{u\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + v\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + w\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right\} \tag{15}$$

$$= -\iiint_\infty \{uF + vG + wH\}\, d\tau.$$

Comparing with § 226 (5) we find a difference in sign, W being mutual potential energy, W_m electrokinetic energy, while the factor $\frac{1}{2}$ is omitted in *mutual* energy.

The integral may now be restricted to the space occupied by currents. The form involving the curl of P, Q, R is that used by Helmholtz*, who writes L, M, N instead of P, Q, R. Replacing P, Q, R by their values (7) we obtain the double volume-integral

$$W = -\iiiint\!\!\iint \left\{(wB' - vC')\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + (uC' - wA')\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + (vA' - uB')\frac{\partial\left(\frac{1}{r}\right)}{\partial z}\right\} d\tau\, d\tau', \tag{16}$$

which differs from the result of substituting (5) in § 226 (5) in the same way as (15), above.

We have thus seen how we may replace every magnet by an apparent current

* Helmholtz, *Ges. Abh.* Bd. I. p. 619.

$$u' = -\frac{1}{4\pi}\Delta F = -\frac{1}{4\pi}\Delta\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right),$$
$$v' = -\frac{1}{4\pi}\Delta G = -\frac{1}{4\pi}\Delta\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right),$$
$$w' = -\frac{1}{4\pi}\Delta H = -\frac{1}{4\pi}\Delta\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

which would produce the same magnetic effect. It was for this reason that Ampère was led to the hypothesis that all magnetism was due to currents of electricity circulating about the molecules of matter.

The above formulæ all refer to currents and magnets placed in a homogeneous medium, and as has been already seen, lose all their simplicity when the inductivity varies. For although we may still calculate the vector potentials due to the induced magnetization, the process will be complicated, and in general impracticable. For this reason, and because both scalar and vector potentials are quantities whose physical significance is much less apparent than that of the strength of the field, Heaviside and Hertz have been led to avoid the employment of potentials, and to deal directly with the electrical and magnetic fields. We have however introduced the vector-potentials here on account of their important mathematical relations, and the fact that they have been so much used by the highest authorities.

230. Magnetic Field due to Current-Sheet. We have found that in a current-sheet the amount of electricity that flows in unit time across a curve connecting any two points in the sheet is equal to the difference of the current-function Ψ at those two points. This quantity is the same whatever the curve connecting them, unless there is an electrode lying between. We shall suppose that a sheet has no electrodes, so that the current flows in closed circuits in the sheet. We may find the magnetic field of such a sheet, at points not lying in the sheet, by the consideration that the strip of the sheet bounded by the curves $\Psi = \text{const.}$ and $\Psi + d\Psi = \text{const.}$, $d\Psi$ being a constant difference in the values of the current-function for the two curves, is equivalent to a linear current of strength $d\Psi$. Such a current, by § 210, is equivalent to a magnetic shell of strength $d\Psi$. The whole current sheet may therefore be replaced by an infinite series of magnetic shells, whose edges only are given, the form of the shells being in-

different, so long as the attracted point lies outside them. These shells may be considered to form a continuous body, which is, being divided into shells, lamellarly magnetized, the potential of magnetization ϕ being equal to the current-function Ψ (§ 124).

The magnetic potential Ω is accordingly at outside points, by § 124 (11),

$$\Omega = -\iint \Psi \frac{\partial \left(\frac{1}{r}\right)}{\partial n} dS.$$

But since the form of the magnetic shells is indifferent, as long as their edges are of the given shape, we may consider them all deformed so as to coincide with the current-sheet, as is illustrated in Fig. 88. The shells overlap each other continuously, so that there are more shells laid on the sheet the greater the values of Ψ. As we cross the sheet, the potential Ω is discontinuous, as in the case of a single magnetic shell. As in that case also, the normal component of the magnetic force, being continuous for all the shells, is continuous on crossing the sheet. The tangential component in the direction of the lines of flow is also continuous, but, as we found at the end of § 228, the component perpendicular to them experiences a discontinuity equal to 4π times the current-density, that is $4\pi\partial\Psi/\partial n_{\Psi}$. This may also be very simply obtained by taking the line-integral of magnetic force around any circuit composed of two infinitely near portions lying on opposite sides of the current-sheet and coinciding with an *electrical* equipotential line, the integral being equal to 4π times the difference in the values of the current-function at the two points where the circuit cuts the sheet.

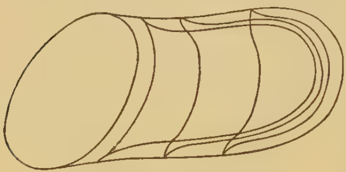

FIG. 88.

231. Examples. Coefficients of Induction. Toroidal and straight coils. We shall now calculate the energy due to currents in a few simple cases. The coefficients of the half-squares and products of the current-strengths in the expression for the electrokinetic energy, are called, for reasons to be explained in the next chapter, *coefficients of induction*, or more briefly, *inductances*, distinguishing coefficients of half-squares by the name self-inductance, coefficients of products by the name mutual inductance.

Any self or mutual inductance is the magnetic flux through a circuit due to unit current in its own or another circuit respectively.

We shall first consider a solid of revolution bounded by a surface generated by revolving any closed plane curve about an axis in its plane not cutting it. Such a solid may be called a *tore*. If the tore be uniformly wound with wire carrying a current, so that every winding lies very nearly in a plane passing through the axis of revolution, Fig. 89, we may very approximately consider the layer of wire as a current sheet, the difference of value of the current function between any two points being mI, where I is the current in the wire, and m is the number of turns of wire between the points. By reason of symmetry the lines of magnetic force must be circles whose planes are perpendicular to the axis of revolution, and whose centers lie on the axis. Consequently the strength of the field H is a function only of the distance ρ from the axis, and the line integral of the field-strength around any line of force is equal to the constant value of H on that line times the circumference of the circle. If n be the total number of turns of wire on the tore, any circle lying in the substance of the tore is linked with the current n times in the same direction, so that the value of the above line-integral is

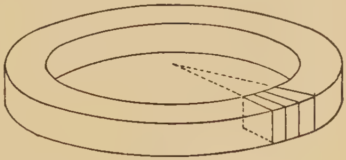

Fig. 89.

(1) $$4n\pi I = 2\pi\rho H.$$

This gives as the value of the force for internal points

(2) $$H = \frac{2nI}{\rho}.$$

A circle lying outside the tore, however, is not linked at all with the current, so that the line integral is zero, and therefore the force H must be zero. Such a closed coil or toroidal current-sheet accordingly emits no tubes of force, but all its tubes lie within the doubly-connected space of the tore. The force accordingly has a discontinuity at the sheet equal to 4π times $\partial\Psi/\partial n$, which is the amount of current crossing unit of length of a circle coinciding with a line of force, or $nI/2\pi\rho$. If the tore be filled with a homogeneous medium of magnetic inductivity μ the induction at any point will be $\mu H = 2n\mu I/\rho$. This whole reasoning

has supposed that the external medium is homogeneous, but since there is no field there, the *value* of the inductivity is immaterial. If z, ρ be rectangular coordinates parallel and perpendicular to the axis in the plane of any orthogonal cross-section of the tore, the whole flux of induction through the section is

$$(3) \qquad \iint \mu H dS = 2n\mu I \iint \frac{d\rho dz}{\rho} = 2n\mu I \int \log\left(\frac{\rho_2}{\rho_1}\right) dz,$$

ρ_1 and ρ_2 being the least and greatest values of ρ on the contour of the section for a given value of z, and being given as functions of z by the equation of the contour.

If another circuit be wound in any manner about the tore, embracing it n' times, the flux through it is n' times that just found, and the mutual electrokinetic energy of a current I_2 in it and a current I_1 in the former winding is, according to the last sentence of § 226,

$$(4) \qquad T = 2nn'\mu I_1 I_2 \int \log\left(\frac{\rho_2}{\rho_1}\right) dz.$$

The mutual inductance of the circuits is accordingly

$$(5) \qquad M_{12} = 2nn'\mu \int \log\left(\frac{\rho_2}{\rho_1}\right) dz.$$

If the second coil coincide with the first the flux through itself is

$$2n^2\mu I^2 \int \log\left(\frac{\rho_2}{\rho_1}\right) dz,$$

so that the self-inductance of the toroidal coil is

$$(6) \qquad L = 2n^2\mu \int \log\left(\frac{\rho_2}{\rho_1}\right) dz.$$

The electrokinetic energy is

$$(7) \qquad T = \frac{1}{8\pi}\iiint \mu H^2 d\tau = \frac{1}{8\pi}\iiint \frac{\mu \,.\, 4n^2 I^2}{\rho^2} 2\pi\rho d\rho dz$$

$$= n^2\mu I^2 \iint \frac{d\rho dz}{\rho} = \tfrac{1}{2} L I^2.$$

For a coil of square cross-section whose side is $2a$ and whose mean radius is R,

$$(8) \qquad L = 4n^2\mu a \log\frac{R+a}{R-a}.$$

For a circular cross-section of radius a

$$(9) \qquad L = 2n^2\mu \int_{-a}^{a} \log \frac{R + \sqrt{a^2 - z^2}}{R - \sqrt{a^2 - z^2}} dz = 4\pi n^2 \mu (R - \sqrt{R^2 - a^2}).$$

If in equation (2) we insert the number of turns of wire per unit of length of the line of force, m, since $n = 2\pi\rho m$,

$$(10) \qquad H = 4\pi m I,$$

or the force depends only on the amount of current per unit of length. In case the radius of the tore is increased indefinitely, so that we get an infinitely long straight coil, m is the number of turns per unit of length of the coil, and we have within a uniform field of the magnitude $4\pi m I$. If any coil of n' turns be wound on outside, the mutual inductance will be

$$4\pi\mu m n'.$$

It is noticeable in all these cases that it is of no importance whether the outer coil is in contact with the inner or not, for in any case it is threaded by the whole flux of force. If there were any field external to the tore, the case would be different. It is however necessary that the tore be entirely filled by the medium of inductivity μ. The formulae of this section are applicable to induction coils and transformers, providing the coils are endless. The line-integral of magnetic force $4\pi n I$ is called the magnetomotive force, and the problem of finding the magnetic induction in the tore is the same as that of finding the current in a tore of conductivity μ in which there is an impressed electromotive force of the amount $4\pi n I$, the lines of flow being circles. In case the cross section of the tore is small compared to its radius, we may neglect the curvature of the coil, and find the reluctance (§ 184), by § 174, so that we have

$$(11) \qquad \text{Induction Flux} = \frac{\text{Magnetomotive force}}{\text{Reluctance}} = \frac{4\pi n I}{\dfrac{l}{\mu S}}.$$

This formula is used in practice in finding the flux in the field magnet of a dynamo-electric machine, although it is accurate only in the case that we have treated, where *all* the tubes of force are encircled by all the current turns, so that the numerator is the same for every tube. Any tube being partly in iron and partly in air, the reluctance of any infinitesimal tube is found by the formula for the resistance of conductors in series, as $\Sigma l/\mu S$.

In order to find the influence of the ends of a uniform straight coil of any cross-section, we may consider that each current turn is replaced by a plane shell, so that the whole current sheet is replaced by a uniformly magnetized cylindrical magnet with intensity of magnetization $\partial\phi/\partial z = mI$. The free surface charges of all the shells accordingly cancel each other except for the two plane ends of the magnet. These ends are single distributions identical with each other except for the difference of sign. If V_1 is the potential at any point due to a uniform single distribution of unit density on the positive end 1, and V_2 that due to an identical distribution on the negative end 2, then at any point outside the region bounded by the cylindrical current sheet and its plane ends, the potential due to the sheet is

$$\Omega = mI\,(V_1 - V_2). \tag{12}$$

We may find the potential at a point *inside* the space in question by the result that for an infinite cylindrical sheet the force is $4\pi mI$, so that if z is measured parallel to the generators of the cylinder in the direction of the force,

$$\Omega = -4\pi mIz \text{ (for the infinite cylinder).} \tag{13}$$

If Ω' is the potential due to all of the infinite coil except the portion which we are considering, we have accordingly

$$\Omega + \Omega' = -4\pi mIz. \tag{14}$$

But the space in question is outside the two magnets replacing the two infinite parts of the sheet, so that for a point between the ends,

$$\Omega' = mI\,(V_2 - V_1),$$

giving

$$\Omega = mI\,(V_1 - V_2 - 4\pi z). \tag{15}$$

Now as we pass one of the ends of the coil the potential V is continuous, being the potential of a single distribution, but its derivative has a discontinuity of amount 4π by § 82, accordingly the potential Ω is discontinuous, but the force is continuous, the discontinuity in changing from the formula (12) to (15) just cancelling the discontinuity in $\partial V/\partial z$. In the case of a circular cylindrical coil, the potentials V_1 and V_2 may be found by the development in spherical harmonics given in § 102, and the devia-

tion from uniformity of the field at any part of the solenoid calculated. In a long solenoid the field is very nearly uniform for a considerable distance from the middle of its length. By differentiating the expressions for V_1 and V_2 with respect to r, the distance from the center of either, multiplying by the element of the area of a sphere of radius r, and integrating, we may find the flux due to either end through a circle perpendicular to and with center in the axis, and hence the correction due to the end to be made in the mutual inductance of the coil with another circuit of a single turn, and thence by another integration with respect to any concentric coil.

232. Pair of Rectangular Circuits. In the case of two linear circuits, we may use Neumann's formula for the mutual inductance

$$M = \iint \frac{\cos(ds_1 ds_2)\, ds_1 ds_2}{r}$$

in those cases which are simple enough for us to effect the integration. If the two circuits are equal rectangles $ABCD$ and $A'B'C'D'$, Fig. 90, of length l_1 and breadth l_2 with corresponding sides parallel, and the lines joining corresponding corners perpendicular to their planes and of length a, then for pairs of sides which are perpendicular the integral vanishes, while for pairs of parallel sides the cosine is either plus or minus unity, according as we consider corresponding or opposite sides in the two rectangles. For the sides AB, $A'B'$ we have

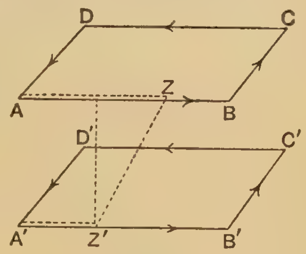

FIG. 90.

$$M_{AB,\,A'B'} = \int_0^{l_1}\int_0^{l_1} \frac{dz\,dz'}{\sqrt{a^2 + (z' - z)^2}} = \int_0^{l_1} \log \frac{\sqrt{a^2 + (l_1 - z)^2} + l_1 - z}{\sqrt{a^2 + z^2} - z}\, dz.$$

The integration of the logarithms in the second integrand may be performed by taking as a new variable the quantity whose logarithm is to be integrated, and then integrating by parts, the result being

$$M_{AB,\,A'B'} = 2\left\{a - \sqrt{a^2 + l_1^2} + l_1 \,.\, \log \frac{l_1 + \sqrt{a^2 + l_1^2}}{a}\right\}.$$

For the pair of sides AB, $C'D'$, substituting the square of their distance apart, $a^2 + l_2^2$, for a^2 and changing the sign, we have

$$M_{AB,\,C'D'} = -2\left\{\sqrt{a^2+l_2^2} - \sqrt{a^2+l_1^2+l_2^2} + l_1 \,.\, \log\frac{l_1+\sqrt{a^2+l_1^2+l_2^2}}{\sqrt{a^2+l_2^2}}\right\}.$$

The portions for the pairs of sides BC, $B'C'$, and BC, $D'A'$, are obtained from these by changing l_1 into l_2. We have then considered just half of the two circuits, so that, adding these four parts and multiplying by two, we obtain the value of the inductance

$$\begin{aligned}M = 8\,\{a - \sqrt{a^2+l_1^2} - \sqrt{a^2+l_2^2} + \sqrt{a^2+l_1^2+l_2^2}\}\\ + 4\left\{l_1 \,.\, \log\frac{(l_1+\sqrt{a^2+l_1^2})}{(l_1+\sqrt{a^2+l_1^2+l_2^2})}\,\frac{\sqrt{a^2+l_2^2}}{a}\right.\\ \left. + l_2 \,.\, \log\frac{(l_2+\sqrt{a^2+l_2^2}}{l_2+\sqrt{a^2+l_1^2+l_2^2}}\,\frac{\sqrt{a^2+l_1^2}}{a}\right\}.\end{aligned}$$

The attraction of the two circuits for each other when traversed by unit current is obtained by differentiating this expression by a.

233. Pair of Parallel Circles. If the circuits are circles of radii R_1, R_2, their planes being perpendicular to the line joining their centers, of length a, we may put

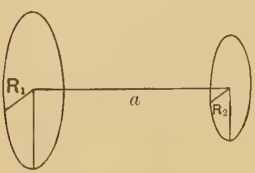

FIG. 91.

$$\begin{aligned} x_1 &= R_1 \cos\phi_1, & x_2 &= R_2\cos\phi_2,\\ y_1 &= R_1\sin\phi_1, & y_2 &= R_2\sin\phi_2,\\ z_1 &= 0, & z_2 &= a,\\ ds_1 &= R_1\,d\phi_1, & ds_2 &= R_2\,d\phi_2,\end{aligned}$$

$$\begin{aligned} r^2 &= a^2 + (R_1\cos\phi_1 - R_2\cos\phi_2)^2 + (R_1\sin\phi_1 - R_2\sin\phi_2)^2\\ &= a^2 + R_1^2 + R_2^2 - 2R_1R_2\cos(\phi_1-\phi_2),\end{aligned}$$

$$M = \int_0^{2\pi}\int_0^{2\pi}\frac{\cos(\phi_1-\phi_2)\,R_1R_2\,d\phi_1\,d\phi_2}{\sqrt{a^2+R_1^2+R_2^2-2R_1R_2\cos(\phi_1-\phi_2)}}.$$

The integration with respect to ϕ_2 amounts merely to multiplication by 2π. If we put

$$\phi_1-\phi_2 = 2\psi-\pi, \quad d(\phi_1-\phi_2) = 2d\psi, \quad \cos(\phi_1-\phi_2) = -\cos 2\psi,$$

$$\frac{4R_1R_2}{a^2+(R_1+R_2)^2} = \kappa^2,$$

the integral becomes

$$M = 8\pi\kappa\sqrt{R_1R_2}\int_0^{\frac{\pi}{2}}\frac{(1-2\sin^2\psi)\,d\psi}{\sqrt{1-\kappa^2\sin^2\psi}},$$

and writing

$$1-2\sin^2\psi=\frac{2}{\kappa^2}(1-\kappa^2\sin^2\psi)+1-\frac{2}{\kappa^2},$$

we have finally

$$M=4\pi\sqrt{R_1R_2}\left\{\frac{2}{\kappa}E(\kappa)+\left(\kappa-\frac{2}{\kappa}\right)F(\kappa)\right\},$$

where E and F are the elliptic integrals

$$E(\kappa)=\int_0^{\frac{\pi}{2}}\sqrt{1-\kappa^2\sin^2\psi}\,d\psi,\quad F(\kappa)=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-\kappa^2\sin^2\psi}}.$$

These definite integrals are functions only of the parameter κ, and their values have been tabulated by Legendre for various values of κ. If we put

$$\kappa^2=\sin^2\gamma,\quad \cos^2\gamma=\frac{a^2+(R_1-R_2)^2}{a^2+(R_1+R_2)^2}=\frac{r_2{}^2}{r_1{}^2},$$

r_1 and r_2 are the maximum and minimum distances of points on the circumferences of the two circles from each other. The expression $M/4\pi\sqrt{R_1R_2}$ being a function only of κ and therefore of γ, has been tabulated by Maxwell as a function of γ. (Treatise, Vol. 2, Art. 701.)

We may also find the value of M in a series of zonal spherical harmonics by means of the series of § 215 by differentiation with respect to r and integration over a spherical segment bounded by the second circle. For a full treatment of the properties of circular coils the reader is referred to Maxwell's Treatise, to Mascart and Joubert, Lessons on Electricity and Magnetism, and to Gray, Absolute Measurements in Electricity and Magnetism, where a great variety of formulae will be found.

234. Non-linear Currents in Parallel Cylinders. If the expressions in the two preceding sections be used to find the self-inductance of a linear circuit we find a difficulty, for on putting $a=0$ in § 232, the expression becomes logarithmically infinite, while on putting $a=0$, $R_1=R_2$ in § 233, κ becomes unity, the elliptic integrals reduce to trigonometric, and $F(\kappa)$ becomes logarithmically infinite ($\log\tan\pi/2$). This is easily seen to be the case for any linear circuit, for if ds_1 and ds_2 traverse the same circuit there is an infinite element in the integrand, and,

considering the element in which it occurs as straight the integral

$$\int \frac{ds}{s}$$

becomes logarithmically infinite. We may see the reason for the self-inductance becoming infinite in another way by considering Biot and Savart's Law, for as we approach a linear conductor the force is inversely proportional to the distance from the conductor. The flux therefore increases like the logarithm of this distance, and is not finite when we approach the linear conductor indefinitely. We may avoid this difficulty by considering conductors of finite cross-section, for in that case the corresponding element of the integral, in which the integrand becomes infinite, $\int \frac{d\tau}{r}$ is not infinite, as was proved for an ordinary potential, § 76.

We shall now consider currents flowing in three-dimensional conductors in the form of cylinders of infinite length whose generators are all parallel. We might treat the problem by the application of the law of Biot and Savart to each infinitesimal tube of flow, but we shall prefer to make use of the general equations § 222 (2), and § 228 (2). It is evident that the lines of force are in planes perpendicular to the conducting cylinders, which we shall take for the XY-plane, so that $N=0$ and the field is independent of the coordinate z. The problem is accordingly a two-dimensional problem, and all the quantities concerned are independent of z. Since $u=v=0$ we have $F=G=0$ so that our equations are

(1) $$4\pi w = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

(2) $$\mu L = \frac{\partial H}{\partial y},$$

(3) $$\mu M = -\frac{\partial H}{\partial x},$$

from which results, if μ is constant,

(4) $$-4\pi\mu w = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}.$$

But this is Poisson's equation for the logarithmic potential, § 91 (10),

so that its integral is

$$(5) \qquad H = C - 2\iint \mu w \log r\,da\,db,$$

where $$r^2 = (x-a)^2 + (y-b)^2,$$

and C is a constant, which, though infinite, does not affect the value of the force.

If the conductors are concentric circular cylindrical tubes and the current-density is uniform, we may find the magnetic force without finding the vector-potential, in the same way as in § 231, for it is evident that the lines of force are all circles in planes perpendicular to the conductors. At points outside the outer tube, at a distance ρ from the axis, the line integral of magnetic force (which we will denote by P instead of H, to prevent confusion with the vector-potential) around a circle is

$$2\pi\rho P = 4\pi I,$$

$$(6) \qquad P = \frac{2I}{\rho},$$

where

$$(7) \qquad I = \iint w\,dx\,dy,$$

is the total current through all the conductors. Accordingly at external points the field is the same as if the current were concentrated in the axis of the conductor. If different tubes are made part of the same circuit, so that all the current flowing in one direction is returned in the other direction by concentric conductors, the total current is equal to zero, and the force is zero at all external points. Such a double tubular conductor accordingly, like a toroidal coil, emits no tubes of magnetic induction. For this reason, when it is wished to protect delicate magnetic instruments from the action of strong currents, the circuit should be formed of concentric conductors. The mutual inductance of any external circuit with such a concentric conductor is accordingly zero, so that, as we shall see in the next chapter, no currents would be induced in the concentric conductors by external currents. Such a conductor would thus be suitable for telephone circuits.

In the space outside the conductors the magnetic potential is

$$(8) \qquad \Omega = 2I\phi = 2I\tan^{-1}\frac{y}{x},$$

which we know to be a harmonic function, as ϕ is conjugate to the function $\log \rho$, both being derived from the function $\log(x+iy)$. In the substance of the conductors, there is no magnetic potential. We may find the force by evaluating the expression for the vector-potential, or as above, except that now the line of force does not surround the whole current, but only a portion of it. If the conductor is a solid cylinder of radius R,

$$2\pi\rho P = 4\pi \,.\, \pi\rho^2 w = 4\pi I \frac{\rho^2}{R^2},$$

$$P = \frac{2I}{R^2}\rho. \tag{9}$$

The integral (5) represents H only when μ has the same constant value everywhere, for if it has discontinuities we must add a part corresponding to the apparent current as shown in § 228. In the case just treated, however, the apparent current vanishes, for the induction is tangent to the surfaces of the conductors.

In the general case, if μ is constant in the space outside of the conductors there is a magnetic potential, and the equations (2) and (3) become

$$\begin{aligned} -\frac{\partial(\mu\Omega)}{\partial x} &= \frac{\partial H}{\partial y} \\ -\frac{\partial(\mu\Omega)}{\partial y} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{10}$$

showing that the function $\mu\Omega$ is conjugate to the vector-potential H, which is accordingly the flux-function for the magnetic induction. The method of functions of a complex variable is accordingly applicable to problems connected with the field of cylindrical conductors. For instance, Fig. 65 represents for external points the lines of force and equipotential lines of the field due to two circular cylinders carrying equal currents in opposite directions. No one of the circles in the figure however represents either of the conductors, whose centers are at the points $\pm a$. The surface of a cylindrical conductor is tangent to lines of force only when it is alone in the field, or accompanied by concentric conductors. Within conductors, although there is no magnetic potential, equations (2) and (3) show that H is still the flux-function for the induction.

If S is the area of the cross-section of any conductor, the vector-potential at any point, whether external or internal, is by

(5) equal to

$$C - 2\mu w S \log \bar{r} = C - 2\mu I \log \bar{r},$$

where $\bar{r}$ is defined by the equation

$$S \log \bar{r} = \iint \log r\, da\, db. \tag{11}$$

But from the interpretation of a definite integral as a mean, § 23, we see that $\log \bar{r}$ is the arithmetical mean of the logarithms of the distances of all the points of the cross-section from the fixed point x, y. Now defining the geometric mean of n quantities as the nth root of their product, we see that $\bar{r}$ is the geometric mean of the distances of the points of the area from the point x, y, for its logarithm is the arithmetical mean of their logarithms. If $\bar{r}_1$ and $\bar{r}_2$ be the geometric mean distances of a point from two areas S_1 and S_2, $\bar{r}_3$ the geometric mean distance of the point from both areas taken together, we have by the definition, (11),

$$(S_1 + S_2) \log \bar{r}_3 = S_1 \log \bar{r}_1 + S_2 \log \bar{r}_2. \tag{12}$$

By means of this principle we may find the geometric mean distance from a complex figure if we know it for the various parts of the figure. This method is due to Maxwell*. We shall first find the geometrical mean distance from a circular ring of infinitesimal width. Let ρ be the radius, ϵ the width of the ring, and h the distance of the given point from its center. Inserting polar coordinates in the equation (11),

$$2\pi\rho\epsilon \log \bar{r} = \int_0^{2\pi} \tfrac{1}{2} \log (h^2 + \rho^2 - 2h\rho \cos \phi)\, \rho\epsilon\, d\phi. \tag{13}$$

This integral assumes different forms according as h is greater or less than ρ. Taking out from the parenthesis the square of the greater of these, and integrating, we get

$$\log \bar{r} = \log h + \frac{1}{4\pi} J\left(\frac{\rho}{h}\right), \quad h > \rho, \tag{14}$$

or

$$\log \bar{r} = \log \rho + \frac{1}{4\pi} J\left(\frac{h}{\rho}\right), \quad \rho > h, \tag{15}$$

where J is the definite integral

$$J(\alpha) = \int_0^{2\pi} \log (1 + \alpha^2 - 2\alpha \cos \phi)\, d\phi,$$

* *Trans. Roy. Soc. Edinburgh*, 1871—2.

which is a function only of the parameter α, and we are to put $\alpha = \rho/h$ when $h > \rho$ and $\alpha = h/\rho$ when $h < \rho$. We can easily show that $J(\alpha) = 0$ if $\alpha < 1$. For

$$J(\alpha) = J_1(\alpha) + J_2(\alpha), \text{ where}$$

$$J_1(\alpha) = \int_0^\pi \log(1 + \alpha^2 - 2\alpha\cos\phi)\, d\phi,$$

$$J_2(\alpha) = \int_\pi^{2\pi} \log(1 + \alpha^2 - 2\alpha\cos\phi)\, d\phi.$$

Substituting $\phi = \pi + \phi'$ gives

$$J_2(\alpha) = \int_0^\pi \log(1 + \alpha^2 + 2\alpha\cos\phi')\, d\phi',$$

and since the variable of integration is indifferent, we may drop the accent. The integral being now between the same limits as in J_1 we may add the integrands, giving

$$J(\alpha) = \int_0^\pi \log\{1 + \alpha^4 - 2\alpha^2(2\cos^2\phi - 1)\}\, d\phi.$$

Now substituting $2\phi = \phi'$ we obtain

$$J(\alpha) = \frac{1}{2}\int_0^{2\pi} \log(1 + \alpha^4 - 2\alpha^2\cos\phi')\, d\phi' = \tfrac{1}{2}J(\alpha^2).$$

Repeating the process we get

$$J(\alpha) = \tfrac{1}{2}J(\alpha^2) = \frac{1}{2^2}J(\alpha^4) \ldots = \frac{1}{2^n}J(\alpha^{2n}),$$

and letting n increase indefinitely we obtain, if $\alpha < 1$, $J(\alpha) = 0$ if $J(0)$ is finite. But $J(0) = 0$. We accordingly obtain from (14) and (15) the result that the geometric mean distance from a circular line is, for an outside point, its distance from the center, and for an inside point, the radius of the circle. By means of this result we may find the mean distance from the area of a ring of finite width, of internal radius R_1 and external R_2. For a point outside the ring the mean distance is its distance from the center. For a point in the space within the ring, by (11) or (12),

$$\pi(R_2^2 - R_1^2)\log\bar{r} = 2\pi\int_{R_1}^{R_2} \log\rho \,.\, \rho\, d\rho$$

$$= \tfrac{1}{2}\pi\{R_2^2(\log R_2^2 - 1) - R_1^2(\log R_1^2 - 1)\},$$

(16) $$\log\bar{r} = \frac{1}{(R_2^2 - R_1^2)}\{R_2^2\log R_2 - R_1^2\log R_1 - \tfrac{1}{2}(R_2^2 - R_1^2)\}.$$

For a point in the area of the ring itself, we must divide the ring into two, one within and one without the given point, so that

$$\pi(R_2^2 - R_1^2)\log\bar{r} = \pi(h^2 - R_1^2)\log h + \pi\{R_2^2\log R_2 - h^2\log h - \tfrac{1}{2}(R_2^2 - h^2)\},$$

(17) $$\log\bar{r} = \frac{1}{R_2^2 - R_1^2}\{R_2^2\log R_2 - R_1^2\log h - \tfrac{1}{2}(R_2^2 - h^2)\}.$$

The vector-potential is always, for uniform flow,

$$H = C - 2\mu I\log\bar{r},$$

and since this is the flux-function for the induction, by reason of the equation

$$\mu P = -\mu L\cos(\rho y) + \mu M\cos(\rho x) = -\frac{\partial H}{\partial y}\frac{\partial y}{\partial \rho} - \frac{\partial H}{\partial x}\frac{\partial x}{\partial \rho} = -\frac{\partial H}{\partial \rho},$$

we obtain the induction perpendicular to the radius, by differentiating according to $-h$, so that

(18) $$\mu P = \frac{2\mu I}{R_2^2 - R_1^2}\left(h - \frac{R_1^2}{h}\right),$$

which agrees with the result (9), in which R_1 is equal to zero.

The electrokinetic energy of the system of currents is, by § 226 (5),

$$T = \frac{1}{2}\iiint H w\, d\tau,$$

and inserting the value of H from (5),

(19) $$T = \frac{1}{2}\iiint Cw\, da\, db\, dz - \iiiint\!\!\int \mu w w'\log r\, da\, db\, da'\, db'\, dz,$$

$$r^2 = (a - a')^2 + (b - b')^2.$$

If we integrate with respect to z from $-\infty$ to ∞, we obtain an infinite result for the energy, but for a finite length l the energy is proportional to l, so that the energy per unit of length of the conductors T/l is given by the above expressions omitting the integration with respect to z. Each point of integration a, b and a', b' is to traverse the cross-sections of *all* the conductors. The first integral, containing the constant C, disappears, since to every current there is a return current, in each of which the same value of C appears, while for the two cross-sections the integral

$$\iint w\, da\, db,$$

representing the total current in the positive direction, is zero. If the flow is uniform in all the conductors, for any two conductors there will be a term

$$-\mu I_1 I_2 \log \bar{r}_{12},$$

where

$$S_1 S_2 \log \bar{r}_{12} = \iiiint \log r\, da\, db\, da'\, db',$$

and $\bar{r}_{12}$ is the geometric mean distance of all *pairs* of points in the two cross-sections. For instance let us consider a single circuit made up of conductors whose cross-section is denoted by S_1 and conductors carrying the return current whose cross-section is S_2. We then have

$$T = \frac{1}{2}\iint_{S_1} w' H' da' db' + \frac{1}{2}\iint_{S_2} w' H' da' db',$$

while if we divide H into two parts, H_1 due to the conductor 1, and H_2 due to the conductor 2,

$$H_1 = C_1 - 2\iint_{S_1} \mu w \log r\, da\, db,$$

$$H_2 = C_2 - 2\iint_{S_2} \mu w \log r\, da\, db,$$

T becomes the sum of the integrals

$$(20) \qquad T = \frac{C_1}{2}\iint_{S_1} w' da' db' - \iiiint_{S_1 S_1} \mu w w' \log r\; da\, db\, da'\, db'$$

$$+ \frac{C_1}{2}\iint_{S_2} w' da' db' - \iiiint_{S_1 S_2} \mu w w' \log r\; da\, db\, da'\, db'$$

$$+ \frac{C_2}{2}\iint_{S_1} w' da' db' - \iiiint_{S_2 S_1} \mu w w' \log r\; da\, db\, da'\, db'$$

$$+ \frac{C_2}{2}\iint_{S_2} w' da' db' - \iiiint_{S_2 S_2} \mu w w' \log r\; da\, db\, da'\, db'.$$

The first and third integrals, being the constant $C_1/2$ multiplied by the direct and return currents respectively, cancel each other, and so do the fifth and seventh. The fourth and sixth are equal to each other, and their sum is

$$-2\mu I_1 I_2 \log \bar{r}_{12} = 2\mu I^2 \log \bar{r}_{12},$$

where $\bar{r}_{12}$ is the mean distance between points of the two cross-sections. The second integral is

$$-\mu I^2 \log \bar{r}_{11},$$

where $\bar{r}_{11}$ is the mean distance between pairs of points of the section S_1, and the eighth integral is the corresponding quantity for the area S_2. Accordingly the self-inductance of the whole circuit per unit of length is

$$(21) \quad \frac{L}{l} = \mu\,(4 \log \bar{r}_{12} - 2 \log \bar{r}_{11} - 2 \log \bar{r}_{22}) = 2\mu \log \frac{\bar{r}_{12}^{\,2}}{\bar{r}_{11}\bar{r}_{22}}.$$

If the conductors are circular cylinders, we may use the formulae already found. If both the direct and return conductors are single cylinders external to each other, their axes being a distance d apart, $\bar{r}_{12} = d$. For infinitely thin tubes of radii R and R', $\bar{r}_{11} = R$, $\bar{r}_{22} = R'$, so that

$$(22) \quad \frac{L}{l} = 2\mu \log \frac{d^2}{RR'}.$$

For tubes of radii R_1, R_2 and R_1', R_2', integrating (17) over the area of the ring,

$$\pi(R_2^2 - R_1^2)\log \bar{r}_{11} = \frac{2\pi}{R_2^2 - R_1^2}\int_{R_1}^{R_2}\{R_2^2 \log R_2 - R_1^2 \log \rho - \tfrac{1}{2}(R_2^2 - \rho^2)\}\,\rho\,d\rho$$

$$= \frac{2\pi}{R_2^2 - R_1^2}\left\{R_2^2(\log R_2 - \tfrac{1}{2})\frac{(R_2^2 - R_1^2)}{2} - \frac{R_1^2}{4}\{R_2^2(\log R_2^2 - 1) - R_1^2(\log R_1^2 - 1)\} + \tfrac{1}{8}(R_2^4 - R_1^4)\right\},$$

and making reductions

$$(23) \quad \log \bar{r}_{11} = \frac{3R_1^2 - R_2^2}{4(R_2^2 - R_1^2)} + \frac{R_1^4}{(R_2^2 - R_1^2)^2}\log\frac{R_1}{R_2} + \log R_2.$$

From this we obtain $\bar{r}_{22}$ by replacing R_1, R_2 by R_1', R_2', so that we obtain for the self-inductance of the circuit

$$(24) \quad \frac{L}{l} = 2\mu\left\{\log\frac{d^2}{R_2R_2'} + \frac{R_1^4}{(R_2^2 - R_1^2)^2}\log\frac{R_2}{R_1} + \frac{R_1'^4}{(R_2'^2 - R_1'^2)^2}\log\frac{R_2'}{R_1'} + \frac{R_2^2 - 3R_1^2}{4(R_2^2 - R_1^2)} + \frac{R_2'^2 - 3R_1'^2}{4(R_2'^2 - R_1'^2)}\right\}.$$

For solid wires, since $\lim\limits_{R=0}(R \log R) = 0$, this becomes

$$(25) \quad \frac{L}{l} = 2\mu\left(\log\frac{d^2}{RR'} + \frac{1}{2}\right).$$

The repulsion between the wires per unit length is

$$\frac{\partial T}{\partial d} = \tfrac{1}{2} I^2 \frac{\partial (L/l)}{\partial d} = \frac{2\mu I^2}{d}.$$

By means of the principles here stated, Maxwell has calculated the inductances of coils of wire, by supposing the diameter of the coil to be so great in comparison with the distance apart of the different turns that the coil may be treated like a group of straight conductors. The treatment of the magnetic field due to currents even in straight conductors whose inductivity is different from that of the surrounding medium, except in the case of concentric cylinders, is a problem of considerable complexity, and the results given by Maxwell, for the case of two wires, Art. 685, are only approximately correct.

CHAPTER XII.

INDUCTION OF CURRENTS.

235. **Systems of Currents as Cyclic Systems.** The phenomena of the induction of electric currents by changes in the magnetic field were discovered by Faraday in 1831*. The results obtained experimentally by Faraday were deduced mathematically from the law of Lenz (see below), and from Ampère's results regarding magnetic shells, together with the principle of Conservation of Energy by F. E. Neumann† in 1845. The credit is due to Maxwell‡ of having had the idea of treating a system of currents and the magnetic field belonging to them as a mechanical system, subject to the ordinary laws of motion, and of thus deducing the equations of induction from the generalized equations of Lagrange and Hamilton. The particular class of systems to which currents may be assimilated is that studied by Helmholtz under the name of cyclic systems, a detailed treatment of which has been given in Chapter III.

We have seen in the last chapter that if the strengths of a system of currents be maintained constant, the currents tend to move in such a way that the energy of the field produced by them tends to increase. This energy is a homogeneous quadratic function of the strengths of the various currents, the coefficients, which we have called inductances, being determined by the form and relative position of the circuits, and the nature of the medium in which they are situated. The medium being specified, these geometrical specifications of the circuits may be made by giving

* Faraday, *Experimental Researches in Electricity*, Vol. I. p. 1.

† F. E. Neumann, "Allgemeine Gesetze der inducirten Ströme," *Abh. Berl. Akad.*, 1845.

‡ Maxwell, "A Dynamical Theory of the Electromagnetic Field," *Phil. Trans.* CLV. 1864. *Sci. Papers*, Vol. I. p. 526.

a certain finite or infinite number of geometrical parameters q_s. The electromagnetic forces due to the action of the currents may be equilibrated by the action of certain impressed forces P_s, and these forces may be determined as partial derivatives with respect to the parameters of the potential energy, or of the energy of the field. These impressed forces we shall call the positional forces, and since they are the negatives of the electromagnetic forces already found, we have for any positional force P_s,

$$P_s = \frac{\partial W}{\partial q_s}. \tag{1}$$

In order to specify the action of the system completely, we must give, beside the values of the parameters q_s, only the values of the current-strength in every current-tube. If the currents are distributed in three dimensions, this necessitates an infinite number, but if there are a finite number of *linear* conductors, only a finite number of electrical parameters I_s. The energy of the field is expressed as a homogeneous quadratic function of these electrical parameters, the coefficients being functions of the positional parameters, whose velocities do not occur. If we consider the negative energy of the field $-W$ as, instead of the negative *potential* energy, the *electrokinetic* energy of the field, the current strengths being considered as cyclic *velocities*, the analogy to a mechanical cyclic system is complete. The cyclic coordinates $\bar{q}_s$, being the time-integrals of the currents, represent the total amounts of electricity that have traversed the respective circuits since a fixed epoch. Since neither these coordinates, nor the velocities of the positional coordinates occur in the expression for the electrokinetic energy, all the conditions for a cyclic system are fulfilled. A restriction must, however, be made, which is of no importance in practice, namely that the velocities of the positional coordinates must be small compared with a certain velocity, which in this case is the velocity $\mathbf{v}$, the ratio of the two units of electricity. For the case of all ordinary velocities, however, the electrokinetic energy is accurately represented in the form already found.

If we have n linear currents, the electrokinetic energy is

$$\begin{aligned} T = &\tfrac{1}{2}L_1 I_1^2 + M_{12} I_1 I_2 \ldots\ldots + M_{1n} I_1 I_n \\ &+ \tfrac{1}{2}L_2 I_2^2 + M_{23} I_2 I_3 \ldots\ldots + M_{2n} I_2 I_n \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \tfrac{1}{2}L_n I_n^2. \end{aligned} \tag{2}$$

where the coefficients L, M have the form obtained in § 220 if a single homogeneous medium is present, and in any case may be defined as magnetic fluxes as in § 231. The electrokinetic momentum of any circuit,

$$(3) \qquad \bar{p}_s = \frac{\partial T}{\partial I_s} = M_{1s} I_1 + \ldots + L_s I_s \ldots + M_{ns} I_n,$$

may be defined as the total flux of magnetic induction through that circuit in the positive direction due to all the currents. We have already found for any positional force, equation (1),

$$(4) \qquad P_s = -\frac{\partial T}{\partial q_s} = -\left\{\tfrac{1}{2} I_1^2 \frac{\partial L_1}{\partial q_s} + I_1 I_2 \frac{\partial M_{12}}{\partial q_s} \ldots\ldots\right\}.$$

The force $\bar{P}_s$ belonging to any cyclic coordinate $\bar{q}_s$ consists of the impressed electromotive force E_s, due to chemical, thermal, or other action, and the dissipative term given by Joule's law, $-R_s I_s$, where R_s is the resistance of the circuit. Accordingly we have

$$(5) \qquad \bar{P}_s = E_s - R_s I_s = \frac{d\bar{p}_s}{dt}$$
$$= \frac{d}{dt}\{M_{1s} I_1 + \ldots + L_s I_s \ldots + M_{ns} I_n\}.$$

If we write this in the form

$$(6) \qquad I_s = \frac{E_s - \dfrac{d\bar{p}_s}{dt}}{R_s}$$

we see that the current in any circuit may be calculated by Ohm's Law provided that we consider acting beside the electromotive force E_s an additional electromotive force $-d\bar{p}_s/dt$. This is called the *electromotive force of induction*, and from the above definition of $\bar{p}_s$ we see that it is equal to the time-rate of diminution of the flux of magnetic induction through the circuit in the positive direction. The law of induction was announced in virtually this form by Faraday*, and was obtained from theoretical considerations involving the idea of work by Neumann†, Helmholtz‡, and Kelvin§. The above equation is the general equation of an

* *Exp. Res.* §§ 114, 3082.

† Neumann, *loc. cit.*

‡ Helmholtz, *Ueber die Erhaltung der Kraft.* Berlin, 1847. *Wiss. Abh.*, Bd. I. p. 12.

§ *B.A. Report*, 1848. *Math. and Phys. Papers*, Vol. I. p. 91.

electric current, and includes the steady state as a particular case, for if the currents do not vary with the time, and there is no motion of any circuit, every $\bar{p}_s$ is constant, and we have for each circuit,

$$I_s = \frac{E_s}{R_s},$$

the usual form of Ohm's Law. The statement is frequently made that Ohm's Law does not hold for induced currents—this is a misconception, for in the statement of Ohm's Law we should include electromotive forces of all kinds, including those due to induction.

If permanent magnets are present there will be terms in T where each current is multiplied by the flux through it due to magnets. These terms will be of the first order in the currents, so that T will not be homogeneous, and we have the case mentioned in § 66—each magnet acting like a concealed current. We have in the previous chapter considered the possible replacement of a magnet by currents, so that we may consider magnets replaced by "concealed" or "apparent" currents of unchangeable strength.

236. Isocyclic and Adiabatic Changes. An adiabatic variation, being defined by the constancy of cyclic momenta, will take place when in each circuit the electromotive force E_s is just large enough to maintain the current in the circuit steady, namely $E_s = R_s I_s$. If the current is varying, this necessitates the variation of E_s. Such changes seldom occur in practice. Isocyclic motions are such that all the currents remain unchanged. The electrokinetic momenta may be varied by motion or deformation of the circuits, involving change of the values of the parameters q_s, or by motion of permanent magnets. The simplest phenomena to observe experimentally, and those first discovered, are of this class.

We may now apply to a system of currents the theorems which have been demonstrated in §§ 69, 70. In particular may be noticed the two theorems of § 70, which may be thus stated.

I. In any motion of currents or magnets during which the strengths of all the currents are unchanged, the work done *by* the impressed electromotive forces E_s is equal to twice the work done

against the positional forces, plus the amount of energy dissipated as heat. For by § 70 (5) and (8)

$$\delta\bar{A} = 2\delta T = -2\delta A = \Sigma \bar{P}_s \delta\bar{q}_s = \Sigma E_s I_s \delta t - \Sigma R_s I_s^2 \delta t,$$

and therefore

$$\Sigma E_s I_s \delta t = \Sigma R_s I_s^2 \delta t - 2\delta A.$$

This theorem was stated by Lord Kelvin in 1860*.

II. Lenz's Law. In any system of conductors, induced currents due to motion of the conductors are so directed as to oppose the motion. This law, stated by Lenz† in 1834, was, together with Ampère's results, the basis of Neumann's deduction of the laws of induction.

Particular Cases of Induction in Linear Conductors.

237. Effect of sudden change of Electromotive Force or Resistance.

(1) Single Circuit. Let us first consider a single circuit of resistance R_0, containing a constant impressed electromotive force E_0, and accordingly traversed by the steady current

$$I_0 = E_0 / R_0.$$

Let now the electromotive force or the resistance, or both, be suddenly changed to new values E_1, R_1. The current now varies from the initial value I_0, in accordance with the differential equation (5) § 235, which becomes

$$L\frac{dI}{dt} + R_1 I = E_1. \tag{1}$$

If we subtract I_1, the steady value of the current under the new circumstances, from the total current, the difference

$$I - I_1 = I - E_1/R_1 = I^{(i)}$$

is called the *induced* or *extra*-current. The differential equation thus becomes

$$L\frac{dI^{(i)}}{dt} + R_1 I^{(i)} = 0, \tag{2}$$

whose integral is

$$I^{(i)} = Ae^{-\frac{R}{L}t},$$

* Nichol's *Cyclopædia*, Article "Magnetism, Dynamical Relations of." *Reprint of Papers on Electrostatics and Magnetism*, § 571.

† Lenz, *Pogg. Ann.* 31, p. 439, 1834.

so that the induced current dies away in geometrical ratio as the time increases in arithmetical progression. Since after an infinite interval of time the total current has attained the steady value I_1, the value of the constant A is determined, and we have

$$(3) \qquad I = I_1 + (I_0 - I_1)\, e^{-\frac{R}{L}t}, \qquad I^{(i)} = (I_0 - I_1)\, e^{-\frac{R}{L}t}.$$

The induced current is always in such a direction as to oppose the change in the total current. The effect of self-induction is accordingly to make changes of strength less sudden. It is to be noticed that the induced current varies in the same manner as the current charging a condenser through a circuit without self-induction, as treated in § 207 (17). We shall here, as there, call the time in which the current decreases in the ratio $1/e$ the relaxation-time,

$$\tau = L/R.$$

Both in the case of the condenser and in the present case increasing the capacity or the self-induction increases the relaxation-time, but whereas in the former case increasing the resistance increases the relaxation-time in the latter it produces the opposite effect.

In practical cases the relaxation-time is usually very short, so that the induced current disappears almost entirely in a very short time. Under these circumstances the total quantity of electricity that has passed may be measured by a ballistic galvanometer. For as the force exerted by the current on a magnet is proportional to the strength of the current, the total quantity passing, or the time integral $\int_0^t I\,dt$, is proportional to the time integral of the force on the magnet, or to the *momentum* imparted to the magnet. If this momentum is all imparted before the magnet has had time to move, it may be easily shown that it may be measured by the first swing of the magnet. The quantity passing in a time t is

$$\int_0^t I\,dt = \int_0^t \{I_1 + (I_0 - I_1)\, e^{-\frac{t}{\tau}}\}\, dt = I_1 t - \tau\,(I_0 - I_1)\,(e^{-\frac{t}{\tau}} - 1).$$

This formula was verified by Helmholtz* in 1851. The total

* Helmholtz, "Ueber die Dauer und den Verlauf der durch Stromesschwankungen inducirten elektrischen Ströme," *Pogg. Ann.* Bd. 83, p. 505. *Wiss. Abh.* Bd. 1, p. 429.

quantity due to the induction current is

$$\int_0^\infty I^{(i)}\,dt = \tau\,(I_0 - I_1).$$

If the current be passed through an electrodynamometer, that is an instrument containing a fixed and a movable coil, the mechanical action between them is proportional to the square of the current and the momentum imparted to the movable coil is proportional to the time integral of the square of the current. The effect due to the whole induced current is

$$\int_0^\infty I^{(i)2}\,dt = (I_0 - I_1)^2 \int_0^\infty e^{-\frac{2t}{\tau}}\,dt = \frac{\tau}{2}\,(I_0 - I_1)^2.$$

These two integrals have the same values that would be obtained from a steady current of strength $(I_0 - I_1)/2$ passing for a time 2τ.

(2) Two Circuits. In the case of two circuits which are closed at the same instant, or which have their electromotive forces or resistances suddenly changed simultaneously, we have during the subsequent period the differential equations

$$\text{(4)}\qquad \begin{aligned} L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} + R_1 I_1 &= E_1,\\ M\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + R_2 I_2 &= E_2. \end{aligned}$$

Again calling the final steady currents $I_1^{(1)} = E_1/R_1$, $I_2^{(1)} = E_2/R_2$, we have for the induced currents $I_1^{(i)} = I_1 - I_1^{(1)}$, $I_2^{(i)} = I_2 - I_2^{(1)}$,

$$\text{(5)}\qquad \begin{aligned} L_1\frac{dI_1^{(i)}}{dt} + M\frac{dI_2^{(i)}}{dt} + R_1 I_1^{(i)} &= 0,\\ M\frac{dI_2^{(i)}}{dt} + L_2\frac{dI_2^{(i)}}{dt} + R_2 I_2^{(i)} &= 0. \end{aligned}$$

These equations are typical of all those in this chapter, and are readily integrated by the assumption

$$I_1^{(i)} = Ae^{\lambda t}, \quad I_2^{(i)} = Be^{\lambda t},$$

where A, B and λ are constants to be determined. Inserting these values in the differential equations (5), the factor $e^{\lambda t}$ appearing in every term may be omitted, giving us the simultaneous equations

$$\text{(6)}\qquad \begin{aligned} (L_1\lambda + R_1)\,A + M\lambda B &= 0,\\ M\lambda A + (L_2\lambda + R_2)\,B &= 0. \end{aligned}$$

These equations can be satisfied for values of A and B differing from zero only if the determinant of the coefficients,

$$\begin{vmatrix} L_1\lambda + R_1, & M\lambda \\ M\lambda \quad , & L_2\lambda + R_2 \end{vmatrix}$$

vanishes. But this being expanded gives us the equation

(7) $$(L_1L_2 - M^2)\lambda^2 + (R_2L_1 + R_1L_2)\lambda + R_1R_2 = 0,$$

a quadratic to determine λ. If we call its roots λ_1 and λ_2, we have

(8) $$\lambda_1 = \frac{-(R_2L_1 + R_1L_2) + \sqrt{(R_2L_1 + R_1L_2)^2 - 4R_1R_2(L_1L_2 - M^2)}}{2(L_1L_2 - M^2)},$$
$$\lambda_2 = \frac{-(R_2L_1 + R_1L_2) - \sqrt{(R_2L_1 + R_1L_2) - 4R_1R_2(L_1L_2 - M^2)}}{2(L_1L_2 - M^2)}.$$

Both roots are real, for we can write the quantity under the radical sign

$$(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2,$$

both terms of which are positive. Both roots are also negative, for since the electrokinetic energy

$$T = \tfrac{1}{2}L_1I_1^2 + MI_1I_2 + \tfrac{1}{2}L_2I_2^2,$$

is intrinsically positive, we must have

$$L_1L_2 - M^2 > 0.$$

Having found the value of λ either of the equations (6) will give us the ratio of the constants A, B. If we choose the value λ_1 the first equation gives

(9) $$\frac{B_1}{A_1} = -\frac{L_1\lambda_1 + R_1}{M\lambda_1}.$$

If we choose the value λ_2 we obtain a different ratio

(10) $$\frac{B_2}{A_2} = -\frac{L_1\lambda_2 + R_1}{M\lambda_2}.$$

The theory of linear differential equations shows that the sum of particular solutions is a solution, and that the general solution is given by

$$I_1^{(i)} = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t},$$
$$I_2^{(i)} = B_1e^{\lambda_1 t} + B_2e^{\lambda_2 t},$$

where the constants A_1, B_1, A_2, B_2 are connected by the equations (9) and (10). We may now determine the absolute values of these

constants by means of the initial values of the currents I_1 and I_2. These being $I_1^{(0)}$ and $I_2^{(0)}$ we have for the *induced* currents when $t=0$,

$$(11) \qquad \begin{aligned} I_1^{(i0)} &= I_1^{(0)} - I_1^{(1)} = A_1 + A_2, \\ I_2^{(i0)} &= I_2^{(0)} - I_2^{(1)} = B_1 + B_2. \end{aligned}$$

These equations with (9) and (10) determine the four constants, so that the solution is complete. The most important case is that in which there is no electromotive force in one circuit, while the other circuit originally open, and containing an electromotive force E, is suddenly closed. The latter circuit is called the primary, and will be taken as that denoted by the suffix 1, the former the secondary, with the suffix 2. We accordingly have

$$I_1^{(0)} = I_2^{(0)} = I_2^{(1)} = 0, \quad I_1^{(1)} = E/R_1$$

and

$$(12) \qquad \begin{aligned} I_1 = \frac{E}{R_1}\Bigg\{1 &- \frac{1}{2}\left(\frac{R_2L_1 - R_1L_2}{\sqrt{(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2}} + 1\right) e^{\lambda_1 t} \\ &+ \frac{1}{2}\left(\frac{R_2L_1 - R_1L_2}{\sqrt{(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2}} - 1\right) e^{\lambda_2 t}\Bigg\}, \\ I_2 = \frac{-EM}{\sqrt{(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2}}&\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right). \end{aligned}$$

Since λ_1 and λ_2 are negative, the induced currents die away as the time goes on. The function

$$I = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

vanishes when

$$t = \frac{1}{\lambda_1 - \lambda_2} \log\left(-\frac{C_2}{C_1}\right),$$

has a maximum or minimum when

$$t = \frac{1}{\lambda_1 - \lambda_2} \log\left(-\frac{\lambda_2 C_2}{\lambda_1 C_1}\right),$$

and the curve representing it has a point of inflexion for

$$t = \frac{1}{\lambda_1 - \lambda_2} \log\left(-\frac{\lambda_2^2 C_2}{\lambda_1^2 C_1}\right).$$

These three points are equidistant, and, since λ_1 and λ_2 have the same sign, are real if C_1 and C_2 have opposite signs. This is the case for the secondary current I_2, but the primary current has both coefficients negative, and consequently has no maximum nor

inflexion, but rises continuously, the appearance of its representative curve to the eye being the same as in the case of a single circuit. The growth of the currents is represented in Fig. 92.

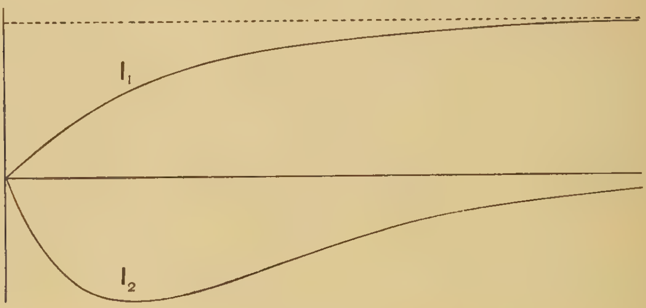

FIG. 92.

The total quantity flowing in the secondary is

$$(13)\quad \int_0^\infty I_2 dt = \frac{EM}{\sqrt{(R_2L_1 - R_1L_2)^2 + 4R_1R_2M^2}}\left\{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right\} = -\frac{EM}{R_1R_2}.$$

This result may also be obtained by direct integration of the second of equations (4), with $E_2 = 0$,

$$(14)\quad \begin{aligned} M(I_1^{(1)} - I_1^{(0)}) + L_2(I_2^{(1)} - I_2^{(0)}) + R_2\int_0^\infty I_2 dt = 0, \\ \int_0^\infty I_2 dt = -\frac{M(I_1^{(1)} - I_1^{(0)})}{R_2}. \end{aligned}$$

To find the effect of breaking the primary current, we have $I_1^{(0)} = E/R_1$, $I_1^{(1)} = 0$, so that the whole quantity passing in the secondary on breaking is the same as on making. This is one of Faraday's fundamental results. The manner of variation of the secondary is, on the contrary, very different from that on making. After the break we have to consider the primary circuit as suppressed, so that the secondary is to be considered by itself, and varies according to equation (3) above, where I_1, the final value, is zero, and I_0, the initial value, is to be found from the above value of the time-integral of the secondary,

$$(15)\quad \begin{aligned} \int_0^\infty I_2 dt = I_0\int_0^\infty e^{-\frac{R_2}{L_2}t}\,dt = \frac{L_2I_0}{R_2} = \frac{E}{R_1}\frac{M}{R_2}, \\ I_0 = \frac{E}{R_1}\frac{M}{L_2}. \end{aligned}$$

The fact that the secondary jumps abruptly from zero to its maximum value I_0 at starting may be reached from considering the preceding case with $R_1 = \infty$. The time between the secondary's taking the value zero and attaining its maximum and the time from then to the inflexion is $(\log \lambda_2/\lambda_1)/(\lambda_1 - \lambda_2)$, which is less the greater R_1, vanishing for $R_1 = \infty$.

The effects here described may be illustrated by means of any of the mechanical models described in § 71. For instance suppose that the mass m_1, Fig. 30, is revolving with a uniform angular velocity, the centrifugal force, which represents the electromagnetic force, being just balanced by an applied force so that the distance of m_1 from the axis remains constant. If m_2 is at rest and we suddenly apply a force to the upper bar so as to increase its angular velocity, the lower bar will begin to turn in the reverse direction, the velocity representing the secondary induced current. If on the other hand the upper bar is suddenly retarded, the lower begins to move forward in the direct sense. Similar effects may be produced by suddenly changing the distance of either m_1 or m_2 from the axis, corresponding to a relative motion of the two circuits, producing a change in the mutual inductance. We have not in this section explicitly considered this case, but since if the change is made suddenly, and the circuits then remain at rest, the differential equations are the same as those we have used, and the solution is obtained from those here given.

238. Periodically-varying Electromotive-force.

(1) Single Circuit. Suppose that in the circuit is included a variable electromotive-force varying proportionately to the cosine of a linear function of the time, as would be the case if a coil of wire should rotate in a uniform magnetic field about an axis in the plane of the coil, and perpendicular to the direction of the field. Then the equation for the current is

$$L\frac{dI}{dt} + RI = E_0 \cos \omega t. \tag{1}$$

A convenient way of treating such an equation is by replacing the trigonometric term $\cos \omega t$ by the exponential $e^{i\omega t}$, whose real part is the trigonometric part in question. The value of I thus obtained will be complex, and its real part will be the solution of the differential equation with the cosine term on the right, while its imaginary part will have as the coefficient of i the solution of

the equation with the sine on the right. In this way by separation of the real and the imaginary we are enabled to use the exponential function, which retains its form on differentiation, while the sine and cosine interchange. Accordingly writing the equation

$$(2) \qquad L\frac{dI}{dt} + RI = E_0 e^{i\omega t},$$

we may get a particular solution by assuming $I = Ae^{i\omega t}$, inserting which in the equation gives, on removing the factor $e^{i\omega t}$,

$$(3) \qquad (Li\omega + R)A = E_0.$$

This determines the complex constant A as

$$A = \frac{E_0}{Li\omega + R} = \frac{E_0(R - Li\omega)}{L^2\omega^2 + R^2},$$

so that the solution of the equation (2) is

$$Ae^{i\omega t} = \frac{E_0(R - Li\omega)(\cos\omega t + i\sin\omega t)}{L^2\omega^2 + R^2}.$$

Taking the real part we obtain for the solution of the equation (1),

$$I = \frac{E_0(R\cos\omega t + L\omega\sin\omega t)}{L^2\omega^2 + R^2}.$$

This assumes a more convenient form if we determine two constants α and J so that

$$\frac{R}{L^2\omega^2 + R^2} = \frac{\cos\alpha}{J}, \quad \frac{L\omega}{L^2\omega^2 + R^2} = \frac{\sin\alpha}{J},$$

giving

$$(4) \qquad \tan\alpha = \frac{L\omega}{R}, \quad J = (L^2\omega^2 + R^2)^{\frac{1}{2}},$$

when the solution becomes

$$(5) \qquad I = \frac{E_0}{J}\cos(\omega t - \alpha).$$

We may obtain this result, and at the same time graphically represent the relations of the current and electromotive-force by making use of the fundamental properties of complex quantities. The complex quantity $E_0 e^{i\omega t}$ has the modulus E_0 and the argument ωt, and is therefore represented by a vector of length E_0 making an angle ωt with the real axis, that is a vector revolving about the origin with angular velocity ω. The *projection* of this vector on the real axis represents the impressed electromotive-force in the

circuit. A quantity varying in this manner is said to perform a *harmonic* oscillation, with the *amplitude* E_0.

The electromotive-force takes on all values between E_0 and $-E_0$ and returns to its original value in the time that the vector takes to make a complete revolution, $T = 2\pi/\omega$. The time T is called the *period*, and its reciprocal, the number of periods in unit time, $n = \omega/2\pi$, is called the *frequency*.

In like manner the quantity $Ae^{i\omega t}$ is represented by a vector revolving with the same period, of length equal to the modulus of the complex quantity A. Since the argument of a quotient is equal to the difference of the arguments, the vector representing $Ae^{i\omega t}$ lags behind that representing $E_0 e^{i\omega t}$ by the constant angle

$$\alpha = \arg. \frac{E_0}{A}.$$

But from the equation (3) we find that this ratio is the complex quantity, $R + iL\omega$, whose argument is $\tan^{-1} L\omega/R$. The current, being represented by the projection of the second vector on the real axis, is said to differ in phase from the electromotive-force by the amount α, the difference in this case being a lag. The amplitude of the current, being the modulus of A, is the quotient of the moduli

$$\frac{|E_0|}{|R + iL\omega|} = \frac{E_0}{(R^2 + L^2\omega^2)^{\frac{1}{2}}}.$$

Expressing these results analytically we obtain equation (5).

The quantity J, by which it is necessary to divide the amplitude of the electromotive-force in order to obtain the amplitude of the current, is called, as proposed by Heaviside, the *impedance*. If the circuit has no self-inductance, or if the current is steady ($\omega = 0$), it becomes the resistance.

It has been proposed by Hospitalier* to call the coefficient of i in the ratio E_0/A, the *reactance*.

The mean value of a quantity varying harmonically taken over any exact number of periods is zero, while in virtue of the formulae

$$\frac{1}{T}\int_0^T \cos^2 \omega t\, dt = \frac{1}{T}\int_0^T \sin^2 \omega t\, dt = \frac{1}{2}, \quad \frac{1}{T}\int_0^T \sin \omega t \cos \omega t\, dt = 0,$$

* Hospitalier, *L'Industrie Electrique*, May 10, 1893. See also, Steinmetz and Bedell, *Trans. Am. Inst. El. Eng.* 1894, p. 640.

the mean value of the square of such a quantity is one-half the square of its amplitude, and the mean of the product of two such quantities of the same period and a difference of phase equal to a right angle is zero.

The quadratic mean, or square root of the mean square of a variable current or electromotive-force is called the *effective* or *virtual* current or electromotive-force. Its value in the case of a harmonically-varying quantity is accordingly the amplitude divided by $\sqrt{2}$.

The activity, or power absorbed by the circuit, is

$$EI = \frac{E_0^2}{J} \cos \omega t \cos (\omega t - \alpha),$$

and its mean value, by the above formulae,

$$(6) \qquad \frac{1}{T}\int_0^T EI\,dt = \frac{E_0^2 \cos \alpha}{2J} = \frac{E_0^2 R}{2\,(R^2 + L^2\omega^2)}.$$

To the solution (5) is to be added, in order to obtain the general solution of (1), the solution of the equation with the right hand member equal to zero, obtained in the preceding section, but as the current thereby represented rapidly dies away, the resulting state of the alternating current is that which we have found.

A number of circuits in parallel, to which a single harmonic electromotive-force is applied, receive virtual currents inversely proportional to their respective impedances—if the frequency is great enough the distribution is almost independent of the resistances of the branches, the impedance being sensibly equal to the reactance.

(2) Two Circuits. Suppose that we have two circuits, one of which, the primary, contains a harmonic electromotive-force, while the secondary contains no impressed electromotive-force, except that due to induction. The equations then are

$$(7) \qquad \begin{aligned} L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} + R_1 I_1 &= E_0 \cos \omega t, \\ M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + R_2 I_2 &= 0, \end{aligned}$$

or making use of complex variables as before,

$$(8)\qquad \begin{aligned} L_1\frac{dI_1}{dt}+M\frac{dI_2}{dt}+R_1I_1&=E_0e^{i\omega t}\\ M\frac{dI_1}{dt}+L_2\frac{dI_2}{dt}+R_2I_2&=0.\end{aligned}$$

A particular solution is given as before by putting

$$I_1=Ae^{i\omega t},\quad I_2=Be^{i\omega t},$$

giving

$$(9)\qquad \begin{aligned}(L_1i\omega+R_1)\,A+Mi\omega B&=E_0,\\ Mi\omega A+(L_2i\omega+R_2)\,B&=0.\end{aligned}$$

Eliminating B from these equations we get

$$(10)\qquad \left\{L_1i\omega+R_1+\frac{M^2\omega^2(R_2-L_2i\omega)}{L_2^2\omega^2+R_2^2}\right\}A=E_0.$$

Comparing this with equation (3) above we find that the current in the primary is the same as if the secondary were absent, and the resistance and self-inductance of the primary were R' and L', where

$$(11)\qquad \begin{aligned} R'&=R_1+\frac{R_2M^2\omega^2}{R_2^2+L_2^2\omega^2},\\ L'&=L_1-\frac{L_2M^2\omega^2}{R_2^2+L_2^2\omega^2}.\end{aligned}$$

These results were first given by Maxwell in 1864 in his celebrated paper "A Dynamical Theory of the Electromagnetic Field*." They constitute the basis of the theory of the alternating current transformer.

We see from equations (11) that the effect of the presence of the secondary circuit is to cause an apparent increase of resistance and decrease of self-inductance in the primary. Both of these effects cause a decrease in the angle of lag of the primary current behind the electromotive-force, and accordingly, by (6), an increase of power. Inserting the values (11) in (6) we obtain for the power

$$(12)\qquad \frac{E_0^2R'}{2(R'^2+L'^2\omega^2)}$$

$$=\frac{E_0^2\{R_1R_2^2+\omega^2(R_1L_2^2+R_2M^2)\}}{2\{R_1^2R_2^2+\omega^2(L_1^2R_2^2+L_2^2R_1^2+2R_1R_2M^2)+\omega^4(L_1L_2-M^2)^2\}}.$$

* *Phil. Trans.* Vol. CLV.

As we increase the ratio $L_2\omega/R_2$ the values of the apparent resistance and self-induction approach the limiting values

$$(13)\qquad \begin{aligned} R_\infty' &= R_1 + R_2\frac{M^2}{L_2^2}, \\ L_\infty' &= L_1 - \frac{M^2}{L_2}. \end{aligned}$$

These values are nearly approached in actual transformers, particularly when fully loaded with a number of lamps in parallel in the secondary. Now although we have in general

$$L_1L_2 > M^2,$$

still when the primary and secondary are toroidal coils wound on the same core (§ 231), so that very nearly the whole induction-flux due to either is linked with the other, we have very nearly

$$L_1L_2 - M^2 = 0.$$

The transformer is then said to have no magnetic leakage. In this case the apparent inductance L_∞' is reduced to zero, the current does not lag, and takes on the largest value that it can have, namely

$$I = E_0/R_\infty'.$$

The expression (12) for the power becomes, if we neglect the square of $R_2/L_2\omega$ in comparison with unity,

$$(14)\qquad \frac{E_0^2(R_1 + R_2M^2/L_2^2)}{2\{(R_1 + R_2M^2/L_2^2)^2 + \omega^2(L_1 - M^2/L_2)^2\}},$$

as we see on dividing numerator and denominator by $L_2^2\omega^2$ and then adding and subtracting the term $R_2^2M^4/L_2^4$ in the denominator. If there is no magnetic leakage, this increases as R_2 decreases, until it reaches the limiting value

$$\frac{1}{2}\frac{E_0^2}{R_1},$$

while if there is magnetic leakage, the power absorbed is a maximum when

$$R_1 + R_2\frac{M^2}{L_2^2} = \omega\left(L_1 - \frac{M^2}{L_2}\right),$$

becoming equal to

$$\frac{E_0^2}{4\omega\left(L_1 - \dfrac{M^2}{L_2}\right)},$$

and thence decreasing as R_2 decreases.

The power when R_2 is zero is only

$$\frac{E_0^2 R_1}{2\{R_1^2+\omega^2(L_1-M^2/L_2)^2\}},$$

which, for high frequencies, may be much less than the maximum*, being, when ω is great enough, sensibly equal to the maximum value multiplied by

$$\frac{2R_1}{\omega(L_1-M^2/L_2)}.$$

The second of equations (9) gives

(15) $$\frac{B}{A}=\frac{-Mi\omega}{L_2 i\omega+R_2}.$$

The modulus of the quotient, being the quotient of the moduli,

(16) $$\left|\frac{B}{A}\right|=\frac{M\omega}{\sqrt{L_2^2\omega^2+R_2^2}}$$

shows that the amplitude $I_2^{(0)}$ of the secondary current is equal to the amplitude of the primary $I_1^{(0)}$ multiplied by $M\omega$ divided by the impedance of the secondary. Inserting the values of R', L', from (11) in $I_1^{(0)}$,

(17) $$|A|=I_1^{(0)}=\frac{E_0}{\sqrt{L'^2\omega^2+R'^2}}$$

gives for $I_2^{(0)}$,

(18) $$|B|=I_2^{(0)},$$

$$=\frac{E_0 M\omega}{[(R_1^2R_2^2+\omega^2(L_1^2R_2^2+L_2^2R_1^2+2M^2R_1R_2)+\omega^4(L_1^2L_2^2+M^4-2L_1L_2M^2)]^{\frac{1}{2}}}.$$

In the case of no magnetic leakage this becomes

(19) $$I_2^{(0)}=\frac{E_0\sqrt{L_1L_2}\,.\,\omega}{[R_1^2R_2^2+\omega^2(L_1R_2+L_2R_1)^2]^{\frac{1}{2}}},$$

and if we may neglect $R_1/L_1\omega$ or $R_2/L_2\omega$ in comparison with unity we have the simple form

(20) $$I_2^{(0)}=\frac{E_0}{R_1\sqrt{\dfrac{L_2}{L_1}}+R_2\sqrt{\dfrac{L_1}{L_2}}}.$$

This is the practical equation of the transformer. By § 231, (5) and (6), we have

$$\frac{L_2}{L_1}=\frac{n_2^2}{n_1^2},$$

* J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, p. 409.

where n_1 and n_2 are the numbers of turns in the primary and secondary coils. If n_2/n_1 is so small that its square may be neglected, we have

$$(21) \qquad I_2^{(0)} = \frac{E_0 \dfrac{n_2}{n_1}}{R_2}.$$

The ratio n_2/n_1 is called the ratio of transformation.

The argument of the ratio B/A, being the difference of the arguments of the numerator and denominator, shows that the secondary lags behind the primary current by the phase-angle

$$\frac{\pi}{2} + \tan^{-1} \frac{L_2 \omega}{R_2}$$

which approaches two right angles as the ratio $L_2\omega/R_2$ increases.

The *efficiency* of the transformation, or the ratio of the activity in the secondary $\frac{1}{2} R_2 I_2^{(0)2}$, to that in the primary, is, by (18) and (12),

$$(21') \qquad \frac{R_2 M^2 \omega^2}{R_1 R_2^2 + \omega^2 (R_1 L_2^2 + R_2 M^2)},$$

which, neglecting $R_1 R_2 / M^2 \omega^2$, becomes

$$(22) \qquad \frac{1}{1 + \dfrac{R_1}{R_2} \dfrac{L_2^2}{M^2}};$$

that is, in practical cases, nearly unity.

239. **Circuit containing a Condenser.** In the cases heretofore considered the only energy of the system has been electrokinetic. If the circuits are connected with conductors upon which charges of electricity can accumulate, we shall in addition have electrostatic, or potential energy.

As the simplest case let us consider a single circuit whose ends are connected to a condenser of capacity K. If the charge of one plate of the condenser at any instant is q, then the current flowing into that plate is defined as

$$(1) \qquad I = \frac{dq}{dt}.$$

The electrostatic energy of the system is (§ 143)

$$(2) \qquad W = \frac{1}{2} \frac{q^2}{K},$$

which gives rise to the difference of potential, or electrostatic electromotive-force impressed in the circuit in the direction of the current,

$$(3) \qquad E = -\frac{\partial W}{\partial q} = -\frac{q}{K}.$$

Accordingly the differential equation for the current is

$$(4) \qquad L\frac{dI}{dt} + RI = -\frac{q}{K},$$

from which, substituting from (1), we obtain the equation for the charge,

$$(5) \qquad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{K} = 0.$$

Again, assuming $q = e^{\lambda t}$ we obtain the quadratic for λ

$$(6) \qquad L\lambda^2 + R\lambda + \frac{1}{K} = 0,$$

whose roots are

$$(7) \qquad \begin{aligned} \lambda_1 &= -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{KL}}, \\ \lambda_2 &= -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{KL}}. \end{aligned}$$

We have now to consider two cases.

CASE I. $R^2 > 4L/K$. *Both roots real.* We then have

$$(8) \qquad q = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

and as λ_1 and λ_2 are both negative, the charge, and likewise the current

$$(9) \qquad I = \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t},$$

die gradually away. If there is a permanent impressed electromotive-force E_0 in the circuit, we must add the quantity E_0K to the charge, which, however, does not affect the current.

Determining the constants A and B by the conditions that there is initially neither current nor impressed electromotive-force, and that the initial charge is q_0, we have

$$(10) \qquad q = \frac{q_0}{\lambda_2 - \lambda_1}\{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\},$$

while if there is no initial charge, but an impressed electromotive-force E_0, we obtain

$$(11) \qquad q = E_0 K\left\{1 - \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\right\}.$$

In either case, the curve representing the charge as a function of the time has a point of inflexion distant from the origin by the amount

(12) $$t = \frac{1}{\lambda_1 - \lambda_2} \log \frac{\lambda_2}{\lambda_1},$$

while the curve of current has one at an equal distance farther on. The curves of charge and of current are represented in Fig. 93.

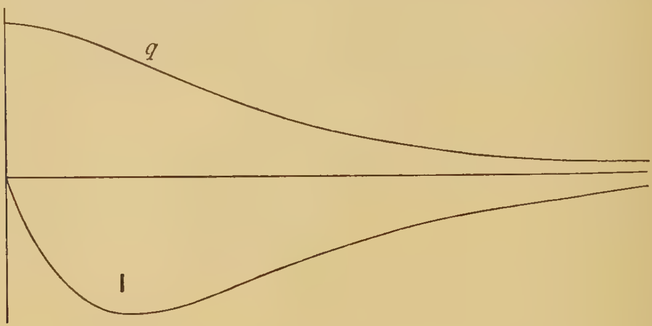

Fig. 93.

Case II. $R^2 < 4L/K$. *Both roots complex.* If we write

$$\mu = -\frac{R}{2L}, \quad \nu = \sqrt{\frac{1}{KL} - \frac{R^2}{4L^2}},$$

we have for the roots

(13) $$\lambda_1 = \mu + i\nu,$$
$$\lambda_2 = \mu - i\nu,$$

and we may write the solution

(14) $$q = e^{\mu t}(A \cos \nu t + B \sin \nu t).$$

In this case the charge not only dies away, but periodically changes sign, performing a *damped* harmonic oscillation of the period

(15) $$T = \frac{2\pi}{\nu} = \frac{2\pi}{\sqrt{\frac{1}{KL} - \frac{R^2}{4L^2}}}.$$

We have for the current

(16) $$I = \frac{dq}{dt} = e^{\mu t}\{(A\mu + B\nu)\cos \nu t + (B\mu - A\nu)\sin \nu t\}.$$

Determining the constants so that the initial current is zero, and the charge q_0, we have

(17) $$q = q_0 e^{\mu t}\left(\cos \nu t - \frac{\mu}{\nu} \sin \nu t\right).$$

This case is represented in Fig. 94. The charge is zero at times such that

$$\nu t = \tan^{-1}\frac{\nu}{\mu} = \theta,$$

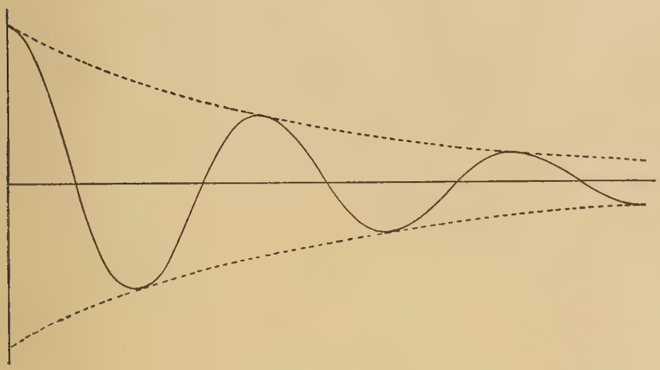

FIG. 94.

which is later than the time of the vanishing of the current by the phase difference θ, which approaches $\pi/2$ the smaller μ.

We may specify the damping, or decrease of the charge or current, by the relaxation-time of the damping factor $e^{\mu t}$, namely $\tau = 2L/R$, or by the logarithmic decrement, that is the logarithm of the ratio of a maximum value to the absolute value of the next following minimum. Since the maximum and minimum values of the parenthesis in (17) are equal and opposite, and separated by intervals of time $T/2 = \pi/\nu$, the ratio of the absolute values of q is $e^{-\frac{\mu\pi}{\nu}}$, and the logarithmic decrement λ,

(18) $$\lambda = -\frac{\mu\pi}{\nu} = \frac{\pi}{\sqrt{\frac{4L}{KR^2} - 1}}.$$

If $R = 0$, there is no damping, $\lambda = 0$ and the period is

(19) $$T_0 = 2\pi\sqrt{KL}.$$

Introducing these values of T_0 and λ we may write (15),

(20) $$T = T_0\sqrt{1 + \frac{\lambda^2}{\pi^2}} = T_0\left(1 + \frac{1}{2}\frac{\lambda^2}{\pi^2} + \dots\right),$$

so that if the damping is small it affects the period only by small quantities of the second order.

We have in this case a type of the very important class of phenomena known as electrical oscillations, of which we shall presently give the general theory. The theory here given was published by Lord Kelvin* in 1855 and by Kirchhoff† in 1864. The theory was confirmed experimentally in a qualitative manner by Feddersen‡ in 1857, by observations on the electric spark arising when a Leyden jar is discharged, and by Helmholtz§ in 1869, and Schiller|| in 1874, under conditions admitting of quantitative results. More exact determinations in absolute measure have been made by Lodge and Glazebrook by a method involving the spark, and by the author, by a method similar to that of Helmholtz.

We have seen that the occurrence of oscillations is due to the presence of both kinetic and potential energy. If there is no kinetic energy, $L=0$, and we reach the case treated in § 207, while if there is no potential energy, we have the case of § 237, to which we may pass by putting $K=\infty$. A mechanical model of an oscillation may be obtained from any mechanical system possessing both potential and kinetic energy, such as a pendulum or a heavy body moved by a spring. The stronger the spring the quicker is the oscillation, so that we may assimilate the reciprocal of the capacity of the condenser to the elasticity of the spring. The self-inductance of the system, on the other hand, is the analogue of the mass, or inertia of the mechanical system. The analogy of the resistance may be obtained by making the system move in a viscous medium, so that the motion is retarded by a force proportional to the velocity.

240. Periodic Electromotive force. Resonance. If into a circuit joined to the plates of a condenser is introduced a harmonically-varying electromotive force, we have for the current, instead of (4) of the preceding section the equation

$$L\frac{dI}{dt}+RI+\frac{1}{K}\int I\,dt=E_0\cos\omega t. \tag{1}$$

* Thomson, "On Transient Electric Currents," *Phil. Mag.* June 1853; *Math. and Physical Papers*, Vol. I. p. 540.

† Kirchhoff, "Zur Theorie der Entladung einer Leydener Flasche," *Pogg. Ann.* Bd. 121, 1864; *Ges. Abh.* p. 168.

‡ Feddersen, "Beiträge zur Kentniss des elektrischen Funkens," Dissertation, Kiel, 1857; *Pogg. Ann.* 103, p. 69.

§ Helmholtz, "Ueber elektrische Oscillationen," *Wissensch. Abh.* Bd. I. p. 531.

|| Schiller, *Pogg. Ann.* 152, p. 535.

Proceeding as in § 238, we write

$$(2)\qquad L\frac{dI}{dt}+RI+\frac{1}{K}\int I\,dt=E_0e^{i\omega t},$$

and assume for the particular solution $I=Ae^{i\omega t}$, which inserted in (2) gives

$$(3)\qquad \left(Li\omega+R+\frac{1}{Ki\omega}\right)A=E_0.$$

From this we get, by comparison with § 238, for the impedance,

$$(4)\qquad J=\left\{R^2+\left(L\omega-\frac{1}{K\omega}\right)^2\right\}^{\frac{1}{2}},$$

and for the lag of the current behind the electromotive force,

$$(5)\qquad \alpha=\tan^{-1}\frac{L\omega-\dfrac{1}{K\omega}}{R},$$

so that the solution of (1) is

$$(6)\qquad I=\frac{E_0\cos(\omega t-\alpha)}{\left\{R^2+\left(L\omega-\dfrac{1}{K\omega}\right)^2\right\}^{\frac{1}{2}}}.$$

In order to obtain the general solution we must add to this result the solution of the equation with $E_0=0$ from the previous section. An oscillation whose period is that of the force, as in our present case, is called a *forced oscillation* or vibration, in contradistinction to the case of the previous section, where, no force being applied, the period is governed by the constants of the system, and the oscillation is called a *free oscillation.* If there is damping, the free oscillation soon dies away, leaving only the forced oscillation. We see by (6) that if there is no condenser, $K=\infty$, we obtain the case of § 238, and the current lags, while if on the other hand $L=0$, the lag is negative, or the current advances by the phase-angle

$$\alpha=\tan^{-1}\frac{1}{K\omega R}.$$

The reason of this is of course that in the differential equation the inductance is multiplied by the derivative, and the capacity-reciprocal by the integral of the current, which, when the electromotive force is an exponential with imaginary exponent, introduce the factor $i\omega$ into the numerator or denominator respectively, producing opposite effects on the argument of A.

Thus the tendency of the inductance and capacity is to neutralise each other's effects. Exact neutralization is produced, so that there is neither lag nor advance, when

$$L\omega = \frac{1}{K\omega}, \quad \omega = \frac{1}{\sqrt{KL}} = \frac{2\pi}{T}.$$

In this case the impedance is the smallest possible, and the magnitude of the current is a maximum, being the same as would be given by Ohm's Law for steady currents with a closed circuit. The period of the electromotive force which gives this result is exactly that of the free vibration which would be natural to the system if there were no damping. Under these circumstances the system is said to be in *resonance* with the force. The magnitude of the current is inversely proportional to the resistance, and if there were no damping would be infinite. For this reason resonant oscillations, either mechanical or electrical, may be very intense. By connecting two similar circuits with two similar Leyden jars, Lodge has caused the oscillatory discharge of one jar to produce such violent resonant oscillations in the other circuit that a considerable spark-discharge is produced. The phenomena of resonance have been demonstrated in a number of interesting papers by Pupin*.

In order to show how the resonance depends on the agreement of the frequency of the impressed force with that of the free vibration, we give in Fig. 95 a graphical representation of the current as a function of the frequency. If we call ω_m the value of ω which gives the maximum current,

$$\omega_m^2 = \frac{1}{KL},$$

the amplitude of I is

$$\frac{E_0}{R\sqrt{1 + \frac{L}{KR^2}\left(\frac{\omega}{\omega_m} - \frac{\omega_m}{\omega}\right)^2}}.$$

In Fig. 95 are plotted the values of the factor of E_0/R as ordinates, the abscissas being those of ω/ω_m. The different curves are, beginning at the outermost, for integral values of the ratio $\sqrt{\frac{L}{K}}\Big/R$ from 1 to 10. The resonance is sharper the larger this ratio.

* Pupin, "Electrical Oscillations of Low Frequency and their Resonance." *Am. Journ. Science*, April, May, 1893.

241. General Theory of Electrical Oscillations. We shall now consider the question of electrical oscillations in the

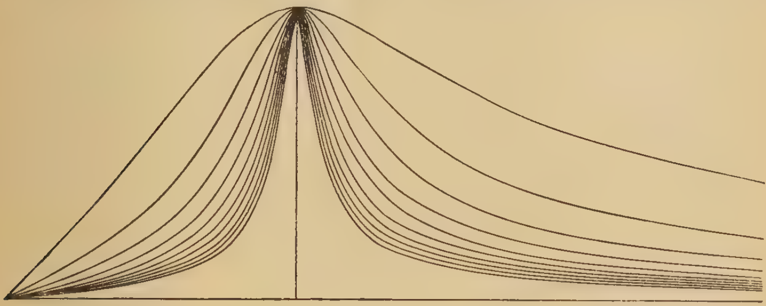

FIG. 95.

most general case of a network of linear conductors, conducted with any number of conductors K which may carry electrostatic charges. These may be grouped in pairs to form condensers, as in the last section, or they may be entirely independent of one another. Of the linear conductors, any one may form a closed circuit unconnected with the others, and affected only by current induction, or may end at points of embranchment with other conductors, or upon any of the conductors K. For brevity we shall call the linear conductors *wires*, and the conductors K *accumulators*. We shall suppose that the net contains p points of embranchment, k of which are connected with accumulators, for all wires which end on the same accumulator are to be considered as meeting in an embranchment. Let the number of wires be l. Then if all the wires form a part of the same net, the number of independent meshes is $l-p+1$, for we see at once that the smallest number of lines that can join p points to form a closed net is p, giving one mesh, and that after the first mesh every additional line adds a mesh*.

For every wire r between points a and b we have an equation

$$(1)\quad M_{r1}\frac{dI_1}{dt}+M_{r2}\frac{dI_2}{dt}+\ldots\ldots+M_{rl}\frac{dI_l}{dt}+R_rI_r=E_{ab}+V_a-V_b,$$

where E_{ab} is the impressed electromotive-force from a to b and V_a and V_b are the potentials of the points a and b. There are l equations of this sort.

* By independent meshes we mean such that circulation about any one is not the resultant of circulation about any number of others. For instance the outer boundary of a plane net is not independent of its meshes.

For every point of embranchment a we have an equation

$$(2) \qquad I_{1a}+I_{2a}+\dots\dots+I_{ta}=\frac{de_a}{dt},$$

the currents being now marked with double suffixes to denote the points between which they run, as in § 171, and e_a denoting the charge of the accumulator connected with the point, or zero if there is no accumulator. These p equations are not all independent, for adding them all together, every current appears in both directions, so that the left-hand side in the sum is identically zero, giving

$$(3) \qquad \frac{de_1}{dt}+\frac{de_2}{dt}+\dots\dots\frac{de_p}{dt}=0,$$

which is merely the statement that the total charge of the system is unaffected by the flow of currents. There are accordingly $p-1$ independent equations (2).

For every accumulator K_a we have an equation, § 138 (10),

$$(4) \qquad V_a=p_{1a}e_1+p_{2a}e_2+\dots\dots+p_{ka}e_k=\frac{\partial W}{\partial e_a}.$$

From the equations (1) the V's may be eliminated by Kirchhoff's principle, § 179. If, traversing any closed circuit, we add the equations (1) for each wire, every V appears with both signs, so that on the right we obtain the sum of the E's around the circuit. We shall thus obtain as many equations as there are independent meshes in the net, $l-p+1$. Other equations may be obtained in the same manner by traversing any unclosed circuit ending on two accumulators. All the potentials at embranchments passed over are eliminated except those of the two ends. The number of equations to be obtained in this manner is one less than the number of accumulators, or $k-1$. We thus obtain in all $l-p+k=n$ equations, and there are the same number of independent variables. We may take as parameters to characterize the system a set of currents, one circulating in each mesh, so that the actual current in any wire is the sum or difference of the currents in the two meshes to which that wire is common. The time-integral of any mesh-current shall be taken for one of the parameters q. Besides the $l-p+1$ q's thus defined, we will choose $k-1$ others, denoting the integral currents along any series of wires joining the accumulators two and two, the whole series forming a chain with two ends. The charge of any accumulator is

thus the difference of the two q's of this sort whose wires it separates. The whole number of q's is now just equal to n, the number of degrees of freedom of the system. The current in any wire is the sum of two or three of the q's with the proper signs, and as the electrokinetic energy is a homogeneous quadratic function of the currents, it becomes one also of the q''s. The derivative $-\frac{d}{dt}\left(\frac{\partial T}{\partial q_s'}\right)$ is the electromotive-force of induction around the circuit s, for

$$\frac{\partial T}{\partial q_s'} = \Sigma_r \frac{\partial T}{\partial I_r} \frac{\partial I_r}{\partial q_s'},$$

and every $\partial I_r/\partial q_s'$ is zero except in the case of the currents which bound the circuit, for any of which $\partial I_r/\partial q_s'$ is either plus or minus unity. The dissipation function, § 64, (7)

$$F = \tfrac{1}{2}\,\{R_1 I_1^2 + R_2 I_2^2 \;\ldots\ldots\; R_2 I_2^2\},$$

becomes also a homogenous quadratic function of the q''s in which the product terms will in general appear. The dissipative force will also be represented by $-\frac{\partial F}{\partial q_s'}$, for

$$\frac{\partial F}{\partial q_s'} = \Sigma_r \frac{\partial F}{\partial I_r} \frac{\partial I_r}{\partial q_s'},$$

which is again the sum of the products RI around the circuit. The terms

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_s'}\right) + \frac{\partial F}{\partial q_s'},$$

are accordingly what we get by adding the equations (1) for all the wires bounding the mesh s.

Since any charge is equal to plus or minus one of the q's of the second sort, or to the difference between two, W, the electrostatic energy, becomes a homogeneous quadratic function of these q's. Again $-\frac{\partial W}{\partial q_s}$ is the electrostatic electromotive-force belonging to q_s, for

$$\frac{\partial W}{\partial q_s} = \Sigma_r \frac{\partial W}{\partial e_r} \frac{\partial e_r}{\partial q_s}.$$

Now by (4), $\frac{\partial W}{\partial e_r} = V_r$, while $\frac{\partial e_r}{\partial q_s}$ is zero except for the accumulators at the beginning and end of q_s, where the derivative has the

values minus one and plus one respectively. We shall write our three functions

$$(5)\qquad \begin{aligned} T &= \tfrac{1}{2}\Sigma_r \Sigma_s M_{rs} q_r' q_s', \\ F &= \tfrac{1}{2}\Sigma_r \Sigma_r R_{rs} q_r' q_s', \\ W &= \tfrac{1}{2}\Sigma_r \Sigma_s p_{rs} q_r q_s, \end{aligned}$$

where the M's are linear combinations of the inductances of the wires, the R's linear combinations of their resistances, and the p's linear combinations of the coefficients of electrostatic potential of the accumulators. The values of the coefficients of the three functions are such that each of the functions is positive for all possible choices of its variables.

We may now apply Lagrange's equations for any parameter q_s.

$$(6)\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_s'}\right) + \frac{\partial F}{\partial q_s'} + \frac{\partial W}{\partial q_s} = E_s,$$

where E_s is the total external electromotive-force around the circuit. Performing the differentiations this becomes

$$(7)\qquad \begin{aligned} &M_{1s}\frac{d^2q_1}{dt^2} + M_{2s}\frac{d^2q_2}{dt^2} + \ldots\ldots M_{ns}\frac{d^2q_n}{dt^2} \\ &\qquad + R_{1s}\frac{dq_1}{dt} + R_{2s}\frac{dq_2}{dt} + \ldots\ldots R_{ns}\frac{dq_n}{dt} \\ &\qquad + p_{1s}q_1 + p_{2s}q_2 \ldots\ldots + p_{ns}q_n = E_s, \end{aligned}$$

a linear differential equation of the second order with constant coefficients. We have one such equation for each parameter q_s.

We shall first find the free oscillations, that is the solutions with every $E_s = 0$. As in the case of the simple examples of § 237, a particular solution may be obtained by assuming for every q_s,

$$(8)\qquad q_s = a_s e^{\lambda t},$$

where λ is the same for all the q's. Inserting these values in (7) we obtain

$$(9)\qquad \begin{aligned} &(M_{11}\lambda^2 + R_{11}\lambda + p_{11})\,a_1 + \ldots\ldots + (M_{1n}\lambda^2 + R_{1n}\lambda + p_{1n})\,a_n = 0, \\ &(M_{21}\lambda^2 + R_{21}\lambda + p_{21})\,a_1 + \ldots\ldots + (M_{2n}\lambda^2 + R_{2n}\lambda + p_{2n})\,a_n = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &(M_{n1}\lambda^2 + R_{n1}\lambda + p_{n1})\,a_1 + \ldots\ldots + (M_{nn}\lambda^2 + R_{nn}\lambda + p_{nn})\,a_n = 0, \end{aligned}$$

a set of linear equations to determine the ratios of the a's when λ is known. If these are to be satisfied by other than zero values of

the a's, however, the determinant of the coefficients must vanish, namely

$$(10)\quad \begin{vmatrix} M_{11}\lambda^2 + R_{11}\lambda + p_{11}, & \ldots\ldots & M_{1n}\lambda^2 + R_{1n}\lambda + p_{12} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ M_{n1}\lambda^2 + R_{n1}\lambda + p_{11}, & \ldots\ldots & M_{nn}\lambda^2 + R_{nn}\lambda + p_{nn} \end{vmatrix} = 0.$$

This is an equation of order $2n$ in λ, from which the odd powers are absent if $F = 0$. We shall denote its roots by

$$\lambda_1, \lambda_2, \ldots\ldots \lambda_{2n}.$$

If we multiply the rth equation (9) by a_r, and take the sum for all r's, we obtain

$$(11)\qquad \lambda^2\Sigma_r\Sigma_s M_{rs}a_r a_s + \lambda\Sigma_r\Sigma_s R_{rs}a_r a_s + \Sigma_r\Sigma_s p_{rs}a_r a_s = 0.$$

The double sum by which λ^2 is multiplied is the value of the function $2T$ when for every q_s' is substituted a_s. We shall denote this by $2T(a)$. Similarly the coefficient of λ is $2F(a)$ and the term independent of λ is $2W(a)$. But by the fundamental property of the three functions, each must be positive. The equation (11),

$$\lambda^2 T(a) + \lambda F(a) + W(a) = 0,$$

shows us at once that λ can not be real and positive, for that would involve the sum of three positive terms being equal to zero.

Secondly, if $F = 0$, that is, if the resistance of every wire is zero,

$$\lambda^2 = -\frac{W(a)}{T(a)},$$

and λ is a pure imaginary. In this case $e^{\lambda t}$ and $e^{-\lambda t}$ are trigonometric functions, representing an undamped oscillation of the same period for all the parameters q.

Thirdly, if F is large enough, λ can be real and negative. In this case each parameter q gradually dies away to zero, the relaxation time being the same for all. This corresponds to Case I of § 239.

Fourthly, if either W or T is zero, instead of a pair of roots we have a single one, which is real and negative, the cases corresponding respectively to § 237 or § 207.

Fifthly, in other cases, that is when neither T, F, nor W vanish, and F is not too large, λ is complex. We shall prove that then the real part of λ is negative.

When any value of λ is determined, the equations (9) determine the quantities a except for a common factor. If complex values enter, since any equation which involves i will also hold good if i be changed to $-i$, changing any root λ to its conjugate λ' causes every a to change to its conjugate a'. We shall denote the a's corresponding to the conjugate roots λ and λ' by a and a', where

$$\lambda = \mu + i\nu, \quad \lambda' = \mu - i\nu,$$

$$a_s = \alpha_s + i\beta_s, \; a_s' = \alpha_s - i\beta_s.$$

Let us now apply the process that gave us equation (11), except that we multiply the equations (9) containing λ by the a''s belonging to λ', obtaining

(12) $\lambda^2 \Sigma_r \Sigma_s M_{rs} a_r a_s' + \lambda \Sigma_r \Sigma_s R_{rs} a_r a_s' + \Sigma_r \Sigma_s p_{rs} a_r a_s' = 0.$

In this equation, any coefficient M_{ab} appears in the terms for which $r=a$, $s=b$ and $r=b$, $s=a$, so that the sum is

$$M_{ab}\,(a_a a_b' + a_b a_a'),$$

or substituting the values of the a's,

$$M_{ab}\,\{(\alpha_a + i\beta_a)(\alpha_b - i\beta_b) + (\alpha_b + i\beta_b)(\alpha_a - i\beta)\} = 2M_{ab}\,(\alpha_a\alpha_b + \beta_a\beta_b).$$

Using a notation similar to that before employed, equation (12) is

(13) $\lambda^2\,\{T(\alpha) + T(\beta)\} + \lambda\,\{F(\alpha) + F(\beta)\} + W(\alpha) + W(\beta) = 0.$

Now performing the same process on the equations (9) with λ', and multiplying by the a's we obtain

(14) $\lambda'^2\,\{T(\alpha) + T(\beta)\} + \lambda'\,\{F(\alpha) + F(\beta)\} + W(\alpha) + W(\beta) = 0,$

so that λ and λ' are roots of the same quadratic. We have therefore for their sum

(15) $$\lambda + \lambda' = 2\mu = -\frac{F(\alpha) + F(\beta)}{T(\alpha) + T(\beta)}.$$

Accordingly μ is negative. The solution therefore represents a damped vibration, as in the second case of § 239, the period and damping being the same for all the q's.

Since for every root λ we obtain a set of values of the a's, we shall distinguish the values for the different roots by a second set of suffixes, so that a_{rs} means the coefficient of e^{λ_s} in the coordinate q_r for the sth period. The theory of differential equations tells us

that for the general solution we must take the sum of the terms $ae^{\lambda t}$ for all the roots, so that we obtain

$$\begin{aligned}
q_1 &= a_{11}e^{\lambda_1 t} + a_{12}e^{\lambda_2 t} + \ldots\ldots + a_{12n}e^{\lambda_{2n} t},\\
q_2 &= a_{21}e^{\lambda_1 t} + a_{22}e^{\lambda_2 t} + \ldots\ldots + a_{22n}e^{\lambda_{2n} t},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
q_n &= a_{n1}e^{\lambda_1 t} + a_{n2}e^{\lambda_2 t} + \ldots\ldots + a_{n2n}e^{\lambda_{2n} t}.
\end{aligned}\tag{16}$$

We may now replace the exponentials by trigonometric terms. The appearance of the terms with conjugate imaginaries

$$a_s e^{\lambda t} + a_s' e^{\lambda' t} = 2e^{\mu t}(\alpha_s \cos \nu t - \beta_s \sin \nu t)$$

leads to the disappearance of imaginaries from the result, so that we obtain,

$$\begin{aligned}
q_1 &= 2\{e^{\mu_1 t}(\alpha_{11}\cos \nu_1 t - \beta_{11}\sin \nu_1 t) + \ldots + e^{\mu_n t}(\alpha_{1n}\cos \nu_n t - \beta_{1n}\sin \nu_n t)\},\\
q_2 &= 2\{e^{\mu_1 t}(\alpha_{21}\cos \nu_1 t - \beta_{21}\sin \nu_1 t) + \ldots + e^{\mu_n t}(\alpha_{2n}\cos \nu_n t - \beta_{2n}\sin \nu_n t)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
q_n &= 2\{e^{\mu_1 t}(\alpha_{n1}\cos \nu_1 t - \beta_{n1}\sin \nu_1 t) + \ldots + e^{\mu_n t}(\alpha_{nn}\cos \nu_n t - \beta_{nn}\sin \nu_n t)\}.
\end{aligned}\tag{17}$$

The ratios of the α's or of the β's in any column are given by the equations (9), being different for the different columns. Since the ratios of the β's of any column are the same as those of the α's, we may otherwise write the equations as

$$\begin{aligned}
q_1 &= 2\{A_{11}e^{\mu_1 t}\cos(\nu_1 t - \gamma_1) + A_{12}e^{\mu_2 t}\cos(\nu_2 t - \gamma_2) \ldots + A_{1n}e^{\mu_n t}\cos(\nu_n t - \gamma_n)\},\\
q_2 &= 2\{A_{21}e^{\mu_1 t}\cos(\nu_1 t - \gamma_1) + A_{22}e^{\mu_2 t}\cos(\nu_2 t - \gamma_2) \ldots + A_{2n}e^{\mu_n t}\cos(\nu_n t - \gamma_n)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
q_n &= 2\{A_{n1}e^{\mu_1 t}\cos(\nu_1 t - \gamma_1) + A_{n2}e^{\mu_2 t}\cos(\nu_2 t - \gamma_2) \ldots + A_{nn}e^{\mu_n t}\cos(\nu_n t - \gamma_n)\},
\end{aligned}\tag{18}$$

where

$$\begin{aligned}
A_{rs}^2 &= \alpha_{rs}^2 + \beta_{rs}^2 = |a_{rs}|^2,\\
\gamma_s &= -\tan^{-1}\frac{\beta_{rs}}{\alpha_{rs}} = -\arg a_{rs}.
\end{aligned}\tag{19}$$

We accordingly may state the general result:—The free vibrations of any electrical system consist, for each electrical coordinate, of the resultant of a number of damped harmonic oscillations of different periods, the number of different terms being n, the number of degrees of freedom of the system. The phase and

damping of any particular simple oscillation are the same for all the coordinates, and the n factors of the amplitudes and the n phases are to be determined from the initial values of the q's and of their first time derivatives.

We will now consider the case of forced vibrations.

On account of the linearity of the equations, if we find a solution $q_r^{(1)}$ for a particular set of values $E_s^{(1)}$ of the right-hand members of our equations (7), and a second solution $q_s^{(2)}$ for a second set $E_s^{(2)}$, then the sum $q_r^{(1)}+q_r^{(2)}$ will be the solution when the right-hand members are $E_s^{(1)}+E_s^{(2)}$. We shall, therefore, consider the effect of each impressed force by itself. Suppose first then that in each circuit there is impressed a harmonic electromotive force, $E_s \cos \omega t$, all of the same period. Then we have the equations of which the sth is

$$\text{(20)} \qquad M_{1s}\frac{d^2q_1}{dt^2}+\ldots+M_{ns}\frac{d^2q_n}{dt^2}+R_{1s}\frac{dq_1}{dt}+\ldots+R_{ns}\frac{dq_n}{dt}$$
$$+p_{1s}q_1+\ldots+p_{ns}q_n=E_se^{i\omega t}.$$

Assuming $q_r = a_s e^{i\omega t}$ these reduce to

$$\text{(21)} \qquad \begin{array}{l} (-M_{11}\omega^2+R_{11}i\omega+p_{11})\,a_1+\ldots+(-M_{1n}\omega^2+R_{1n}i\omega+p_{1n})\,a_n=E_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (-M_{n1}\omega^2+R_{n1}i\omega+p_{n1})\,a_1+\ldots+(-M_{nn}\omega^2+R_{nn}i\omega+p_{nn})\,a_n=E_n \end{array}$$

a set of linear equations to determine the a's.

If we call the determinant of equation (10), $D(\lambda)$, and $D_{rs}(\lambda)$ the minor of the element of the rth column and sth row, we have as the solution of (21),

$$\text{(22)} \qquad a_r=\frac{\Sigma_s D_{rs}(i\omega)\,E_s}{D(i\omega)}.$$

Since $D(\lambda)=0$ is the determinantal equation for the free vibration, whose roots are $\lambda_1, \lambda_2 \ldots \lambda_{2n}$, we have

$$\text{(23)} \qquad D(\lambda)=C(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{2n})=C\Pi_s(\lambda-\lambda_s).$$

Accordingly the denominator $D(i\omega)$ is

$$\text{(24)} \qquad D(i\omega)=C\Pi_s(i\omega-\lambda_s)=C\Pi_s\{-\mu_s+i(\omega-\nu_s)\}.$$

The minors $D_{rs}(i\omega)$ are rational integral functions of $i\omega$, and the numerators are therefore complex quantities, which reduce to real

ones if the R's are zero. Calling the modulus of a numerator B_r, and its argument θ_r,

(25) $$\Sigma_s D_{rs}(i\omega) E_s = B_r e^{i\theta_r},$$

θ_r is a small angle if the resistances are small. We thus have

(26) $$a_r = \frac{B_r e^{i\theta_r}}{C\Pi_s \{-\mu_s + i(\omega - \nu_s)\}} = \frac{B_r e^{i(\theta_r - \Sigma_s \alpha_s)}}{C\Pi_s A_s},$$

where

(27) $$A_s = \{\mu_s^2 + (\omega - \nu_s)^2\}^{\frac{1}{2}}, \quad \tan \alpha_s = -\frac{\omega - \nu_s}{\mu_s}.$$

Retaining now only the real parts, we have for the solution

(28) $$q_r = \frac{B_r \cos(\omega t + \theta_r - \Sigma_s \alpha_s)}{C\Pi_s[(\mu_s^2 + (\omega - \nu_s)^2)^{\frac{1}{2}}]}.$$

Thus if the resistances are small, all the oscillations are in nearly the same phase. If the frequency of the impressed force coincides with that of any one of the free oscillations, $\omega - \nu_s = 0$, and one factor of the denominator reduces to μ_s, so that if the damping of that oscillation is small, the amplitude is very large, or infinite if there is no damping. This is the case of resonance. (Resonance may also be defined in a slightly different manner as occurring when $i\omega$ is one of the roots of the equation $D(\lambda) = 0$ in which all the R's have been put equal to zero. This corresponds with our example in § 240. In practical cases the difference is very small.)

If now we have a system acted on by electromotive forces each one of which is the sum of any number of harmonic components of different periods, any component may cause resonance with any free oscillation of the system, so that resonance may occur in a large number of ways.

242. Examples. Two Circuits. We shall illustrate the principles of the preceding section, aside from the examples that have already been given in §§ 239, 240, involving one degree of freedom, by an example of two circuits. Consider an induction coil in which both the primary and secondary contain a condenser in series. This is the case of the so-called Tesla high-frequency coil, in which a Leyden jar produces an oscillatory discharge through the primary, while the ends of the secondary are usually connected with a small capacity, say a pair of knobs. We shall

take for q_1 and q_2 the charges of the two condensers, so that the currents are

$$(1)\qquad I_1 = \frac{dq_1}{dt}, \qquad I_2 = \frac{dq_2}{dt}.$$

We accordingly have

$$(2)\qquad \begin{aligned} T &= \tfrac{1}{2}L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2}L_2 I_2^2, \\ F &= \tfrac{1}{2}R_1 I_1^2 + \tfrac{1}{2}R_2 I_2^2, \\ W &= \tfrac{1}{2}\frac{q_1^2}{K_1} + \tfrac{1}{2}\frac{q_2^2}{K_2}, \end{aligned}$$

and the differential equations for the free oscillations are

$$(3)\qquad \begin{aligned} L_1 \frac{d^2q_1}{dt^2} + M\frac{d^2q_2}{dt^2} + R_1\frac{dq_1}{dt} + \frac{1}{K_1}q_1 &= 0, \\ M\frac{d^2q_1}{dt^2} + L_2\frac{d^2q_2}{dt^2} + R_2\frac{dq_2}{dt} + \frac{1}{K_2}q_2 &= 0. \end{aligned}$$

The equation for the frequencies is

$$(4)\qquad \begin{vmatrix} L_1\lambda^2 + R_1\lambda + \dfrac{1}{K_1}, & M\lambda^2 \\ M\lambda^2, & L_2\lambda^2 + R_2\lambda + \dfrac{1}{K_2} \end{vmatrix} = 0,$$

or

$$(5)\qquad (L_1L_2 - M^2)\lambda^4 + (L_1R_2 + L_2R_1)\lambda^3 + \left(\frac{L_1}{K_2} + \frac{L_2}{K_1} + R_1R_2\right)\lambda^2 + \left(\frac{R_1}{K_2} + \frac{R_2}{K_1}\right)\lambda + \frac{1}{K_1K_2} = 0.$$

As this equation is of the fourth degree, we shall treat only the case of no damping, which, as we have seen, will not cause a large error in the determination of the frequencies. Putting then $R_1 = R_2 = 0$ the equation becomes

$$(6)\qquad \lambda^4 + \frac{L_1K_1 + L_2K_2}{K_1K_2(L_1L_2 - M^2)}\lambda^2 + \frac{1}{K_1K_2(L_1L_2 - M^2)} = 0,$$

or, as we may otherwise write,

$$(7)\qquad \frac{1}{\lambda^4} + (L_1K_1 + L_2K_2)\frac{1}{\lambda^2} + K_1K_2(L_1L_2 - M^2) = 0.$$

If the two roots of this quadric are

$$\frac{1}{\lambda_1^2},\ \frac{1}{\lambda_2^2},$$

we have for the periods,

$$T = \frac{2\pi i}{\lambda_1}, \quad T' = \frac{2\pi i}{\lambda_2},$$

so that

$$(8) \qquad \begin{aligned} T &= \pi\sqrt{2\{L_1K_1 + L_2K_2 + \sqrt{(L_1K_1 - L_2K_2)^2 + 4K_1K_2M^2}\}}, \\ T' &= \pi\sqrt{2\{L_1K_1 + L_2K_2 - \sqrt{(L_1K_1 - L_2K_2)^2 + 4K_1K_2M^2}\}}. \end{aligned}$$

If we introduce the periods of the two circuits alone,

$$T_1 = 2\pi\sqrt{K_1L_1}, \quad T_2 = 2\pi\sqrt{K_2L_2},$$

and a quantity θ which is nearly a mean proportional between them,

$$\theta = 2\pi\sqrt{M\sqrt{K_1K_2}},$$

these periods become

$$(9) \qquad \begin{aligned} T &= \sqrt{\frac{T_1^2 + T_2^2 + \sqrt{(T_1^2 - T_2^2)^2 + 4\theta^4}}{2}}, \\ T' &= \sqrt{\frac{T_1^2 + T_2^2 - \sqrt{(T_1^2 - T_2^2)^2 + 4\theta^4}}{2}}. \end{aligned}$$

In case $T_1 = T_2$,

$$(10) \qquad \begin{aligned} T^2 &= T_1^2 + \theta^2 = 4\pi^2(K_1L_1 + M\sqrt{K_1K_2}), \\ T'^2 &= T_1^2 - \theta^2 = 4\pi^2(K_2L_2 - M\sqrt{K_1K_2}). \end{aligned}$$

This is a case of so-called resonance, though not the one that we have examined. We see that one of the periods is greater and the other less than the common period of the separate circuits.

If the period of one of the circuits is much greater than that of the other, so that both

$$T_1 > T_2 \text{ and } T_1^2 - T_2^2 > 2\theta^2,$$

we have, developing the square roots by the binomial theorem, the approximation,

$$(11) \qquad \begin{aligned} T^2 &= T_1^2 + \frac{\theta^4}{T_1^2 - T_2^2}, \\ T'^2 &= T_2^2 - \frac{\theta^4}{T_1^2 - T_2^2}. \end{aligned}$$

In this case the longer period is nearly that of the longer individual period, being somewhat longer, while the shorter period is somewhat shorter than the shorter individual period. This is probably

the usual case of the Tesla coil, where only the longer oscillation plays much part. For a further treatment of this example, the reader is referred to articles by Oberbeck* and Blümcke†.

We shall now consider the forced oscillation. Let there be an impressed force $E_0 \cos \omega t$ in the primary circuit, there being none in the secondary. Then we have for the secondary

$$q_2 = a_2 e^{i\omega t},$$

(12) $$a_2 = \frac{E_0 M \omega^2}{(L_1 L_2 - M^2)\,\omega^4 - \left(\frac{L_1}{K_2} + \frac{L_2}{K_1} + R_1 R_2\right)\omega^2 + \frac{1}{K_1 K_2} + i\left\{-(L_1 R_2 + L_2 R_1)\,\omega^3 + \left(\frac{R_1}{K_2} + \frac{R_2}{K_1}\right)\omega\right\}}$$

The amplitude of the secondary current $I_2^{(0)} = \omega \,.\, |\, a_2 \,|$ is

(13) $$I_2^{(0)} = \frac{E_0 M \omega^3}{\left[\left\{(L_1 L_2 - M^2)\,\omega^4 - \left(\frac{L_1}{K_2} + \frac{L_2}{K_1} + R_1 R_2\right)\omega^2 + \frac{1}{K_1 K_2}\right\}^2 + \left\{-(L_1 R_2 + L_2 R_1)\,\omega^3 + \left(\frac{R_1}{K_2} + \frac{R_2}{K_1}\right)\omega\right\}^2\right]^{\frac{1}{2}}}$$

We get resonance when ω^2 is one of the roots of the quadratic

(14) $$(L_1 L_2 - M^2)\,\omega^4 - \left(\frac{L_1}{K_2} + \frac{L_2}{K_1}\right)\omega^2 + \frac{1}{K_1 K_2} = 0.$$

In case there is no condenser in the secondary, we have

$$K_2 = \infty,$$

and there is then but one frequency for resonance,

(15) $$\omega^2 = \frac{L_2}{K_1 (L_1 L_2 - M^2)}.$$

This is the practical case of a transformer or induction coil, and is treated by J. J. Thomson in his *Recent Researches in Electricity and Magnetism*, Chapter VI., to which the student is referred for further examples of this subject. For a treatment at length of the subject of oscillations, the student may consult Rayleigh, *Theory of Sound*, Chapters IV, V. and X.B, and Routh, *Advanced Rigid Dynamics*, Chapter II.

* Oberbeck. "Ueber den Verlauf der electrischen Schwingungen bei den Tesla'schen Versuchen." *Wied. Ann.* 55, p. 623, 1895.

† Blümcke. "Bemerkung zu der Abhandlung des Hrn. A. Oberbeck." *Wied. Ann.* 58, p. 405, 1896.

CHAPTER XIII.

EQUATIONS OF ELECTROMAGNETIC FIELD.

ELECTROMAGNETIC WAVES.

243. Localized Electric Force of Induction. In the preceding chapter we have developed the theory of current induction in linear circuits, on the basis of the treatment of a set of currents as a mechanical cyclic system, and we have thus arrived at equations which are justified by experiment. We have found for the electromotive force of induction in any circuit,

$$(1) \qquad E_i = -\frac{d}{dt}\left(\frac{\partial T}{\partial I}\right) = -\frac{d\bar{p}}{dt},$$

where $\bar{p}$, the electro-kinetic momentum corresponding to the circuit, is by the results of § 226 defined as the total flux of magnetic induction through the circuit, that is the surface integral

$$(2) \qquad \bar{p} = \iint \{\mathfrak{L}\cos(nx) + \mathfrak{M}\cos(ny) + \mathfrak{N}\cos(nz)\}\, dS$$

over any cap bounded by the circuit.

If we consider the electromotive force around the circuit as made up of electric forces acting at each point of the circuit, just as in § 166 we considered the electromotive force due to electrostatic action as the line-integral of the electrostatic field-intensity, we may here consider the electromotive force as a line-integral around the circuit,

$$(3) \qquad E_i = \int (X dx + Y dy + Z dz) = \int F \cos(Fds)\, ds.$$

The vector F whose components are X, Y, Z is a quantity of the same nature as the electric field-intensity, and we shall not in future distinguish whether it is of electrostatic or electrodynamic origin. If we apply Stokes's theorem to the line-integral in (3) we

convert it into a surface-integral which, in virtue of (1), must be equal to the negative time-derivative of the surface-integral in (2).

$$(4)\quad \iint\left\{\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\cos(nx)+\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)\cos(ny)+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\right\}dS$$
$$=-\frac{\partial}{\partial t}\iint\{\mathfrak{L}\cos(nx)+\mathfrak{M}\cos(ny)+\mathfrak{N}\cos(nz)\}\,dS.$$

As we assume that the circuit does not change geometrically with the time the differentiation with respect to t may be passed under the sign of integration, and operates only on the quantities $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$. Since the two surface-integrals may be taken over the same surface, and the equality holds for any portion of surface whatever (as we may choose any cap over any circuit), the integrands are necessarily equal at all points of space, necessitating the equations

$$(5)\quad \begin{aligned} -\frac{\partial\mathfrak{L}}{\partial t}&=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},\\ -\frac{\partial\mathfrak{M}}{\partial t}&=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},\\ -\frac{\partial\mathfrak{N}}{\partial t}&=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}.\end{aligned}$$

These equations, which are more compactly expressed by

$$(6)\quad -\frac{\partial\overline{\mathfrak{B}}}{\partial t}=\text{curl }\overline{F},$$

are the general equations of induction, and are justified because of their leading, by the reverse process, to the equation (1), which is directly verified by experiment. A direct experimental verification of equations (5) has been given but recently.

If we wish to introduce the vector-potential belonging to the magnetic induction, by § 226 we have the alternative expression for $\overline{p}$*,

$$(7)\quad \overline{p}=\int(F\,dx+G\,dy+H\,dz).$$

Comparing this now with the line-integral in (3) gives us

$$(8)\quad \int(X\,dx+Y\,dy+Z\,dz)=-\frac{\partial}{\partial t}\int(F\,dx+G\,dy+H\,dz).$$

* See the definition of $\overline{p}$ following equation (3), p. 469.

From the equality of the line-integrals we must not conclude the equality of the integrands, for the line-integral of any lamellar vector point-function around a closed path vanishes. We accordingly obtain

$$(9)\qquad X = -\frac{\partial F}{\partial t} + X', \quad Y = -\frac{\partial G}{\partial t} + Y', \quad Z = -\frac{\partial H}{\partial t} + Z',$$

where (X', Y', Z') is a lamellar vector. If X, Y, Z denote the whole electric force, when the state of the magnetic field is not changing it becomes the electrostatic force, so that the components X', Y', Z' must be the negative derivatives of the electric potential. Accordingly the equations are

$$(10)\qquad X = -\frac{\partial F}{\partial t} - \frac{\partial V}{\partial x}, \quad Y = -\frac{\partial G}{\partial t} - \frac{\partial V}{\partial y}, \quad Z = -\frac{\partial H}{\partial t} - \frac{\partial V}{\partial z}.$$

These are the equations as given by Maxwell*. We shall however prefer the form (5), not containing either potential, as introduced by Heaviside† and Hertz‡. Since the electrostatic field has no curl, it need not be considered separately in equations (5).

If however there are impressed electromotive forces X', Y', Z' not of electrostatic origin, such as those due to chemical or thermal effects, and X, Y, Z still denote the *total* field, we must replace X, Y, Z in equations (5) by $X - X', Y - Y', Z - Z'$. (Heaviside, Vol. I. p. 449.)

In a closed conductor undergoing electromagnetic induction there are not necessarily differences of electric potential, for

* *Treatise*, Art. 598, equations (B).

† "Electromagnetic Induction and its Propagation." *Electrician*, Feb. 1885, *Papers*, Vol. I., p. 447, eq. (20).

‡ "Die Kräfte elektrischer Schwingungen behandelt nach der Maxwell'schen Theorie." *Wied. Ann.*, 36, p. 1, 1889. Jones's trans., p. 138.

example in the case of a circular ring placed perpendicularly to the force of a varying uniform magnetic field the electric potential is constant. If however the circuit is broken, current flows for a very short time until the electric force vanishes; there is then produced a disturbance of charges producing differences of potential to be calculated from the equations

$$-\frac{\partial V}{\partial x}=\frac{\partial F}{\partial t},\quad -\frac{\partial V}{\partial y}=\frac{\partial G}{\partial t},\quad -\frac{\partial V}{\partial z}=\frac{\partial H}{\partial t}.$$

Conductors connected to the broken ends of the circuit, for instance the plates of an electrometer, will then show a difference of potential.

244. Displacement Currents. If we compare the equations (5) with the equations § 222 (2),

$$(11)\qquad \begin{aligned} 4\pi u &= \frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},\\ 4\pi v &= \frac{\partial L}{\partial z}-\frac{\partial N}{\partial x},\\ 4\pi w &= \frac{\partial M}{\partial x}-\frac{\partial L}{\partial y},\end{aligned}$$

we notice that they are analogous in having the right-hand sides equal to the curl of the electric and magnetic field respectively. We make the analogy still more complete by introducing the conception introduced into the theory by Maxwell of the electrical displacement current *.

Suppose that we have a condenser charged with electricity. There is then a field of electric force, the lines of force running from the positively charged plate to the negative. The electric induction is, by § 182 (16),

$$\mathfrak{F}=4\pi\sigma.$$

If now the plates be connected by a conducting wire, the positive charge passes from the positive plate along the wire, until a state of equilibrium is reached. During this period the electric induction between the condenser plates is diminishing and finally reaches zero. *The hypothesis of Maxwell is that the change of the*

* "A Dynamical Theory of the Electromagnetic Field (11)," *Phil. Trans.* Vol. CLV. 1864.

induction produces the same magnetic effect as would be produced by a current of current-density

$$\bar{q} = \frac{1}{4\pi}\frac{\partial \overline{\mathfrak{F}}}{\partial t}$$

at every point of the field, which together with the current in the wire would form a closed circuit. As the equations § 222 (2) were deduced from the magnetic effect of closed currents, some hypothesis is necessary if we are to deal with unclosed currents, and Maxwell's hypothesis is justified by its remarkable consequences. Since Maxwell calls the vector $\overline{\mathfrak{F}}/4\pi$ the *electrical displacement*, he terms the vector $\dfrac{1}{4\pi}\dfrac{\partial \overline{\mathfrak{F}}}{\partial t}$ the *displacement current.*

The consequence of Maxwell's hypothesis is that in the dielectric we must introduce the components of the displacement current $\dfrac{1}{4\pi}\dfrac{\partial \mathfrak{X}}{\partial t}$, $\dfrac{1}{4\pi}\dfrac{\partial \mathfrak{Y}}{\partial t}$, $\dfrac{1}{4\pi}\dfrac{\partial \mathfrak{Z}}{\partial t}$ in place of u, v, w in the equations (11), giving

$$\text{(12)} \qquad \begin{aligned} \frac{\partial \mathfrak{X}}{\partial t} &= \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \\ \frac{\partial \mathfrak{Y}}{\partial t} &= \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \\ \frac{\partial \mathfrak{Z}}{\partial t} &= \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}. \end{aligned}$$

These equations are now completely analogous to the equations (5) except for the difference of sign on the left, the two sets being represented by

$$\text{(13)} \qquad \begin{aligned} -\frac{\partial \overline{\mathfrak{B}}}{\partial t} &= \operatorname{curl} \bar{F}, \\ \frac{\partial \overline{\mathfrak{F}}}{\partial t} &= \operatorname{curl} \bar{H}. \end{aligned}$$

If the dielectric is conducting, we must introduce both the conduction and the displacement current, so that the equations are

$$\text{(14)} \qquad \begin{aligned} \frac{\partial \mathfrak{X}}{\partial t} + 4\pi u &= \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \\ \frac{\partial \mathfrak{Y}}{\partial t} + 4\pi v &= \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \\ \frac{\partial \mathfrak{Z}}{\partial t} + 4\pi w &= \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}. \end{aligned}$$

Differentiating these equations respectively by x, y, z and adding we obtain

$$(15)\quad \frac{\partial}{\partial x}\left(\frac{1}{4\pi}\frac{\partial \mathfrak{X}}{\partial t}+u\right)+\frac{\partial}{\partial y}\left(\frac{1}{4\pi}\frac{\partial \mathfrak{Y}}{\partial t}+v\right)+\frac{\partial}{\partial z}\left(\frac{1}{4\pi}\frac{\partial \mathfrak{Z}}{\partial t}+w\right)=0,$$

so that the total current is solenoidal, like the flow of an incompressible fluid. Integrating (15) through a portion of space τ bounded by a closed surface S,

$$(16)\quad \frac{1}{4\pi}\frac{\partial}{\partial t}\iiint\left(\frac{\partial \mathfrak{X}}{\partial x}+\frac{\partial \mathfrak{Y}}{\partial y}+\frac{\partial \mathfrak{Z}}{\partial z}\right)d\tau=-\iiint\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)d\tau,$$

$$=\iint\{u\cos(nx)+v\cos(ny)+w\cos(nz)\}\,dS,$$

which by § 182 (17) becomes

$$(17)\quad \frac{\partial}{\partial t}\iiint\rho d\tau=\frac{\partial e}{\partial t}=\iint\{u\cos(nx)+v\cos(ny)+w\cos(nz)\}\,dS.$$

That is, the increase of charge of any portion of space is equal to the electricity brought in by conduction. This agrees perfectly with our previous conceptions. Our statement made in § 129 that *electricity* is not incompressible is also reconcilable with Maxwell's statement that the *total* current, the resultant of the conduction and displacement currents, is like the flow of an incompressible fluid.

By the analogy between the equations (5) and (12) we might call the vector $\frac{1}{4\pi}\frac{\partial \mathfrak{B}}{\partial t}$ the magnetic displacement current. Magnetic conduction-currents do not exist, although they have been introduced into the equations by Heaviside* for the sake of symmetry.

245. Complete System of Equations for Media at rest. We may now collect all the fundamental equations of the theory as it has been developed. Before doing this it will be convenient to make a slight change in our units. It will be recalled that in the whole of Part III since the introduction of the electromagnetic system of units we have considered all quantities, whether electrical or magnetic, to be measured in that system. Up to the present this has been most convenient, and in practical cases dealing with electro-magnetism and electro-magnetic

* "Electromagnetic Induction and its Propagation." *Electrical Papers*, Vol. I. p. 441.

induction this will generally be true. We are now, however, about to consider a new class of phenomena, and it will be convenient to use the Gaussian system, that is, to measure all electrical quantities in electrostatic units, and all magnetic ones in magnetic units. We shall therefore be obliged to reintroduce the factor A, § 210, which will multiply the electric currents, and divide the electrical forces, according to § 212, equations (6) and (9). Equations (14) and (5) thus become*

$$\text{(A)}\quad \begin{aligned} A\frac{\partial \mathfrak{X}}{\partial t} + 4\pi A u &= \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \\ A\frac{\partial \mathfrak{Y}}{\partial t} + 4\pi A v &= \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \\ A\frac{\partial \mathfrak{Z}}{\partial t} + 4\pi A w &= \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}, \end{aligned} \qquad \text{(B)}\quad \begin{aligned} -A\frac{\partial \mathfrak{L}}{\partial t} &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \\ -A\frac{\partial \mathfrak{M}}{\partial t} &= \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \\ -A\frac{\partial \mathfrak{N}}{\partial t} &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}. \end{aligned}$$

These are the equations of cross-connection between the electric and magnetic fields and thus show that in non-conductors the curl of the force of either field determines, or is determined by, the time-variation of the induction of the other. If we know the state of the field at any instant we may accordingly find it at any subsequent instant. For we have the three sets of equations expressing the Fourier-Ohm laws,

$$\text{(C)}\quad \begin{aligned} \mathfrak{X} &= \epsilon X, \\ \mathfrak{Y} &= \epsilon Y, \\ \mathfrak{Z} &= \epsilon Z, \end{aligned} \qquad \text{(D)}\quad \begin{aligned} \mathfrak{L} &= \mu L, \\ \mathfrak{M} &= \mu M, \\ \mathfrak{N} &= \mu N, \end{aligned} \qquad \text{(E)}\quad \begin{aligned} u &= \lambda X, \\ v &= \lambda Y, \\ w &= \lambda Z. \end{aligned}$$

The letter ϵ denotes the electric inductivity, which in Chapter IX, where we did not distinguish electric and magnetic quantities, was denoted by μ. It will be noticed that equations (A) and (B), which are the fundamental equations of the theory, contain no quantities that are intrinsic to the media, but only those quantities which completely specify the electric and magnetic state of the fields. The equations (C), (D), and (E), on the contrary, contain the quantities ϵ, μ and λ, which denote properties of the media. These latter equations are not fundamental to the theory, as they may under certain circumstances be replaced by others. In addition

* In Hertz's papers the right-hand members appear with the opposite sign, since Hertz employs the *left-handed* arrangement of axes. (Cf. Fig. 1.)

we have for the electric and magnetic energies and the dissipativity, or heat generated per unit of time,

$$W = \frac{1}{8\pi}\iiint_\infty (X\mathfrak{X} + Y\mathfrak{Y} + Z\mathfrak{Z})\, d\tau,$$

$$\text{(F)} \qquad T = \frac{1}{8\pi}\iiint_\infty (L\mathfrak{L} + M\mathfrak{M} + N\mathfrak{N})\, d\tau,$$

$$H = \iiint_\infty (Xu + Yv + Zw)\, d\tau.$$

246. Eolotropic Media. The equations (C), (D) and (E) have been established on the supposition that the medium is isotropic, that is that it has the same properties in all directions at any point. In some bodies, such as certain natural crystals, this is not true. The assumption next in order of simplicity to that made in Chapter IX is to assume that the energy per unit volume is a homogeneous quadratic function of the components of the field

$$\frac{1}{8\pi}\{\epsilon_{11}X_1^2 + \epsilon_{22}Y^2 + \epsilon_{33}Z^2 + \epsilon_{23}YZ + \epsilon_{31}ZX + \epsilon_{12}XY\},$$

the six coefficients being properties of the medium. If we then apply the reasoning of § 180, we find that our results are the same as before, providing that we define the inductions by the equations

$$\text{(C}'\text{)} \quad \begin{aligned} \mathfrak{X} &= \epsilon_{11}X + \epsilon_{12}Y + \epsilon_{13}Z, \\ \mathfrak{Y} &= \epsilon_{21}X + \epsilon_{22}Y + \epsilon_{23}Z, \\ \mathfrak{Z} &= \epsilon_{31}X + \epsilon_{32}Y + \epsilon_{33}Z, \end{aligned} \qquad \text{(D}'\text{)} \quad \begin{aligned} \mathfrak{L} &= \mu_{11}L + \mu_{12}M + \mu_{13}N, \\ \mathfrak{M} &= \mu_{21}L + \mu_{22}M + \mu_{23}N, \\ \mathfrak{N} &= \mu_{31}L + \mu_{32}M + \mu_{33}N, \end{aligned}$$

where

$$\epsilon_{rs} = \epsilon_{sr}; \quad \mu_{rs} = \mu_{sr}.$$

The inductions thus defined have all the properties that we have hitherto predicated with regard to them. It has been pointed out by Pupin* that these are not the only possible generalizations of the equations (C) and (D).

Media which are not isotropic are called *eolotropic*. A body may also be eolotropic with respect to conduction, in which case

$$\text{(E}'\text{)} \quad \begin{aligned} u &= \lambda_{11}X + \lambda_{12}Y + \lambda_{13}Z, \\ v &= \lambda_{21}X + \lambda_{22}Y + \lambda_{23}Z, \\ w &= \lambda_{31}X + \lambda_{32}Y + \lambda_{33}Z. \end{aligned}$$

* Pupin. "Studies in the Electro-magnetic Theory." *American Journal of Science*, Vol. L., p. 326, 1895.

We shall in the future, as we have done in the past, consider only isotropic bodies.

247. Consequences of the Equations of the Field. Propagation. If we differentiate the equations (A) respectively by x, y, z and add, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{X}}{\partial x}+\frac{\partial \mathfrak{Y}}{\partial y}+\frac{\partial \mathfrak{Z}}{\partial z}\right)+4\pi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0,$$

the consequences of which we have discussed in Chapter X. If the medium is an insulator, the relaxation-time is infinite, and

$$\frac{\partial \mathfrak{X}}{\partial x}+\frac{\partial \mathfrak{Y}}{\partial y}+\frac{\partial \mathfrak{Z}}{\partial z}$$

is independent of the time.

Applying the same process to the equations (B), we obtain

$$\frac{\partial \mathfrak{L}}{\partial x}+\frac{\partial \mathfrak{M}}{\partial y}+\frac{\partial \mathfrak{N}}{\partial z}$$

independent of the time, and the value of this divergence is zero, except in intrinsic magnets (§ 201).

We shall now deduce the more important consequences of the equations, proceeding from the simpler to the more complicated cases. We shall first, therefore, consider the phenomena in insulators, in which the equations (A) and (B) are exactly symmetrical. On account of the dual nature of the relations of the two fields it follows at once that every effect of electrodynamic induction in producing electromotive forces has an analogous effect in the production of magnetomotive forces by electric displacement currents. For instance a closed iron ring placed in an electrostatic field varying with the time would become magnetized. Effects of this sort have not yet been observed, on account of the extreme smallness of the factor A, by which the displacement current is multiplied. For the same reason, electrostatic forces produced in insulators by the variation of magnetic fields have not been successfully observed, although the attempt has been made by Lodge*. The justification of the equations (A) has been given by other results.

* Lodge. "On an Electrostatic Field produced by varying Magnetic Induction." *Phil. Mag.* (5) 27, p. 469, 1889.

If we perform upon the equations (13) the operation of curl, which is typified by the result of differentiating the third of equations (B) by y and subtracting it from the second differentiated by z we obtain, after adding and subtracting $\frac{\partial^2 X}{\partial x^2}$,

$$(1) \qquad A\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right) = \Delta X - \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right).$$

Now supposing the medium to be homogeneous, that is ϵ and μ constant, making use of the equations (C) and (D), and supposing there is originally no electrification, we have

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

and making use of the first of equations (A) we transform (1) into

$$A^2\mu\epsilon\frac{\partial^2 X}{\partial t^2} = \Delta X.$$

Proceeding in like manner we obtain for the other components,

$$A^2\mu\epsilon\frac{\partial^2 Y}{\partial t^2} = \Delta Y, \quad A^2\mu\epsilon\frac{\partial^2 Z}{\partial t^2} = \Delta Z,$$

$$(2) \qquad A^2\mu\epsilon\frac{\partial^2 L}{\partial t^2} = \Delta L, \quad A^2\mu\epsilon\frac{\partial^2 M}{\partial t^2} = \Delta M,$$

$$A^2\mu\epsilon\frac{\partial^2 N}{dt^2} = \Delta N.$$

We thus find that in insulators each component of the two fields satisfies a differential equation of the form

$$(3) \qquad \frac{\partial^2 \phi}{\partial t^2} = a^2\Delta\phi,$$

where $$a = 1/A\sqrt{\mu\epsilon}.$$

Since this is an equation of great importance in mathematical physics, we shall investigate its general solution. Let us multiply both sides of the equation by the element of volume $d\tau$ and integrate throughout the volume bounded by a closed surface S, applying the divergence theorem to the right-hand member,

$$(4) \qquad \iiint\frac{\partial^2 \phi}{\partial t}\,d\tau = a^2\iiint\Delta\phi\,d\tau = -a^2\iint\frac{\partial \phi}{\partial n}\,dS.$$

If the surface S is a sphere of radius r with its center at the point P, we have

$$(5)\qquad -\iint\frac{\partial\phi}{\partial n}\,dS=\iint\frac{\partial\phi}{\partial r}\,r^2d\omega=r^2\frac{\partial}{\partial r}\iint\phi_r d\omega,$$

where by ϕ_r we denote the values of ϕ at points on the surface of the sphere of radius r, with center P.

Introducing polar coordinates into the left-hand side of equation (4) also we may write it

$$(6)\qquad \frac{\partial^2}{\partial t^2}\iiint\phi d\tau=\frac{\partial^2}{\partial t^2}\iint d\omega\left(\int_0^r\phi_r r^2dr\right).$$

Now differentiating this and the transformed right-hand member (5) by the upper limit r changes our equation (3) into

$$(7)\qquad r^2\frac{\partial^2}{\partial t^2}\iint\phi_r d\omega=a^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\iint\phi_r d\omega\right).$$

The surface integral

$$\iint\phi_r d\omega,$$

which appears on both sides is 4π times the mean value of the function ϕ on the surface of the sphere of radius r. Calling this mean value $\bar{\phi}_r$ we have the equation

$$(8)\qquad r^2\frac{\partial^2\bar{\phi}_r}{\partial t^2}=a^2\frac{\partial}{\partial r}\left(r^2\frac{\partial\bar{\phi}_r}{\partial r}\right),$$

which, on performing the differentiations and dividing by r, may be written

$$(9)\qquad \frac{\partial^2(r\bar{\phi}_r)}{\partial t^2}=a^2\frac{\partial^2(r\bar{\phi}_r)}{\partial r^2}.$$

If we now introduce two new independent variables

$$u=at+r,\quad v=at-r,$$

we have, putting $r\bar{\phi}_r=\psi$,

$$\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\psi}{\partial v}\frac{\partial v}{\partial t}=a\left(\frac{\partial\psi}{\partial u}+\frac{\partial\psi}{\partial v}\right),$$

$$\frac{\partial\psi}{\partial r}=\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial r}+\frac{\partial\psi}{\partial v}\frac{\partial v}{\partial r}=\frac{\partial\psi}{\partial u}-\frac{\partial\psi}{\partial v},$$

$$\frac{\partial^2\psi}{\partial t^2}=a^2\left(\frac{\partial^2\psi}{\partial u^2}+2\frac{\partial^2\psi}{\partial u\partial v}+\frac{\partial^2\psi}{\partial v^2}\right),$$

$$\frac{\partial^2\psi}{\partial r^2}=\frac{\partial^2\psi}{\partial u^2}-2\frac{\partial^2\psi}{\partial u\partial v}+\frac{\partial^2\psi}{\partial v^2},$$

so that our equation (9) becomes

$$\frac{\partial^2 \psi}{\partial u \partial v} = 0. \tag{10}$$

Of this equation the general solution is

$$\psi = g_1(u) + g_2(v), \tag{11}$$

where g_1 and g_2 are perfectly arbitrary functions of their arguments. We consequently have for the solution of (9),

$$r\bar{\phi}_r = g_1(at + r) + g_2(at - r). \tag{12}$$

When $r = 0$ we have

$$0 = g_1(at) + g_2(at),$$

and this being true for all values of t the functions g_1, g_2 are not independent, but one is the negative of the other, whatever the value of the argument. Putting then

$$g_1 = g, \quad g_2 = -g,$$

we have

$$r\bar{\phi}_r = g(at + r) - g(at - r). \tag{13}$$

Differentiating by r,

$$\bar{\phi}_r + r\frac{\partial \bar{\phi}_r}{\partial r} = g'(at + r) + g'(at - r), \tag{14}$$

and again putting $r = 0$, we obtain

$$\bar{\phi}_{r=0} = 2g'(at). \tag{15}$$

But $\bar{\phi}_r$ is the mean value of ϕ over the surface of a sphere of radius r with center at P, and the mean value over a sphere of radius zero is the value at P itself. Accordingly

$$\phi_P = 2g'(at). \tag{16}$$

Now differentiating (13) by r and t,

$$\frac{\partial}{\partial r}(r\bar{\phi}_r) = g'(at + r) + g'(at - r),$$

$$\frac{\partial}{\partial t}(r\bar{\phi}_r) = a\left\{g'(at + r) - g'(at - r)\right\},$$

so that

$$\frac{\partial(r\bar{\phi}_r)}{\partial r} + \frac{1}{a}\frac{\partial(r\bar{\phi}_r)}{\partial t} = 2g'(at + r), \tag{17}$$

and for $t = 0$,

$$\left[\frac{\partial(r\bar{\phi}_r)}{\partial r} + \frac{1}{a}\frac{\partial(r\bar{\phi}_r)}{\partial t}\right]_{t=0} = 2g'(r). \tag{18}$$

Now inserting the value of $\bar{\phi}_r$,

$$(19)\qquad 2g'(r)=\left[\frac{\partial}{\partial r}\left(\frac{r}{4\pi}\iint\phi_r d\omega\right)+\frac{1}{a}\left(\frac{r}{4\pi}\iint\frac{\partial\phi_r}{\partial t}\,d\omega\right)\right]_{t=0}.$$

Suppose that for a certain initial instant, for which we shall take $t=0$, the values of the function ϕ and of its time derivative $\frac{\partial\phi}{\partial t}$ are given as functions of a point in space,

$$(20)\qquad [\phi]_{t=0}=F(x,y,z),\quad \left[\frac{\partial\phi}{\partial t}\right]_{t=0}=f(x,y,z).$$

Inserting in the equation (19) it becomes

$$(21)\qquad 2g'(r)=\frac{\partial}{\partial r}\left(\frac{r}{4\pi}\iint F_r d\omega\right)+\frac{r}{4\pi a}\iint f_r d\omega,$$

but when $r=at$ the value of the left-hand side is by (16) equal to ϕ_P.

Accordingly we have finally,

$$(22)\qquad \phi_P=\frac{1}{4\pi}\left[\frac{\partial}{\partial(at)}\left(at\iint F_{at}d\omega\right)+t\iint f_{at}d\omega\right].$$

This solution was given by Poisson*. It shows that the value of ϕ at all times may be calculated for every point P if we know the mean value of $\partial\phi/\partial t$ at a time earlier by the interval at for all points on the surface of a sphere of radius at about P, as well as the rate of variation of the mean value of ϕ as the radius of the sphere is altered. Suppose that initially ϕ and $\frac{\partial\phi}{\partial t}$ are both zero except for a certain region whose nearest point lies at a distance r_1 from P and whose farthest at a distance r_2. Then as long as $t<r_1/a$ the mean value of ϕ on the sphere of radius at is zero, and after $t>r_2/a$ as well. Accordingly there is no disturbance except between the times r_1/a and r_2/a, or the quantity ϕ is *propagated in all directions with the velocity a*. It may be easily shown that ϕ_P is finite if F and f are finite everywhere.

We might have obtained the same result in a more simple manner by transforming $\Delta\phi$ to polar coordinates in equation (3), making ϕ independent of the angular coordinates, when the equation becomes

$$(23)\qquad \frac{\partial^2\phi}{\partial t^2}=a^2\left(\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}\right),$$

* *Nouveaux Mémoires de l'Académie des Sciences*, t. III.

or
$$\frac{\partial^2 (r\phi)}{\partial t^2} = a^2 \frac{\partial^2 (r\phi)}{\partial r^2}.$$

Of this a solution is

(24)
$$\phi = \frac{g(at - r)}{r},$$

as has been shown (12). Accordingly for all points and times for which $at - r$ has the same particular value we have the same value of $r\phi$, or a particular value of ϕ travels outwards with the velocity a. The value of ϕ is inversely proportional to the distance r traversed. The solution makes the value of ϕ infinite at the point from which it is propagated. This is only an apparent difficulty, for just as the potential due to a single mass-point is infinite at the point, but is never infinite when the mass is continuously distributed with finite density, so here if we consider a finite region in which ϕ is not zero, the infinite value will not occur, as is shown by our general solution (22).

We see, accordingly, that a state of electric or magnetic field existing in any region of space has its action propagated with the finite velocity $a = 1/A\sqrt{\mu\epsilon}$ in all directions, and inasmuch as by the equations (A), (B), the time-variation of one field is proportional to the curl of the other, the second term of (22) shows that a curl of one field in any part of space causes a propagation of a field of the other kind.

The conclusion that electrical and magnetic actions are propagated with a finite velocity is the great and remarkable consequence of Maxwell's theory, and was enunciated by him in 1864 in his celebrated paper on the Dynamical Theory of the Electromagnetic Field. From this, and the other consequences of the equations, he was led to enunciate the theory that light was an electromagnetic phenomenon. In fact, the equation (3) is, as we have shown, the equation of wave motion, and is the basis of any undulatory theory, whether of light or of sound. The manner in which the equations give us a theory suitable for light and not for sound will be discussed in § 249.

Since the velocity of propagation is $1/A\sqrt{\epsilon\mu}$, in air the velocity should be $1/A = \mathbf{v}$, or the velocity which corresponds to the ratio of the two electrical units of quantity. Determinations of this purely electrical quantity, as refinements in measurement increased, gave results showing a surprising agreement with the

determinations of the velocity of light, so that many German authors are accustomed to speak of A as the reciprocal of the velocity of light. It seems preferable, however, to keep the definition of A and $\mathbf{v}$ purely electrical, as we have given it in § 212.

A further confirmation of the electromagnetic theory of light was sought in the fact that the index of refraction, being inversely proportional to the velocity, should in non-magnetic bodies, for which $\mu = 1$, be proportional to the square root of the electric inductivity. This relation was experimentally verified for a sufficient number of transparent dielectrics to make it appear that the agreement was not accidental, although many exceptions were found.

Nevertheless, although these considerations made the electromagnetic nature of light very probable, the theory of propagation of actual electrical disturbances with finite velocity remained unverified by experiment until 1887, when Hertz began the publication of his remarkable researches*, which have since carried conviction of the truth of Maxwell's theory of electricity and magnetism to the most conservative parts of the scientific world. For an account of them the reader is referred to Hertz's collected papers on "Die Ausbreitung der elektrischen Kraft," or to the English translation by D. E. Jones.

248. Transfer of Energy. Poynting's Theorem. We shall now form the equation of activity for any portion τ of the field. If $E = W + T$ be the total energy, H the dissipativity, we have

$$(1) \quad \frac{\partial E}{\partial t} + H = \frac{1}{8\pi}\frac{\partial}{\partial t}\iiint(\mathfrak{X}X + \mathfrak{Y}Y + \mathfrak{Z}Z + \mathfrak{L}L + \mathfrak{M}M + \mathfrak{N}N)\, d\tau$$
$$+ \iiint (Xu + Yv + Zw)\, d\tau.$$

Since ϵ and μ do not vary with the time we have, by (C), (D),

$$\frac{\partial}{\partial t}(\mathfrak{X}X) = \mathfrak{X}\frac{\partial X}{\partial t} + X\frac{\partial \mathfrak{X}}{\partial t} = 2X\frac{\partial \mathfrak{X}}{\partial t},$$
$$\frac{\partial}{\partial t}(\mathfrak{L}L) = \mathfrak{L}\frac{\partial L}{\partial t} + L\frac{\partial \mathfrak{L}}{\partial t} = 2L\frac{\partial \mathfrak{L}}{\partial t}.$$

* "Ueber die Ausbreitungsgeschwindigkeit der elektrodynamischen Wirkungen." *Wied. Ann.* 34, p. 551, 1888, trans. p. 107.

Inserting these and the corresponding values in the integrand, and replacing the time derivatives by the curl-components from equations (A) and (B), we have

$$(2)\quad \frac{\partial E}{\partial t}+H$$

$$=\frac{1}{4\pi A}\iiint\left[X\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}\right)+Y\left(\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}\right)+Z\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)\right.$$
$$\left.-\left\{L\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)+M\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)+N\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)\right\}\right]d\tau.$$

Now integrating by the theorem of § 226 (4), this becomes

$$(3)\quad \frac{\partial E}{\partial t}+H=\frac{1}{4\pi A}\iint\{(YN-ZM)\cos(nx)+(ZL-XN)\cos(ny)$$
$$+(XM-YL)\cos(nz)\}\,dS.$$

We have supposed that there are no intrinsic electromotive forces, but if there are, the components X, Y, Z in (1) must be replaced by $X-X'$, $Y-Y'$, $Z-Z'$ (§ 243), except in the last integral representing the dissipativity, consequently that integral will not be entirely cancelled as above, but there will remain the term

$$(4)\qquad \iiint(X'u+Y'v+Z'w)\,d\tau=\mathfrak{A},$$

representing the activity of the impressed forces, in addition to the surface-integral. If we extend the integral in (3) to a space to whose boundaries electric and magnetic actions do not extend, since the integrand in the surface-integral vanishes we have

$$(5)\qquad \frac{\partial E}{\partial t}+H=\mathfrak{A}$$

as the equation of activity (cf. § 64 (6)), whose H is the present E, while H of (5) is the $2F$ of § 64, (8).)

If the fields are not zero at the surface S, the equation (3) shows that the energy in the volume will be accounted for by supposing that a quantity of energy

$$\{(YN-ZM)\cos(nx)+(ZL-XN)\cos(ny)+(XM-YL)\cos(nz)\}/4\pi A$$

per unit of surface S enters the volume τ in unit time. We may therefore call the vector

$$\overline{R}=\mathbf{V}FH/4\pi A,$$

whose components are

$$R_x = (YN - ZM)/4\pi A,$$
$$R_y = (ZL - XN)/4\pi A,$$
$$R_z = (XM - YL)/4\pi A,$$

the *energy-current-density*.

The equation (3) accordingly states that the quantity of energy R is transferred per unit of time across the unit of surface tangent to the direction of both the electric and the magnetic force. This is Poynting's* remarkable theorem.

It has been remarked by J. J. Thomson, Heaviside and Hertz that this determination of the energy current is not the only possible one, since we may add to the above vector any solenoidal vector without changing the surface-integral in (3). Hertz has also pointed out that, as this makes energy flow at all points where fields of both kinds exist, it involves the continuous flow of energy (in closed tubes) when a magnetic pole and an electrified point exist in each other's presence. In many cases, however, the notion of the motion of the energy here given is a very fruitful one. It has been further developed in several papers by Wien†. The vector R is sometimes called the *radiant vector*.

249. Plane Waves. Let us again consider a perfect insulator. The equations § 247 (2) are all satisfied by any function of the argument $s = lx + my + nz - at$, where $a = 1/A\sqrt{\epsilon\mu}$.

(1) $$\phi = \phi\,(lx + my + nz - at).$$

For we have

(2) $$\frac{\partial\phi}{\partial x} = l\phi'(s),\quad \frac{\partial\phi}{\partial y} = m\phi'(s),\quad \frac{\partial\phi}{\partial z} = n\phi'(s),\quad \frac{\partial\phi}{\partial t} = -a\phi'(s),$$
$$\frac{\partial^2\phi}{\partial x^2} = l^2\phi''(s),\ \frac{\partial^2\phi}{\partial y^2} = m^2\phi''(s),\ \frac{\partial^2\phi}{\partial z^2} = n^2\phi''(s),\ \frac{\partial^2\phi}{\partial t^2} = a^2\phi''(s),$$

$$\Delta\phi = (l^2 + m^2 + n^2)\,\phi''(s),$$

and therefore

(3) $$\frac{\partial^2\phi}{\partial t^2} = a^2\Delta\phi,$$

* Poynting, *Phil. Trans.* 2, p. 343, 1884.

† Wien. "Ueber den Begriff der Localisirung der Energie." *Wied. Ann.* 45, p. 685, 1892. "Ueber die Bewegung der Kraftlinien im electromagnetischen Felde." *Wied. Ann.* 46, p. 352, 1892.

if we take

$$(4) \qquad l^2+m^2+n^2=1.$$

The quantities X, Y, Z, L, M, N accordingly have the same set of values for all points for which

$$(5) \qquad lx+my+nz-at=\text{const.}$$

But this is the equation of a plane whose normal has the direction cosines l, m, n, and whose distance from the origin is $at+\text{const.}$ The plane is accordingly travelling in the direction of its normal with the velocity $a=1/A\sqrt{\mu\epsilon}$. The disturbance is accordingly a plane electromagnetic wave, whatever the nature of the function ϕ.

The six functions ϕ are not independent. For let

$$(6) \qquad X=\phi_1,\ Y=\phi_2,\ Z=\phi_3,\ L=\psi_1,\ M=\psi_2,\ N=\psi_3.$$

Then inserting these in the equations (A) and (B) we have

$$(7) \qquad \begin{aligned} -\sqrt{\frac{\epsilon}{\mu}}\,\phi_1' &= m\psi_3'-n\psi_2', \\ -\sqrt{\frac{\epsilon}{\mu}}\,\phi_2' &= n\psi_1'-l\psi_3', \\ -\sqrt{\frac{\epsilon}{\mu}}\,\phi_3' &= l\psi_2'-m\psi_1', \end{aligned} \qquad (8) \qquad \begin{aligned} \sqrt{\frac{\mu}{\epsilon}}\,\psi_1' &= m\phi_3'-n\phi_2', \\ \sqrt{\frac{\mu}{\epsilon}}\,\psi_2' &= n\phi_1'-l\phi_3', \\ \sqrt{\frac{\mu}{\epsilon}}\,\psi_3' &= l\phi_2'-m\phi_1', \end{aligned}$$

a system of linear equations to determine the ratios of ϕ''s and ψ''s.

Multiplying the equations (7) or (8) in order by l, m, n, and adding either set, we obtain

$$(9) \qquad l\phi_1'+m\phi_2'+n\phi_3'=0,\ l\psi_1'+m\psi_2'+n\psi_3'=0.$$

These are two differential equations with regard to the variable s, integrating which gives

$$l\phi_1+m\phi_2+n\phi_3=C_1,\ l\psi_1+m\psi_2+n\psi_3=C_2,$$

that is

$$(10) \qquad lX+mY+nZ=C_1,\ lL+mM+nN=C_2.$$

This shows that the component of either field resolved parallel to the direction of propagation is constant as we travel in that direction as well as in the plane of the wave, and is therefore

constant throughout space. But such a constant field is not propagated at all, and we shall therefore disregard it, and put both constants equal to zero. Both fields are consequently perpendicular to the direction of propagation. It is for this reason that Maxwell's theory is appropriate for an explanation of light, which, as the phenomena of polarization show, is due to *transverse* undulations.

Although the forces of the two fields lie in the wave-plane, and are constant over any particular wave-plane, it does not follow that their directions are the same in all wave-planes, that is for different values of s. We shall however assume that their *directions* are the same in all wave-planes, and we will call the direction cosines of F, α_1, β_1, γ_1, and of H, α_2, β_2, γ_2. Such a wave is said to be plane-polarized. Then we have

$$(11)\quad X=\alpha_1\phi,\ Y=\beta_1\phi,\ Z=\gamma_1\phi,\ L=\alpha_2\psi,\ M=\beta_2\psi,\ N=\gamma_2\psi,$$

and our equations (7) and (8) take the form

$$(12)\quad \begin{aligned} -\sqrt{\frac{\epsilon}{\mu}}\,\alpha_1\phi' &= (m\gamma_2-n\beta_2)\,\psi',\\ -\sqrt{\frac{\epsilon}{\mu}}\,\beta_1\phi' &= (n\alpha_2-l\gamma_2)\,\psi',\\ -\sqrt{\frac{\epsilon}{\mu}}\,\gamma_1\phi' &= (l\beta_2-m\alpha_2)\,\psi'. \end{aligned} \qquad (13)\quad \begin{aligned} \sqrt{\frac{\mu}{\epsilon}}\,\alpha_2\psi' &= (m\gamma_1-n\beta_1)\,\phi',\\ \sqrt{\frac{\mu}{\epsilon}}\,\beta_2\psi' &= (n\alpha_1-l\gamma_1)\,\phi',\\ \sqrt{\frac{\mu}{\epsilon}}\,\gamma_2\psi' &= (l\beta_1-m\alpha_1)\,\phi'. \end{aligned}$$

Multiplying the equations of the first set respectively by α_2, β_2, γ_2 and adding, we get

$$(14)\qquad \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0,$$

or the electric and magnetic forces are mutually perpendicular, as well as perpendicular to the direction of propagation. There are accordingly two directions, either of which might be chosen to define the plane of polarization, and it rests with experiment to decide between them.

Squaring and adding either equations (12) or equations (13) gives

$$\epsilon\phi'^2=\mu\psi'^2.$$

Extracting the square root and integrating,

$$\sqrt{\epsilon}\,\phi=\sqrt{\mu}\,\psi=f(s),$$

putting the integration constant equal to zero for the same reason as before. Accordingly the fields are

$$X = \frac{\alpha_1 f(s)}{\sqrt{\epsilon}}, \quad Y = \frac{\beta_1 f(s)}{\sqrt{\epsilon}}, \quad Z = \frac{\gamma_1 f(s)}{\sqrt{\epsilon}},$$

$$L = \frac{\alpha_2 f(s)}{\sqrt{\mu}}, \quad M = \frac{\beta_2 f(s)}{\sqrt{\mu}}, \quad N = \frac{\gamma_2 f(s)}{\sqrt{\mu}}.$$

The two fields are accordingly propagated together.

Comparing the energies of unit volume we find

$$W = \frac{\epsilon}{8\pi}(X^2 + Y^2 + Z^2) = \frac{(f(s))^2}{8\pi} = \frac{\mu}{8\pi}(L^2 + M^2 + N^2) = T,$$

or the energy is equally divided between the two fields.

The radiant vector $\mathbf{V}FH$ is of course in the direction of propagation.

250. Propagation in a Conductor. In § 247 in deducing the equations of propagation we have supposed the conductivity to be zero. If we do not make this assumption in substituting the value of the curl-components on the left of equation (1) we obtain

(1) $$A^2\mu\left(\epsilon\frac{\partial^2 X}{\partial t^2} + 4\pi\lambda\frac{\partial X}{\partial t}\right) = \Delta X,$$

and in like manner all the components of both fields satisfy the equation

(2) $$A^2\left(\mu\epsilon\frac{\partial^2 \phi}{\partial t^2} + 4\pi\lambda\mu\frac{\partial \phi}{\partial t}\right) = \Delta\phi.$$

The general solution of this equation has been given by Boussinesq, but it is too complicated to be given here. A particularly interesting special case will be treated below. In the mean time we shall content ourselves with the consideration of disturbances which are harmonic functions of the time. For this purpose we shall assume

(3) $$\phi = e^{i\omega t}\, U(x, y, z),$$

when our equation becomes

(4) $$A^2(-\mu\epsilon\omega^2 + 4\pi\lambda\mu\omega i)\, U = \Delta U.$$

The equation

(5) $$\Delta U = k^2 U$$

has been made the subject of a treatise by Pockels. We shall here consider only the case in which U depends on a single rectangular coordinate x, the circumstances being the same all over each plane perpendicular to the X-axis. In a conducting dielectric the value of k^2 is complex. In metals we know nothing regarding the value of the electric inductivity ϵ, for whereas electrostatic phenomena may be explained by supposing it to be infinite, in variable states this is far from being the case. In fact in all experiments that have been performed with electric waves thus far the value of ω has not been great enough to make the influence of the term containing ϵ appreciable in comparison with that containing λ. (See § 206.) We shall therefore neglect the first term, so that our equation of propagation is

$$(6) \qquad 4\pi A^2\lambda\mu\frac{\partial\phi}{\partial t} = \Delta\phi.$$

This is the equation for the conduction of heat, as given by Fourier. We shall consider it in some detail below (§ 254), but shall now return to the consideration of the equation

$$(7) \qquad \frac{\partial^2 U}{\partial x^2} = k^2 U,$$

in which k^2 is the pure imaginary

$$(8) \qquad k^2 = 4\pi A^2\lambda\mu\omega i.$$

The solution of equation (7) is

$$(9) \qquad U = C_1 e^{kx} + C_2 e^{-kx}.$$

Since we have $\sqrt{i} = (1+i)/\sqrt{2}$ the value of k is

$$(10) \qquad k = \pm A\sqrt{2\pi\lambda\mu\omega}\,.(1+i).$$

Accordingly the real and imaginary parts of

$$e^{i(\omega t \pm A\sqrt{2\pi\lambda\mu\omega}\,.\,x)}\; e^{\pm A\sqrt{2\pi\lambda\mu\omega}\,.\,x}$$

furnish us with particular solutions of (6). We thus obtain

$$(11) \qquad \begin{aligned}
\phi &= e^{-A\sqrt{2\pi\lambda\mu\omega}\,.\,x}\cos(\omega t - A\sqrt{2\pi\lambda\mu\omega}\,.\,x),\\
\phi &= e^{-A\sqrt{2\pi\lambda\mu\omega}\,.\,x}\sin(\omega t - A\sqrt{2\pi\lambda\mu\omega}\,.\,x),\\
\phi &= e^{A\sqrt{2\pi\lambda\mu\omega}\,.\,x}\cos(\omega t + A\sqrt{2\pi\lambda\mu\omega}\,.\,x),\\
\phi &= e^{A\sqrt{2\pi\lambda\mu\omega}\,.\,x}\sin(\omega t + A\sqrt{2\pi\lambda\mu\omega}\,.\,x).
\end{aligned}$$

Of these the first two, being of the form $e^{-\frac{x}{d}}f(x-at)$, represent damped waves travelling in the direction of increasing x with the velocity $a=\frac{1}{A}\sqrt{\frac{\omega}{2\pi\lambda\mu}}$. The periodic factor at any time repeats its values when x is increased by the distance $l=\frac{1}{A}\sqrt{\frac{2\pi}{\lambda\mu\omega}}$, which is called the *wave-length,* the frequency being $n=\omega/2\pi$. The damping factor $e^{-A\sqrt{2\pi\lambda\mu\omega}.x}$ which causes the amplitude of the wave to decrease in geometrical progression as the distance travelled increases in arithmetical, has the relaxation-distance, or the distance in which the amplitude diminishes in the ratio $1/e$,

$$d=\frac{1}{A\sqrt{2\pi\lambda\mu\omega}}.$$

The last two particular solutions represent waves travelling in the opposite direction with the same velocity and damping.

Since the velocity depends on the frequency, there is no definite velocity of propagation in a conductor. On account of the damping, harmonic disturbances of high frequency rapidly die out, consequently alternating fields penetrate but a short distance into conductors. This was shown by Maxwell*, but its importance was brought out by the researches of Heaviside†, Lord Rayleigh‡ and Hertz§.

We shall now consider the relations between the two fields. If the components X, Y, Z, L, M, N are equal respectively to the complex constants $A_1, A_2, A_3, B_1, B_2, B_3$, each multiplied by $e^{i\omega t+kx}$, inserting in the equations (A) and (B) we obtain

$$\begin{aligned} &4\pi A\lambda A_1=0, && -A\mu i\omega B_1=0,\\ (12)\qquad &4\pi A\lambda A_2=-kB_3, && -A\mu i\omega B_2=-kA_3,\\ &4\pi A\lambda A_3=kB_2, && -A\mu i\omega B_3=kA_2. \end{aligned}$$

Eliminating A_2/B_3 or A_3/B_2 we obtain the value for k already found. We thus see that the directional relations of the fields

* *Treatise*, Art. 689.

† "The Induction of Currents in Cores." *Electrician*, 1884. Papers, Vol. I., p. 353.

‡ "On the Self-induction and Resistance of Straight Conductors." *Phil. Mag.* 21, p. 381, 1886.

§ "Ueber die Fortleitung elektrischer Wellen durch Drähte." *Wied. Ann.* 37, p. 395, 1889. Jones's trans. p. 160.

and the direction of propagation are the same as in insulators, but the ratio of the two fields is

(13) $$\frac{H}{F} = \frac{4\pi A\lambda}{k} = \sqrt{\frac{2\pi\lambda}{\mu\omega}}(1 - i).$$

The magnetic field accordingly lags in phase by one-eighth of a period behind the electric, while in an insulator the fields have the same phase.

251. Reflection of Waves by a Conductor. We shall now consider the effect of a train of plane waves in an insulator striking the plane surface of a conductor which is parallel to their plane. We shall suppose the conductor to extend to infinity in one direction. Let us take the plane $x = 0$ as the face of the conductor. Let the waves be harmonic in the insulator, for which $x < 0$, and let the electric force be parallel to the Y-axis, the magnetic to the Z-axis. Then in the wave approaching the conductor we have

(1) $$Y = \frac{C_1}{\sqrt{\epsilon}} e^{i\omega\left(t - \frac{x}{a}\right)}, \qquad N = \frac{C_1}{\sqrt{\mu}} e^{i\omega\left(t - \frac{x}{a}\right)}.$$

In the conductor we have

(2) $$Y = C_2 e^{i\omega t - kx}, \qquad N = C_2\sqrt{\frac{2\pi\lambda}{\mu_1\omega}}(1 - i)\, e^{i\omega t - kx}.$$

At the plane $x = 0$ the boundary conditions to be satisfied are that the tangential components of both forces and the normal components of both inductions are continuous. The latter components being zero we have only the first two conditions to satisfy. There are not enough constants to enable us to satisfy them both, it is accordingly necessary to add to the terms representing the disturbance in the insulator other terms representing a wave travelling in the opposite direction, or a *reflected* wave. We therefore take

(3) $$Y = \frac{C_1}{\sqrt{\epsilon}} e^{i\omega\left(t - \frac{x}{a}\right)} + \frac{C_1'}{\sqrt{\epsilon}} e^{i\omega\left(t + \frac{x}{a}\right)}, \qquad N = \frac{C_1}{\sqrt{\mu}} e^{i\omega\left(t - \frac{x}{a}\right)} - \frac{C_1'}{\sqrt{\mu}} e^{i\omega\left(t + \frac{x}{a}\right)},$$

in the insulator. Our boundary conditions are then, dividing by $e^{i\omega t}$,

$$
(4)\qquad
\begin{aligned}
\frac{C_1 + C_1'}{\sqrt{\epsilon}} &= C_2,\\
\frac{C_1 - C_1'}{\sqrt{\mu}} &= C_2\sqrt{\frac{2\pi\lambda}{\mu_1\omega}}(1-i).
\end{aligned}
$$

Accordingly we have

$$
(5)\qquad
\begin{aligned}
\frac{C_1'}{C_1} &= \frac{1-\sqrt{\dfrac{2\pi\lambda\mu}{\omega\epsilon\mu_1}}(1-i)}{1+\sqrt{\dfrac{2\pi\lambda\mu}{\omega\epsilon\mu_1}}(1-i)},\\
\frac{C_2}{C_1} &= \frac{2}{\sqrt{\epsilon}+\sqrt{\dfrac{2\pi\lambda\mu}{\omega\mu_1}}(1-i)}.
\end{aligned}
$$

Since these two ratios are complex, at the surface of the conductor there is a difference of phase between the direct, reflected and transmitted waves. As we increase the conductivity of the conductor the ratio C_1'/C_1 approaches the value -1, in which case the electric force vanishes at the boundary, which is a *node* for the electric field, while the magnetic field is a maximum. On the other hand, as we increase the frequency of the oscillation or the magnetic inductivity of the conductor the ratio C_1'/C_1 approaches $+1$, or the magnetic force vanishes at the boundary, while the electric is a maximum. If we put

$$
r = \sqrt{\frac{2\pi\lambda\mu}{\omega\epsilon\mu_1}}
$$

we find for the ratios of the amplitudes of the reflected and transmitted to that of the direct waves

$$
\begin{aligned}
\left|\frac{C_1'}{C_1}\right| &= \left\{\frac{(1-r)^2+r^2}{(1+r)^2+r^2}\right\}^{\frac{1}{2}},\\
\left|\frac{C_2}{C_1}\right| &= \frac{2}{\sqrt{\epsilon}\,[(1+r)^2+r^2]^{\frac{1}{2}}}.
\end{aligned}
$$

Accordingly whether r be very great or very small the reflected waves tend to become as great as the direct, while the larger r the less is the magnitude of the transmitted waves.

The experiments of Hertz* confirmed the above results, the boundary of the conductor being more nearly a node for the electric than for the magnetic force. If the amplitude of the reflected waves approaches that of the direct ones, the two systems will interfere and produce a set of standing waves, with nodes at regular distances from the conductor. This field was explored by Hertz by means of a resonator, composed of a circle of wire with its ends terminating in two small balls near together, constituting a condenser. This system has a certain period of its own, and what has been said in § 240 applies to it. It was tuned to the period of the waves, and being placed anywhere in the field would have currents induced in it by the harmonic electromotive forces of the field.

Thus where the force is a maximum sparks appear between the balls of the resonator, disappearing at the nodes. For the further theory of the resonator the reader is referred to Poincaré, *Les Oscillations Electriques*, J. J. Thomson, *Recent Researches in Electricity and Magnetism*, and Drude, *Physik des Aethers*.

252. Spherical Oscillator. We have hitherto considered waves in insulators, without considering how they are produced. In the experiments of Hertz the waves were produced by disturbing the charges in a peculiar dumb-bell-shaped conductor, and allowing oscillations to set in, which were propagated outward through the air. A satisfactory theory of the oscillations in Hertz's oscillator has not been given. We may state the general problem: given a charge disposed in any manner not in equilibrium upon a conductor of any form, find the nature of the oscillations that ensue while the conductor is settling down to its state of equilibrium, together with the fields to which they give rise. This problem is a very complicated one, and has been solved for very few cases. We shall consider the case of a conducting sphere, the charge upon which is that induced by placing the sphere in a steady uniform electric field. The field is supposed suddenly destroyed, and the charge then oscillates until equilibrium is attained.

We shall suppose $\epsilon = \mu = 1$.

* "Ueber elektrodynamische Wellen im Luftraume und deren Reflexion." *Wied. Ann.* 34, p. 609, 1888.

Since both fields are propagated according to the equation

$$\frac{\partial^2\phi}{\partial t^2} = a^2\Delta\phi,$$

and since any derivative of a solution is also a solution, we may take

$$\text{(1)}\qquad \begin{aligned} X &= \frac{\partial^2\phi}{\partial x\partial z}, & L &= A\frac{\partial^2\phi}{\partial y\partial t}, \\ Y &= \frac{\partial^2\phi}{\partial y\partial z}, & M &= -A\frac{\partial^2\phi}{\partial x\partial t}, \\ Z &= -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -\Delta\phi + \frac{\partial^2\phi}{\partial z^2}, & N &= 0, \end{aligned}$$

which satisfy the solenoidal condition and the equations (A) and (B). If we assume ϕ to be a function only of r and t, we have for a diverging wave

$$\text{(2)}\qquad \phi = \frac{f(r - at)}{r}.$$

Differentiating ϕ by the coordinates we obtain

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z} = \frac{\partial\phi}{\partial r}\frac{z}{r},$$

$$\text{(3)}\qquad \begin{aligned} \frac{\partial^2\phi}{\partial x\partial z} &= z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{\partial r}{\partial x} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{xz}{r}, \\ \frac{\partial^2\phi}{\partial y\partial z} &= z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{\partial r}{\partial y} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{yz}{r}, \\ \frac{\partial^2\phi}{\partial z^2} &= z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{\partial r}{\partial z} + \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{z^2}{r} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \end{aligned}$$

so that the forces are

$$\text{(4)}\qquad \begin{aligned} X &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{xz}{r}, \\ Y &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{yz}{r}, \\ Z &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\frac{z^2}{r} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \Delta\phi. \end{aligned}$$

The field is thus the resultant of two parts, the first, equal to

$$z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right),$$

parallel to the radius, and the second, equal to

$$\frac{1}{r}\frac{\partial\phi}{\partial r} - \Delta\phi,$$

parallel to the Z-axis. At the surface of the sphere, $r = R$, if the conductivity is large, the lines of force are normal to the surface, so that this second component vanishes, and we have

$$\frac{1}{r}\frac{\partial\phi}{\partial r} = \Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r},$$

that is

(5) $$\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right]_{r=R} = 0.$$

When $t = 0$, the electric forces are derivable from a potential, which is, by (1), equal to $-\frac{\partial\phi}{\partial z}$ (since $\Delta\phi = 0$). But by § 194 (7) the potential is, in the case supposed, proportional to

$$\frac{z}{r^3} = -\frac{\partial}{\partial z}\left(\frac{1}{r}\right).$$

Consequently initially

$$\phi = \frac{C}{r}. \qquad (r > R)$$

Introducing the value of ϕ from (2) this gives

(6) $$f(r) = C.$$

Consequently the function f is constant for all values of its argument greater than R. Hence the value of ϕ,

$$\phi = \frac{f(r - at)}{r},$$

remains equal to C/r so long as $r - at > R$, and the field remains unchanged. When $t = (r - R)/a$, the wave arrives at the distance r, and to determine the field at subsequent instants we must determine the values of f for values of its argument less than R.

Let us make use of equation (5).

Differentiating ϕ by r gives

(7) $$\frac{\partial\phi}{\partial r} = \frac{f'(r - at)}{r} - \frac{f(r - at)}{r^2},$$
$$\frac{\partial^2\phi}{\partial r^2} = \frac{f''(r - at)}{r} - \frac{2f'(r - at)}{r^2} + \frac{2f(r - at)}{r^3},$$

and inserting these in the equation (5) we have

$$\text{(8)} \qquad \frac{f''(R-at)}{R} - \frac{f'(R-at)}{R^2} + \frac{f(R-at)}{R^3} = 0.$$

This is an ordinary differential equation, which, if we put

$$u = R - at,$$

becomes

$$\text{(9)} \qquad \frac{d^2 f}{du^2} - \frac{1}{R}\frac{df}{du} + \frac{f}{R^2} = 0.$$

This has the solution $f = e^{\lambda u}$, where

$$\text{(10)} \qquad \lambda^2 - \frac{1}{R}\lambda + \frac{1}{R^2} = 0, \quad \lambda = \frac{1 \pm i\sqrt{3}}{2R}.$$

From this we obtain the general solution

$$f(u) = e^{\frac{u}{2R}}\left(A \cos \frac{\sqrt{3}}{2R} u + B \sin \frac{\sqrt{3}}{2R} u\right),$$

$$\phi = \frac{e^{\frac{r-at}{2R}}\left(A \cos \frac{\sqrt{3}}{2R}(r-at) + B \sin \frac{\sqrt{3}}{2R}(r-at)\right)}{r},$$

representing a damped harmonic spherical wave of wave-length

$$l = 4\pi R/\sqrt{3} = 7{\cdot}255 R.$$

The logarithmic decrement is

$$\pi/\sqrt{3} = 1{\cdot}814,$$

so that the oscillation almost ceases after a complete vibration. This extreme damping is due to the radiation of the energy, and not at all to the dissipation in the conductor, of which we have taken no account.

The nature of the field radiated by an oscillation of this sort has been discussed by Hertz*, making certain assumptions.

The preceding problem is the simplest that can be proposed to represent a practical case of oscillations in a conductor. The above demonstration is given by Poincaré. It is evident, from the investigation of oscillations in the last chapter, that a system has as many possible periods as there are degrees of freedom. In a conductor of three dimensions the currents have an infinite

* Hertz. "Die Kräfte der elektrischen Schwingungen, behandelt nach der Maxwell'schen Theorie." *Wied. Ann.* 36, p. 1, 1889. Translation, p. 137.

number of degrees of freedom, and there are an infinite number of periods. The above problem corresponds to the lowest possible frequency for a sphere, when the surface-density of the electrification is a zonal surface-harmonic of degree one. For the general treatment of oscillations in spheres and cylinders the reader is referred to J. J. Thomson, *Recent Researches in Electricity and Magnetism.*

253. Waves on Wires. We now come to what is perhaps the most important practical problem connected with electrical waves, namely their propagation along wires, for upon this question depends the theory of telegraphy and telephony. The subject has been treated in great detail by Heaviside*, to whose papers the reader is referred.

We shall suppose that the direct and return conductors are either cylindrical wires parallel to the X-axis, or concentric tubes. Let R be the sum of the resistances of the two wires per unit of length. Let K be the capacity of the pair of conductors per unit of length, L their self-inductance per unit of length. Let the total current in one wire be I and the difference of potential between points on the two wires having the same x-coordinate be V. All these quantities are supposed measured in the electromagnetic system.

We may describe the phenomenon as follows. When an electromotive force is applied between any two corresponding points on the wires, say by connecting them with the poles of a battery, electricity of opposite signs flows out upon the surfaces of the two wires, producing an electric field in the surrounding space. The electrifications then move along the wires, causing a current, thus producing a magnetic field. Both these fields immediately begin to penetrate into the conductor, and are there dissipated into heat. As the electric field, whose lines are in the planes perpendicular to the conductors, rises from zero, it gives rise to displacement currents in the planes perpendicular to the conductors. The magnetic effect of these displacement currents we shall ignore in comparison with those of the conduction currents in the wire. We shall also ignore the penetration of the currents into the conductors, the theory of which would lead us too far for the present purpose. This we may safely do if the

* Heaviside. "Contributions to the Theory of the Propagation of Current in Wires." *Papers*, XX. *et al.*

wires are small enough, or the tubes thin enough, or in any case if the conductivity is great enough. Ignoring the manner of distribution of the current, then, we consider only the total current I in the wire at any point. This is variable from point to point, and is, like V, a function of x and t. We shall suppose that the phenomena are exactly symmetrical in the two conductors as far as the currents go, so that I has equal values with opposite signs at corresponding points in the two conductors.

Let us consider the charge that exists at any instant on the portion of one conductor between the points x and $x + dx$. Since the capacity per unit length is K, the difference of potential V, we have for the charge dq,

$$dq = VK\,dx.$$

If the current flowing in the positive direction at x is I, that at $x + dx$ is

$$I + \frac{\partial I}{\partial x}\,dx,$$

so that the total gain of charge of the element in unit time is

$$-\frac{\partial I}{\partial x}\,dx = \frac{\partial}{\partial t}(dq) = K\frac{\partial V}{\partial t}\,dx.$$

We accordingly obtain the differential equation connecting the current and difference of potential,

(1) $$-\frac{\partial I}{\partial x} = K\frac{\partial V}{\partial t}.$$

Considering now the flow of the current, we have in the pair of corresponding elements of the two wires the electrostatic electromotive force $-\frac{\partial V}{\partial x}\,dx$ and the electromotive force of induction $-L\,dx\frac{\partial I}{\partial t}$, the resistance being $R\,dx$. Accordingly the current equation is

(2) $$L\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x}.$$

The equations (1) and (2) are the equations of the problem. Differentiating (1) by t and (2) by x we may eliminate I, obtaining

(3) $$KL\frac{\partial^2 V}{\partial t^2} + KR\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}.$$

Differentiating (1) by x and (2) by t we eliminate V, obtaining

$$KL\frac{\partial^2 I}{\partial t^2}+KR\frac{\partial I}{\partial t}=\frac{\partial^2 I}{\partial x^2}. \tag{4}$$

Consequently both the current and the potential are propagated in accordance with the equation

$$KL\frac{\partial^2 \phi}{\partial t^2}+KR\frac{\partial \phi}{\partial t}=\frac{\partial^2 \phi}{\partial x^2}, \tag{5}$$

which, as it will be observed, is of the same form as the equation § 250 (2), the resistance of the wires here taking the place of the conductivity of the medium. We shall, with Poincaré, refer to the equation (5) as the *Telegraphic Equation*. If the wire is surrounded by a concentric tube, we have (§ 144), inserting the factor A^2 for electromagnetic measure,

$$K=\frac{\epsilon A^2}{2\log\frac{R_2}{R_1}},$$

while if the currents be supposed concentrated in the adjacent surfaces of the tubes, we have by § 234 (21), since $\bar{r}_{12}=\bar{r}_{22}$,

$$L=2\mu\log\frac{R_2}{R_1}.$$

Consequently the coefficient of the first term is

$$KL=\epsilon\mu A^2.$$

If we have two wires of the same diameter, which is small with respect to their distance apart, the formula § 159 (29) becomes

$$K=\frac{\epsilon A^2}{2\log\frac{d^2}{R^2}},$$

while the formula of § 234 (22) gives

$$L=2\mu\log\frac{d^2}{R^2},$$

so that we have the same relation as before. This relation is of course not accidental, depending upon the similar equations satisfied by the electric and magnetic fields between the conductors, relations which are brought out in § 234. The only reason for any deviation from the above relation is that in obtaining L the current density was supposed uniform, while in obtaining K the surface density was not. The theory of electromagnetic waves

in free space however shows us that in all cases we must have the coefficient of $\frac{\partial^2\phi}{\partial t^2}$ equal to $\epsilon\mu A^2$.

If accordingly the resistance of the wires is negligible, the disturbances are propagated with the speed $1/A\sqrt{\epsilon\mu}$ or in air with the velocity $1/A = \mathbf{v}$. The determination of this velocity and the comparison of it with the results of determinations of the ratio of the two units $\mathbf{v}$ is thus a matter of great importance. Determinations of the velocity have been made by Blondlot*, by Trowbridge and Duane†, and by Saunders‡, under the direction of the author. These have given a satisfactory agreement with theory, and are the only direct experimental verifications, for it is to be noticed that in the determinations of Hertz, based upon measurements of wave-lengths, the frequency was not observed, but calculated, thus virtually begging the question.

The result just announced, and the statement that the velocity of the waves should be approximately that of light, were contained in a paper published by Kirchhoff§ in 1857, and were afterwards rediscovered by Heaviside||. The equations (1) and (2) were given by Heaviside|| in 1876, and both researches remained singularly unnoticed until recently.

254. Particular Case. Submarine cable. The first case of the telegraphic equation to be treated was that in which the self-inductance of the circuit is negligible in comparison with its resistance. This is the case in a submarine cable, in which the wire is surrounded by a concentric tube of water, separated from it by a thin layer of dielectric. We may then neglect the first term of (5), making use of the equation

$$KR\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \tag{6}$$

* Blondlot. *Comptes Rend.* 117, p. 543, 1893.

† Trowbridge and Duane. "The Velocity of Electric Waves." *Phil. Mag.* (5), 40, p. 211, 1895.

‡ Saunders. "On the Velocity of Electrical Waves." *Physical Review*, (4), 2, p. 81, 1896.

§ Kirchhoff. "Ueber die Bewegung der Electricität in Drähten." *Pogg. Ann.* Bd. 100, 1857; *Ges. Abh.* p. 131.

|| Heaviside. "On the Extra Current." *Phil. Mag.* Aug. 1876; *Papers*, Vol. I. p. 53.

and obtaining the current from the equation

(7) $$I = -\frac{1}{R}\frac{\partial V}{\partial x}.$$

We thus arrive at the so-called electrostatic theory of propagation, given in 1855 by Lord Kelvin in his paper "On the Theory of the Electric Telegraph*," which established beyond question the feasibility of an Atlantic cable.

As we have seen in § 250, harmonic disturbances are propagated with a velocity proportional to the square root of the frequency. There is therefore no definite velocity of propagation in a cable, and there is liability of signals mixing with each other and losing their character. We are however more concerned with the question of how a single arbitrary short disturbance is propagated. If we consider a cable with different constants, for which

$$K'R'\frac{\partial V'}{\partial t'} = \frac{\partial^2 V'}{\partial x'^2},$$

by changing the variables by multiplying by constant factors we may make one solution do for both. If we put

$$x' = ax, \quad t' = bt,$$

we have

$$\frac{K'R'}{b}\frac{\partial V'}{\partial t} = \frac{1}{a^2}\frac{\partial^2 V'}{\partial x^2},$$

so that $$V'(x', t') = V(x, t),$$

if $$\frac{b}{a^2} = \frac{K'R'}{KR},$$

that is

$$K'R'\frac{x'^2}{t'} = KR\frac{x^2}{t}.$$

Accordingly the time necessary to produce a given potential at a distance x from the origin is proportional to KR multiplied by the square of the distance. We quote Lord Kelvin's words:

"We may be sure beforehand that the American telegraph will succeed, with a battery sufficient to give a sensible current at the remote end, when kept long enough in action; but the time required for each deflection will be sixteen times as long as would be with a wire a quarter of the length, such, for instance,

* *Proc. Roy. Soc.*, May 1855; *Math. and Phys. Papers*, Vol. II. p. 61.

as in the French submarine telegraph to Sardinia and Africa. One very important result is, that by increasing the diameter of the wire and of the gutta-percha covering in proportion to the whole length, the distinctness of utterance will be kept constant; for R varies inversely as the square of the diameter, and K (the electrostatical capacity of the unit of length) is unchanged when the diameters of the wire and the covering are altered in the same proportion."

(The so-called "KR-Law" has been applied to the theory of telephony on long-distance land-lines, to which it is not at all applicable, as has been shown by Heaviside. The use made of this law by the chief electrician of the English telegraphs would have prevented long distance telephony in England, even had there been any long distances.)

Guided by the conclusion announced above, we shall insert a new variable, $u = x/\sqrt{t}$, and attempt to satisfy the equation (6) by a function of u alone.

We have

(8)
$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial V}{\partial u}\frac{x}{\sqrt{t^3}},$$
$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial V}{\partial u}\frac{1}{\sqrt{t}},$$
$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{\sqrt{t}}\frac{\partial^2 V}{\partial u^2}\frac{\partial u}{\partial x} = \frac{1}{t}\frac{\partial^2 V}{\partial u^2},$$

so that our equation becomes

(9)
$$-\frac{KR}{2}u\frac{dV}{du} = \frac{d^2 V}{du^2},$$

or

(10)
$$\frac{d}{du}\left(\log\frac{dV}{du}\right) = -\frac{KR}{2}u.$$

The integral of this equation is

(11)
$$\log\frac{dV}{du} = -\frac{KR}{4}u^2 + \text{const.},\quad \frac{dV}{du} = Ce^{-\frac{KR}{4}u^2},$$

or, integrating from 0 to $u = x/\sqrt{t}$,

(12)
$$V = C\int_0^{x/\sqrt{t}} e^{-\frac{KR}{4}u^2}\,du.$$

If we now write u instead of $u\sqrt{KR}/2$ this is

$$(13) \qquad V = C' \int_0^{\frac{x}{2}\sqrt{\frac{KR}{t}}} e^{-u^2}\, du.$$

This definite integral is a function of its upper limit, and therefore of x and t, satisfying equation (6). For $x > 0$ and $t = 0$ the value of the integral is $\sqrt{\pi}/2$*. As we may add any constant to V, we will put

$$(14) \qquad V = V_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2}\sqrt{\frac{KR}{t}}} e^{-u^2}\, du\right).$$

Thus for $x > 0$ and $t = 0$, $V = 0$. For $x = 0$, $t > 0$, the value of the definite integral is zero, so that $V = V_0$. Consequently the solution (14) represents the result of connecting one end of the cable with a constant battery, and leaving it permanently connected.

The definite integral in (14) is the transcendent known as the probability-integral, for which numerical tables have been calculated. From these the values of V have been plotted, showing the potential at the different points on the cable, Fig. 96, the different curves being for times 1, 2, 3, 4, 5 times KR. It is to be noticed that however small the interval of time from the instant of connecting the battery, the disturbance is felt somewhat at all points, however remote. Thus the velocity of propagation would be infinite, if we could speak of a velocity. This shows that the

* This may be shown as follows. We have

$$J = \int_0^\infty e^{-x^2}\, dx = \int_0^\infty e^{-y^2}\, dy.$$

Consequently

$$J^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

Changing to polar coordinates,

$$J^2 = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-\rho^2} \rho\, d\rho\, d\phi = \frac{\pi}{2}\int_0^\infty e^{-\rho^2} \rho\, d\rho$$

$$= -\frac{\pi}{4}\left[e^{-\rho^2}\right]_0^\infty = \frac{\pi}{4}.$$

Therefore $$J = \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-u^2}\, du.$$

theory is only an approximation, for it is hardly imaginable that the velocity should be greater than in free space.

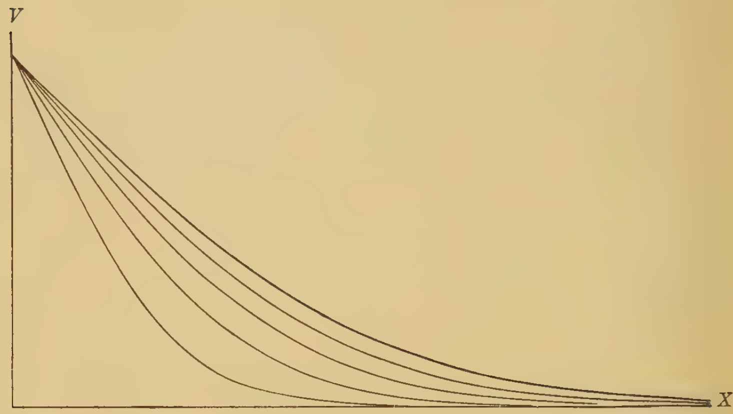

FIG. 96.

The current is obtained, according to (7), by differentiating with regard to x. We must therefore differentiate the integral by its upper limit, multiplying this by its derivative by x. Accordingly

$$(15) \qquad I = V_0 \sqrt{\frac{K}{\pi R}} \frac{e^{-\frac{KR}{4}\frac{x^2}{t}}}{\sqrt{t}}.$$

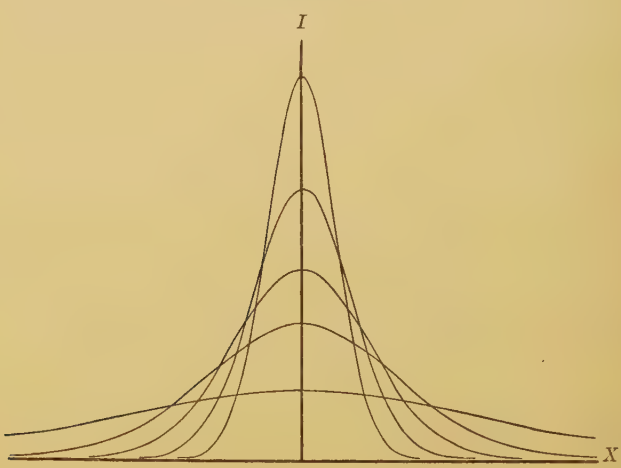

FIG. 97.

The values of the current for different points are shown in Fig. 97 for times $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 4 times KR. When $x=0$ we have

$$I = V_0 \sqrt{\frac{K}{\pi R}} \frac{1}{\sqrt{t}},$$

so that when $t=0$ the value of the current is infinite, instantaneously.

The rise of the potential at any given point other than $x=0$ is shown by the outside curve in Fig. 98, taken from Lord

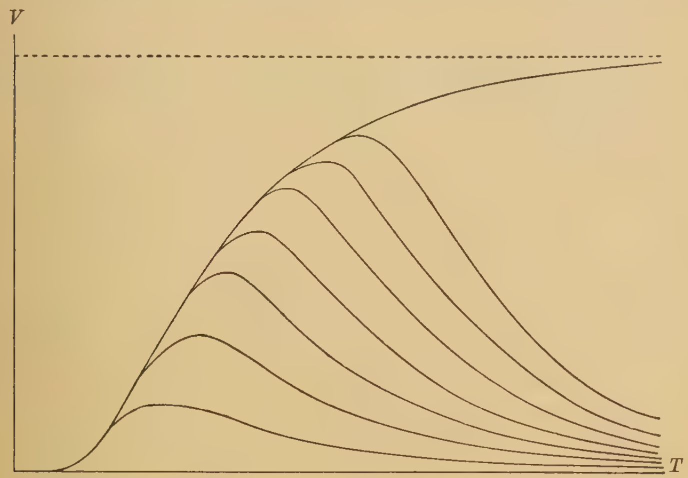

FIG. 98.

Kelvin's paper, the abscissas representing the time, the ordinates the potential.

The potential rises asymptotically to its value at the origin, but the current rises to a maximum, which occurs at the time $t = KRx^2/2$, and then dies away to zero. The maximum values of I are inversely proportional to the distance from the origin, and to the resistance R.

Since our differential equations are linear, disturbances due to different initial states are merely added. If the battery is connected, instead of permanently, for a definite time τ, and the cable then put to earth, the effect is the same as if, in the preceding case, after a time τ we permanently apply the potential $-V_0$ at the origin. If the preceding solution be called $V(t)$ the present will be $V(t) - V(t-\tau)$, consequently we may obtain the graphical

representation by taking the difference of ordinates of the outside curve in Fig. 98 and the same curve pushed to the right the distance τ. The other curves in Fig. 98 represent the potential at x when the battery is applied for times 1, 2, 3, 4, 5, 6, 7, times KRx^2.

Since any derivative of a solution of (6) is a solution, the derivative of (14) by x is also a solution, and

$$V = q_0 \sqrt{\frac{R}{\pi K}} \frac{e^{-\frac{KR}{4}\frac{x^2}{t}}}{\sqrt{t}}$$

represents the result of instantaneously connecting a battery and then insulating the end of the cable. The distribution at any time is of course shown in Fig. 97, and while V is initially infinite at the origin, the total charge

$$q = K\int_0^\infty V dx = q_0$$

is finite, and remains constant throughout.

255. General case of Telegraphic Equation. The telegraphic equation (5) has been treated by Heaviside, Poincaré*, Picard†, and Boussinesq‡. We shall give the solution of Boussinesq, not only because he has given the general solution of the more general equation § 250 (2), but because his method obtains the solution by an ingenious artifice from Poisson's solution § 247 (22), and the knowledge of other methods required by the processes of Poincaré and Picard is unnecessary.

Let us put

$$\frac{1}{KL} = a^2, \quad \frac{R}{L} = 2b,$$

so that our equation is

$$(1) \qquad \frac{\partial^2 V}{\partial t^2} + 2b\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2},$$

* Poincaré. "Sur la propagation de l'électricité." *Comptes Rendus*, 117, p. 1027, 1893.

† Picard. "Sur l'équation aux dérivées partielles qui se rencontre dans la théorie de la propagation de l'électricité." *Comptes Rendus*, 118, p. 16, 1894.

‡ Boussinesq. "Intégration de l'équation du son pour un fluide indéfini." *Comptes Rendus*, 118, p. 162, 1894.

and let us suppose that initially the state of the line is given, that is the potential and current are given at all points, by

$$(2)\qquad \left.\begin{aligned} &V = F(x), \quad I = G(x), \\ &\frac{\partial V}{\partial t} = -\frac{1}{K}\frac{\partial I}{\partial x} = -\frac{1}{K} G'(x) = f(x), \end{aligned}\right\} t = 0.$$

Let us transform the equation by putting

$$(3)\qquad V = e^{pt} u.$$

Accordingly

$$(4)\qquad \begin{aligned} \frac{\partial V}{\partial t} &= \left(\frac{\partial u}{\partial t} + pu\right) e^{pt}, \\ \frac{\partial^2 V}{\partial t^2} &= \left(\frac{\partial^2 u}{\partial t^2} + 2p\frac{\partial u}{\partial t} + p^2 u\right) e^{pt}, \\ \frac{\partial^2 V}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} e^{pt}, \end{aligned}$$

so that if we put $p = -b$ the equation becomes

$$(5)\qquad \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b^2 u.$$

The initial conditions now become

$$(6)\qquad \left.\begin{aligned} &u = F(x), \\ &\frac{\partial u}{\partial t} = f(x) + bF(x) = g(x), \end{aligned}\right\} t = 0.$$

The method of Boussinesq may be applied to the more general equation

$$(7)\qquad \frac{\partial^2 u}{\partial t^2} = a^2 \left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} + b^2 u,$$

which we shall accordingly consider, putting whenever we choose u independent of y. The artifice employed is the introduction into the function u of one more degree of freedom, by making it depend upon another parameter z which is finally to be given any *constant* value we please. Let then u satisfy the auxiliary equation

$$(8)\qquad a^2 \frac{\partial^2 u}{\partial z^2} = b^2 u,$$

so that the equation (7) becomes, taking this into account,

$$(9)\qquad \frac{\partial^2 u}{\partial t^2} = a^2 \left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right\}.$$

We will suppose that the initial conditions are now,

$$\left.\begin{aligned} u &= \Phi(x, y, z), \\ \frac{\partial u}{\partial t} &= \phi(x, y, z), \end{aligned}\right\} t = 0. \tag{10}$$

Then the solution of (9) is, by § 247 (22), inserting explicitly the rectangular coordinates and the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, of the radius r in the functions Φ_{at}, ϕ_{at},

$$\begin{aligned} u(x, y, z) = \frac{1}{4\pi}\frac{\partial}{\partial t}\iint t\Phi(x+at\cos\alpha, y+at\cos\beta, z+at\cos\gamma)\,d\omega \\ + \frac{1}{4\pi}\iint t\phi(x+at\cos\alpha, y+at\cos\beta, z+at\cos\gamma)\,d\omega. \end{aligned} \tag{11}$$

The equation (8) holds for all values of t. Solutions of it are

$$\begin{aligned} \Phi(x, y, z) &= F(x, y)\cos\frac{ib}{a}z + H(x, y)\sin\frac{ib}{a}z, \\ \phi(x, y, z) &= g(x, y)\cos\frac{ib}{a}z + h(x, y)\sin\frac{ib}{a}z, \end{aligned} \tag{12}$$

and for $z = 0$ these reduce to $\Phi = F$, $\phi = f$, so that for $t = 0$ we have the proper values of u, $\frac{\partial u}{\partial t}$. We therefore insert the values (12) in the integral (11). Now as we integrate over the whole sphere, the sine terms, being odd functions of z, disappear, while the cosine terms, being even functions, give us double the value that we should get by integrating over the hemisphere for which $z > 0$.

Accordingly, giving z the constant value zero,

$$\begin{aligned} u = \frac{1}{2\pi}\frac{\partial}{\partial t}\iint tF(x+at\cos\alpha, y+at\cos\beta)\cos(ibt\cos\gamma)\,d\omega \\ + \frac{1}{2\pi}\iint tg(x+at\cos\alpha, y+at\cos\beta)\cos(ibt\cos\gamma)\,d\omega. \end{aligned} \tag{13}$$

In the telegraphic equation F and g are independent of y, and are therefore constant on all small circles of the sphere normal to the X-axis. We will therefore employ polar coordinates, α the angle made by r with the X-axis, and χ the angle that the plane of r and the X-axis makes with the XZ-plane. Then

$$\begin{aligned} \cos\gamma &= \sin\alpha\cos\chi, \\ d\omega &= \sin\alpha\, d\alpha\, d\chi, \end{aligned} \tag{14}$$

and

(15) $$u=\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tF(x+at\cos\alpha)\cos(ibt\sin\alpha\cos\chi)\sin\alpha\, d\alpha\, d\chi$$

$$+\frac{1}{2\pi}\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tg(x+at\cos\alpha)\cos(ibt\sin\alpha\cos\chi)\sin\alpha\, d\alpha\, d\chi.$$

Let us put

$$ibt\sin\alpha=\zeta.$$

The definite integral

(16) $$I_0(\zeta)=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\zeta\cos\chi)\,d\chi,$$

is one of the set

(17) $$I_p(\zeta)=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2p}\chi\cos(\zeta\cos\chi)\,d\chi,$$

which may be evaluated by developing the cosine into a series and integrating each term. We thus obtain

(18) $$I_p(\zeta)=\frac{1}{\pi}\sum_{q=0}^{q=\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(-1)^q}{(2q)!}\sin^{2p}\chi\,\zeta^{2q}\cos^{2q}\chi\,d\chi,$$

an infinite power-series in ζ, the coefficient of ζ^{2q} being

$$\frac{(-1)^q}{(2q)!}I_{pq},$$

where

(19) $$I_{pq}=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2p}\chi\cos^{2q}\chi\,d\chi,$$

an integral which we can evaluate. Integrating by parts, writing the integrand $(\sin^{2p}\chi\cos\chi)\cos^{2q-1}\chi$,

(20) $$\pi I_{pq}=\frac{\sin^{2p+1}\chi\cos^{2q-1}\chi}{2p+1}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{2q-1}{2p+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2(p+1)}\chi\cos^{2(q-1)}\chi\,d\chi$$

$$=\frac{2q-1}{2p+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2p}\chi(1-\cos^2\chi)\cos^{2(q-1)}\chi\,d\chi$$

$$=\frac{2q-1}{2p+1}\pi(I_{p,\,q-1}-I_{pq}),$$

from which follows the reduction-formula

$$(21) \qquad I_{pq} = \frac{2q-1}{2(p+q)} I_{p,\, q-1}.$$

Integrating (19) again by parts, writing the integrand

$$\sin^{2p-1}\chi\,(\cos^{2q}\chi \sin\chi),$$

$$\pi I_{pq} = -\frac{\cos^{2q+1}\chi \sin^{2p-1}\chi}{2q+1}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{2p-1}{2q+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2(q+1)}\chi \sin^{2(p-1)}\chi\, d\chi$$

$$(22) \qquad = \frac{2p-1}{2q+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2q}(1-\sin^2\chi)\sin^{2(p-1)} d\chi$$

$$= \frac{2p-1}{2q+1}\pi(I_{p-1,\, q} - I_{pq}),$$

giving the reduction-formula,

$$(23) \qquad I_{pq} = \frac{2p-1}{2(p+q)} I_{p-1,\, q}.$$

By q successive applications of the reduction-formula (21) and p of the formula (23) we get

$$(24) \quad I_{pq} = \frac{(2q-1)(2q-3)\ldots 1 \,.\, (2p-1)(2p-3)\ldots 1}{2(p+q)\,2(p+q-1)\ldots 2(p+1)\,2p \,.\, 2(p-1)\ldots 2} I_{00}.$$

But we have

$$I_{00} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\chi = 1.$$

Accordingly, introducing even factors into numerator and denominator,

$$(25) \qquad I_{pq} = \frac{(2p)!\,(2q)!}{2^{2(p+q)}\, p!\, q!\, (p+q)!},$$

and our integral (17) becomes

$$(26) \qquad I_p(\zeta) = \sum_{q=0}^{q=\infty} \frac{(-1)^q (2p)!}{2^{2(p+q)}\, p!\; q!\, (p+q)!} \zeta^{2q}.$$

If we multiply this by $\dfrac{p!\,(2\zeta)^p}{(2p)!} = \dfrac{\zeta^p}{1\,.\,3\,.\,5\ldots(2p-1)}$, the series

$$(27) \quad J_p(\zeta) = \frac{p!\,(2\zeta)^p}{(2p)!} I_p(\zeta) = \left(\frac{\zeta}{2}\right)^p \sum_{q=0}^{q=\infty} \frac{(-1)^q}{q!\,(p+q)!}\left(\frac{\zeta}{2}\right)^{2q},$$

is called the Bessel's function of order p, and tables of its values have been calculated for real and pure imaginary values of the argument ζ*. Our integral (16) is therefore equal to

$$J_0(\zeta) = J_0(ibt \sin \alpha)$$

and (15) is

$$(28) \qquad u = \frac{1}{2}\frac{\partial}{\partial t}\int_0^\pi t\, F(x + at \cos\alpha)\, J_0(ibt \sin\alpha) \sin\alpha\, d\alpha$$
$$+ \frac{1}{2}\int_0^\pi t\, g(x + at\cos\alpha)\, J_0(ibt \sin\alpha) \sin\alpha\, d\alpha.$$

The Bessel's function $i^{-p}J_p(ix)$ of a pure imaginary argument is usually denoted by $I_p(x)$ (*not* the I_p of the preceding). Putting then

$$at\cos\alpha = \lambda, \quad -at\sin\alpha = d\lambda, \quad \zeta = ib\sqrt{t^2 - \lambda^2/a^2},$$

our solution becomes

$$(29) \qquad u = \frac{1}{2a}\frac{\partial}{\partial t}\int_{-at}^{at} F(x+\lambda)\, I_0(b\sqrt{t^2 - \lambda^2/a^2})\, d\lambda$$
$$+ \frac{1}{2a}\int_{-at}^{at} g(x+\lambda)\, I_0(b\sqrt{t^2 - \lambda^2/a^2})\, d\lambda.$$

Performing the differentiation by t (§ 26), since $I_0(0) = 1$

$$u = \frac{1}{2}\{F(x+at) + F(x-at)\}$$
$$(30) \qquad + \frac{1}{2a}\int_{-at}^{at} F(x+\lambda)\, \frac{\partial}{\partial t} I_0(b\sqrt{t^2 - \lambda^2/a^2})\, d\lambda$$
$$+ \frac{1}{2a}\int_{-at}^{at} g(x+\lambda)\, I_0(b\sqrt{t^2 - \lambda^2/a^2})\, d\lambda.$$

This solution was obtained by Heaviside† and by Poincaré, by entirely different processes.

We shall now suppose that initially there is no current in the line, and that the potential is zero except between two points

$$x_1 < x_2.$$

That is

$$(31) \qquad \begin{aligned} &F(x) = 0, \text{ except when } x_1 < x < x_2, \\ &\left.\begin{aligned} G(x) &= 0 \\ g(x) &= bF(x) \end{aligned}\right\} \text{ for all } x. \end{aligned}$$

* See Gray and Mathews, *Treatise on Bessel Functions.*

† "The General Solution of Maxwell's Electromagnetic Equations in a Homogeneous Isotropic Medium." *Phil. Mag.* Jan. 1889, p. 30; *Papers*, Vol. II. p. 478, eq. (40).

Then for

$$x - x_2 > at \text{ or } x_1 - x > at$$

all the functions occurring have the value zero, therefore V is zero. This at once distinguishes the solution from that for the case of the cable, for the disturbance does not arrive at x until the time

$$t = \frac{x - x_2}{a} \text{ on the right,}$$

or

$$t = \frac{x_1 - x}{a} \text{ on the left.}$$

When

$$x - x_1 > at > x - x_2$$

we have

$$\begin{aligned} V &= \frac{e^{-bt}}{2}\left[F(x - at) + \frac{1}{a}\int_{-at}^{-(x-x_2)} F(x+\lambda)\left(b + \frac{\partial}{\partial t}\right) I_0 d\lambda\right] \\ &= \frac{e^{-bt}}{2}\left[F(x - at) + \frac{1}{a}\int_{x-x_2}^{at} F(x-\lambda)\left(b + \frac{\partial}{\partial t}\right) I_0 d\lambda\right]. \end{aligned} \tag{32}$$

This represents the disturbance while the wave is passing over, for a point on the right. In like manner for a point on the left,

for

$$x_1 - x < at < x_2 - x,$$

$$V = \frac{e^{-bt}}{2}\left[F(x + at) + \int_{x_1-x}^{at} F(x+\lambda)\left(b + \frac{\partial}{\partial t}\right) I_0 d\lambda\right]. \tag{33}$$

Finally, at later instants,

$$x - x_1 < at \text{ or } x_2 - x < at,$$

$$V = \frac{e^{-bt}}{2}\int_{x-x_2}^{x-x_1} F(x-\lambda)\left(b + \frac{\partial}{\partial t}\right) I_0 d\lambda. \tag{34}$$

This represents the disturbance after the wave, travelling with velocity a, has passed on. Accordingly the solution, while representing a wave travelling with the velocity a, as in free space, differs from that case in that there remains a residue, or *tail* to the wave, which does not fall to zero however great the time. The exponential factor shows that the disturbance, both in the wave proper, and in the tail, is continually becoming attenuated.

Thus when successive impulses are transmitted, each leaves a tail, which interferes with all the succeeding waves, and the possibility of telephonic speaking depends not only on the attenuation and distortion with the distance, but on the magnitude of the tail of the wave. The tail also explains the discrepancies that existed between the results of the attempts made to determine

the velocity of electric waves by means of telegraph lines, what was generally observed being more probably the maximum disturbance than the front of the wave.

In order to give a concrete idea of the nature of the propagation, and to afford a means of comparison with the electrostatic theory, we shall suppose that the function V is constant and equal to V_0, from x_1 to x_2. We shall also change our units of time and length, by taking the relaxation-time $\tau = 1/b$ for the unit of time, and the relaxation distance, $d = a\tau = a/b$, as the unit of length. Accordingly putting

$$t' = \frac{t}{\tau} = bt, \quad x_2' = \frac{x - x_2}{d} = \frac{b(x - x_2)}{a}, \quad \frac{b\lambda}{a} = \mu,$$

we have

$$V = \frac{V_0 e^{-t'}}{2}\left[1 + \int_{x'}^{t'} \left\{1 + \frac{\partial}{\partial t'}\right\} I_0 \sqrt{t'^2 - \mu^2}\, d\mu\right] \tag{35}$$

for a point on the right while the wave is passing over. This equation was given by Heaviside in 1888, who carefully refrained from giving his method of deduction, remarking "since, although they were very laboriously worked out by myself, yet as mathematical solutions, are more likely to have been given before in some other physical problem than to be new*."

Inasmuch as not only Heaviside's results but any others were overlooked by the three French mathematicians quoted, who published results six years later, we may conclude that in the English writer modesty and original productiveness were more strongly developed than historical research. (This modesty is not maintained on the same plane throughout.)

Inserting in the value of V the series for I_0 (dropping accents),

$$I_0 \sqrt{t^2 - \mu^2} = \sum_{q=0}^{q=\infty} \frac{(t^2 - \mu^2)^q}{2^{2q}(q\,!)^2}, \tag{36}$$

and developing each term by the binomial theorem, we obtain

$$I_0 \sqrt{t^2 - \mu^2} = \sum_{q=0}^{q=\infty} \sum_{p=0}^{p=q} \frac{(-1)^p\, t^{2(q-p)}\, \mu^{2p}}{2^{2q}\, q\,!\, p\,!\,(q-p)\,!}. \tag{37}$$

* "Electromagnetic Waves." *Phil. Mag.* 1888; *Papers*, Vol. II. p. 373, eq. (52).

Changing the order of summation and putting

$$q = q' + p,$$

(38) $$I_0 \sqrt{t^2 - \mu^2} = \sum_{p=0}^{p=\infty} \sum_{q'=0}^{q'=\infty} \frac{(-1)^p t^{2q'} \mu^{2p}}{2^{2(q'+p)} (q'+p)!\, p!\, q'!},$$

which by the definition of I_p* is equal to

(39) $$\sum_{p=0}^{p=\infty} \frac{(-1)^p \mu^{2p}}{2^p p!\, t^p} I_p(t).$$

Differentiating by t and adding to itself,

(40) $$\left(1 + \frac{\partial}{\partial t}\right) I_0 = \sum_{p=0}^{p=\infty} \frac{(-1)^p \mu^{2p}}{2^p p!} \left\{\frac{I_p(t)}{t^p} + \frac{I_p'(t)}{t^p} - \frac{pI_p(t)}{t^{p+1}}\right\},$$

which by the formula connecting the derivative of I_p with the contiguous function I_{p+1},

$$I_p'(t) = \frac{p}{t} I_p(t) + I_{p+1}(t)$$

[Gray and Mathews (141)] may be written

(41) $$\sum_{p=0}^{p=\infty} \frac{(-1)^p \mu^{2p}}{2^p p!} \frac{\{I_p(t) + I_{p+1}(t)\}}{t^p}.$$

Inserting in the integral (35), and integrating from x to t,

(42) $$V = \frac{V_0 e^{-t}}{2} \left[1 + \sum_0^\infty \frac{(-1)^p t^{p+1}}{2^p p!(2p+1)} \{I_p(t) + I_{p+1}(t)\} - \sum_0^\infty \frac{(-1)^p x^{2p+1}}{2^p p!\,(2p+1)} \frac{\{I_p(t) + I_{p+1}(t)\}}{t^p}\right].$$

The terms free from x may, by writing out the sums for I_p, and collecting terms, be shown to be equal to e^t.

We, therefore, finally obtain for V

(43) $$V = \frac{V_0}{2}\left[1 - e^{-t} \sum_0^\infty (-1)^p \frac{1.3.5\ldots 2p-1}{(2p+1)!} \frac{\{I_p(t) + I_{p+1}(t)\}}{t^p} x^{2p+1}\right].$$

From this the values of V have been calculated and plotted by Mr W. P. Boynton in Fig. 99, which shows the distribution of potential along the line to the right, for times $t = 1, 2, 3, 4, 5$ times

* From (27) we have for *the present* $I_p(\zeta)$,

$$I_p(\zeta) = i^{-p} J_p(i\zeta) = \left(\frac{\zeta}{2}\right)^p \sum_{q=0}^{q=\infty} \frac{1}{q!\,(p+q)!} \left(\frac{\zeta}{2}\right)^{2q}.$$

the relaxation-time. This may be compared with Fig. 96 showing the electrostatic theory. The rise of potential at particular points,

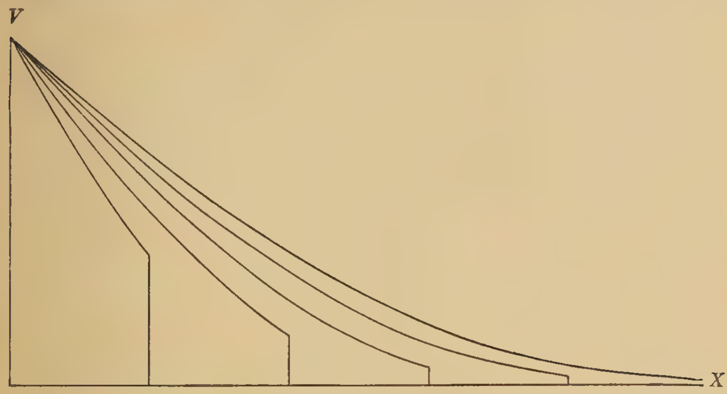

FIG. 99.

as a function of the time, is shown in Fig. 100, which is the analogue of the outer curve in Fig. 98. The different curves

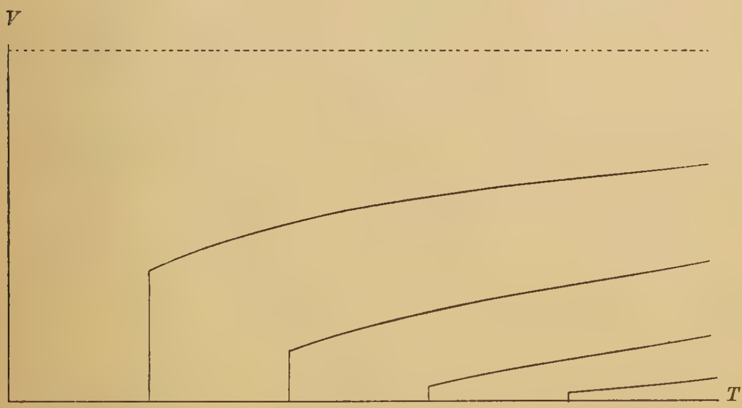

FIG. 100.

are for the points at 1, 2, 3, 4 times the relaxation-distance from the start.

The potential produced by connecting the battery for a definite time, and then removing it may, as before, be obtained by taking the differences of two curves relatively displaced. In this way the effect of the initial potential shown by the rectangle in *a* is shown in *b*, *c*, *d*, *e*, Fig. 101 for the times ·2, ·4, ·6, 1·6 times the relaxation-

time, the dotted lines showing the wave as it would be if there were no resistance. The last figure, f, Fig. 101, shows the effect of shortening the duration of a signal, the tail left being noticeably smaller.

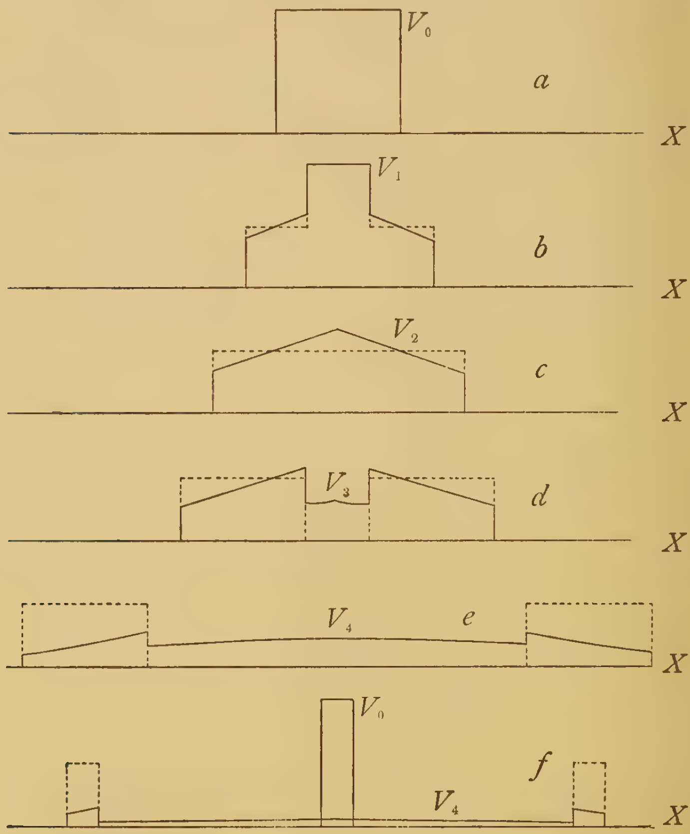

FIG. 101.

From these figures we may obtain an idea of the distance to which telephony is possible, if we know the relaxation-distance of the line. With ordinary land-lines the relaxation-distance is of the order of several hundred kilometers. This has made speaking possible between Boston and Chicago. Obviously it is of importance to make the relaxation-distance $d = \mathbf{v} . 2L/R$ as great as possible by making the distance between the wires great, and using large copper wires.

256. Terminal Conditions. In the preceding examples we have considered the line to be without end. In many practical

cases we wish to know what goes on in a line of finite length, when the ends are connected to any electromagnetic systems whatever, both when the systems are left to themselves and when electromotive-forces are applied. Space is lacking for more than the briefest possible treatment of this matter, which is very fully treated in Heaviside's papers on wires to which reference has already been made.

The method of procedure is the same in every case.

We shall make use of the equations

(1) $$KL\frac{\partial^2 V}{\partial t^2}+KR\frac{\partial V}{\partial t}=\frac{\partial^2 V}{\partial x^2},$$

(2) $$-\frac{\partial I}{\partial x}=K\frac{\partial V}{\partial t}.$$

Let us seek particular solutions of the form

(3) $$V=e^{\lambda t}u(x),\ \ I=e^{\lambda t}w(x).$$

Inserting in (1) we have

(4) $$(KL\lambda^2+KR\lambda)\,u=\frac{d^2u}{dx^2},$$

and if we put

(5) $$KL\lambda^2+KR\lambda=-\mu^2,$$

we have the equation for u,

(6) $$\frac{d^2u}{dx^2}+\mu^2u=0.$$

The solution of this is

(7) $$u=A\cos\mu x+B\sin\mu x,$$

where A and B are constants to be determined. From (2) we obtain

(8) $$-\frac{dw}{dx}=K\lambda u=K\lambda(A\cos\mu x+B\sin\mu x),$$
$$w=\frac{K\lambda}{\mu}(B\cos\mu x-A\sin\mu x).$$

The functions (3) are solutions of the differential equations whatever the value of λ. The values of λ that are admissible are determined by the terminal conditions. We shall take as an example one of the simplest cases possible. Let us suppose that at one end, where $x=l$, the two wires are connected, while at the

other they are connected with the plates of a condenser of capacity K_0. The conditions are then

(9)
$$V = 0, \quad x = l,$$
$$K_0 \frac{dV}{dt} = -I, \; x = 0.$$

Applying these to the solutions (3), (7), (8), we have

(10)
$$A \cos \mu l + B \sin \mu l = 0,$$
$$K_0 \lambda A = -\frac{K\lambda}{\mu} B.$$

Eliminating A/B we obtain

(11)
$$\mu \tan \mu l = \frac{K}{K_0},$$

a transcendental equation to determine μ. This has an infinite number of roots, which may be real or complex. When these are determined, λ is determined for each root by the equation (5). Thus we find that there are an infinite number of possible periods for the free vibration, corresponding to the n periods for a system with n degrees of freedom. The equation (11) corresponds to the determinantal equation of § 241 (10). The ratio B/A is determined by (10). The determination of the absolute value of the coefficients depends on the initial conditions.

Having found an infinite number of particular solutions, any root μ_s being distinguished by its suffix, the general solution is

(12)
$$V = \Sigma_s e^{\lambda_s t} (A_s \cos \mu_s x + B_s \sin \mu_s x),$$

where we sum for all the roots. If the potential is initially given by

(13)
$$V = F(x), \; t = 0,$$

we must have

(14)
$$F(x) = \Sigma_s (A_s \cos \mu_s x + B_s \sin \mu_s x).$$

The problem to be solved is then that of developing an arbitrary function of x in a trigonometric series of the form (14) where the μ_s's are the roots of a certain transcendental equation, namely (11). The problem is in general of considerable complexity, and we shall content ourselves with referring to Heaviside, who has treated it at great length.

If there is no condenser at $x = 0$, but the circuit is open, equation (11) is

$$\tan \mu l = \infty,$$

and the roots are

$$(14) \qquad \mu_s = \frac{(2s+1)}{2}\frac{\pi}{l}.$$

The series (14) then becomes

$$(15) \qquad F(x) = \Sigma_s \left(A_s \cos\frac{(2s+1)}{2}\frac{\pi x}{l} + B_s \sin\frac{(2s+1)}{2}\frac{\pi x}{l}\right),$$

a Fourier's series, with the even terms omitted. If $R=0$ we see by (5) that λ is a pure imaginary, so that all the oscillations are harmonic.

The wave-length L is

$$(16) \qquad L = \frac{4l}{2s+1}, \quad l = (2s+1)\frac{L}{4},$$

or the length of the wires is an odd number of quarter wave-lengths.

If on the other hand the capacity K_0 is infinite, we get

$$(17) \qquad \mu \tan \mu l = 0,$$

which is the same as if we had considered the circuit closed at the origin also, putting $V=0$, for $x=0$, from which (7) and the first of (10) would give

$$\sin \mu l = 0,$$

$$(18) \qquad \mu_s = \frac{s\pi}{l}, \quad L = \frac{2l}{s}.$$

The length of the line is then a multiple of a half wave-length. The two cases correspond to the cases of an organ pipe open at one end and closed at the other, or closed at both ends.

These conclusions have been verified by experiment. The above theory applies, for instance, to the experiments of Saunders cited above.

The method employed in this example is typical of the general process, for the terminal conditions, of whatever nature, are given in the form of an ordinary linear differential equation in the time, involving the derivatives of V and I. Applying this to our assumed solutions (3) introduces algebraic functions of λ, so that, eliminating by means of (5), we obtain an equation of the form

$$(19) \qquad \tan \mu l = \phi(\mu),$$

where ϕ is an algebraic function. The case we have considered is the simplest case of this transcendental equation.

257. Equations for Bodies in Motion. All the equations of this chapter have been deduced on the supposition that all the media were at rest. In deducing their extension to the case of media in motion we shall follow the method of Hertz, as given in the last and crowning paper of his remarkable researches*.

We shall suppose the media to be moving at every point with velocities v whose components at any point are α, β, γ. The medium is not supposed necessarily to be moving like a rigid body—it may be deformed in any manner. At the surface of separation between two media, although the velocity may be discontinuous its normal component must be continuous, in order to preclude the occurrence of vacant spaces. The fundamental assumption made by Hertz is that as the medium moves or is distorted, the lines of force are carried by the medium so as to pass through the same material points. That is, this would be the effect of the motion if it were the only influence at work to change the field. Besides this, we have the usual effects that appear in bodies at rest.

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, represent the field at any point at rest with respect to the coordinate-axes. The total change in $\mathfrak{X}$ at a point in motion will depend on several causes, the first being the change that is instantaneously taking place at the fixed point through which the material point happens to be passing. This we shall denote by $\frac{\partial \mathfrak{X}}{\partial t}$. Secondly the point is displaced to new parts of the field where the forces are different. The sum of these two parts we shall call

$$(1) \quad \frac{d\mathfrak{X}}{dt} = \frac{\partial \mathfrak{X}}{\partial t} + \frac{\partial \mathfrak{X}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathfrak{X}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathfrak{X}}{\partial z}\frac{dz}{dt} = \frac{\partial \mathfrak{X}}{\partial t} + \alpha\frac{\partial \mathfrak{X}}{\partial x} + \beta\frac{\partial \mathfrak{X}}{\partial y} + \gamma\frac{\partial \mathfrak{X}}{\partial z}.$$

If a small element normal to the X-axis of area dS were displaced parallel to itself, the flux through it would vary as just stated. But if the element rotates, it takes in new amounts of flux. At the start the flux through it was $\mathfrak{X}dS$, but as it turns, it acquires a projection normal to the Y-axis at the rate $\frac{\partial \alpha}{\partial y}$, consequently its flux in the positive direction *decreases* from this

* "Ueber die Grundgleichungen der Elektrodynamik für bewegte Körper." *Wied. Ann.* 41, p. 369, 1890; Trans. p. 241.

cause at the rate $\mathfrak{Y}\frac{\partial\alpha}{\partial y}$, and in the same manner from its Z-projection at the rate $\mathfrak{Z}\frac{\partial\alpha}{\partial z}$. But the area of the X-projection of the element is also increasing, at the rate $\frac{\partial\beta}{\partial y}$ in the Y-direction and $\frac{\partial\gamma}{\partial z}$ in the Z-direction. From this cause the flux increases at the rate

$$\mathfrak{X}\left\{\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right\}.$$

We have therefore to replace the term $\frac{\partial\mathfrak{X}}{\partial t}$ in equations (A) by the sum

$$(2)\qquad \frac{\partial\mathfrak{X}}{\partial t}+\alpha\frac{\partial\mathfrak{X}}{\partial x}+\beta\frac{\partial\mathfrak{X}}{\partial y}+\gamma\frac{\partial\mathfrak{X}}{\partial z}+\mathfrak{X}\left\{\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right\}-\left\{\mathfrak{Y}\frac{\partial\alpha}{\partial y}+\mathfrak{Z}\frac{\partial\alpha}{\partial z}\right\}$$
$$=\frac{\partial\mathfrak{X}}{\partial t}+\frac{\partial}{\partial y}(\beta\mathfrak{X}-\alpha\mathfrak{Y})-\frac{\partial}{\partial z}(\alpha\mathfrak{Z}-\gamma\mathfrak{X})+\alpha\left\{\frac{\partial\mathfrak{X}}{\partial x}+\frac{\partial\mathfrak{Y}}{\partial y}+\frac{\partial\mathfrak{Z}}{\partial z}\right\}.$$

We have thus added in virtue of the motion two parts, the first of which is the component of the curl of the vector whose components are

$$\gamma\mathfrak{Y}-\beta\mathfrak{Z},\ \alpha\mathfrak{Z}-\gamma\mathfrak{X},\ \beta\mathfrak{X}-\alpha\mathfrak{Y},$$

that is the vector product of the induction of the field and the velocity. The last term is the component of the velocity times the divergence of the induction. We may therefore abbreviate our equations which replace both (A) and (B) thus

$$(\mathrm{A}'')\qquad A\left\{\frac{\partial\mathfrak{F}}{\partial t}+\operatorname{curl}\mathbf{V}\mathfrak{F}v+v\operatorname{div}\mathfrak{F}+4\pi q\right\}=\operatorname{curl}H,$$

$$(\mathrm{B}'')\qquad -A\left\{\frac{\partial\mathfrak{B}}{\partial t}+\operatorname{curl}\mathbf{V}\mathfrak{B}v+v\operatorname{div}\mathfrak{B}\right\}=\operatorname{curl}F.$$

We may interpret the meaning of the new terms physically thus. In the equation (B″) the term

$$-A\operatorname{curl}\mathbf{V}\mathfrak{B}v$$

produces a part of curl F. Two vectors having the same curl differ only by a lamellar vector (§ 223). Consequently the motion gives rise to an electromotive force $-A\mathbf{V}\mathfrak{B}v=A\mathbf{V}v\mathfrak{B}$ perpendicular to the magnetic field and to the direction of the motion. This is the ordinary electromotive force of induction, and its magnitude may be specified as equal in any element of conductor

to the number of unit tubes of induction cut perpendicularly by the element in unit time.

On the left of (A″) appears the term

$$A\left\{\frac{\partial \mathfrak{F}}{\partial t} + \text{curl } \mathbf{V}\mathfrak{F}v + v \text{ div } \mathfrak{F} + 4\pi q\right\},$$

consisting of, first, the ordinary conduction current q, as in § 222 (2), second, the displacement current, as in equations (A), third, a new term equal to the product of the velocity by the density of electric charge. In other words a charge in motion produces the same effect as a current. This is verified by Rowland's celebrated experiment*. The corresponding quantity in equation (B″) shows the known electromotive forces produced by the motion of magnets, and also explains the phenomena of unipolar induction. It is to be noticed that the theory makes the magnets carry their lines of force with them.

The complete symmetry between electrical and magnetic phenomena may now be said to have been completely verified by experiment, with the exception of the second term of equation (A″) which would show the existence of a magnetic field due to the motion of insulating bodies in an electric field. The existence of such magnetic fields is made probable by an experiment made by Röntgen†.

The method of deducing the equations (A″) and (B″) may be applied even when our axes of coordinates are in motion, if α, β, γ be the velocities relative to the axes, and ϵ, μ, λ refer to the points fixed with respect to the moving axes. Thus the mutual actions of bodies depend only on their relative motion. Some simple considerations of this nature elucidating the phenomena of unipolar induction are found in a paper by the author in the *Electrical World*‡, the statements there made being borne out by experiments by Lecher§.

* Helmholtz, "Bericht betreffend Versuche über die elektromagnetische Wirkung elektrischer Convection, ausgeführt von H. A. Rowland." *Pogg. Ann.* 148, p. 487, 1876; *Wiss. Abh.* Bd. I. p. 791.

† "Ueber die durch Bewegung eines im homogenen elektrischen Felde befindlichen Dielektricums hervorgerufene electrodynamische Kraft." *Wied. Ann.* 35, p. 264, 1888.

‡ "Unipolar Induction and Current without difference of Potential." *Elec. World* (N.Y.), 23, pp. 491, 523, Apr. 14—21, 1894.

§ "Eine Studie über unipolare Induction." *Wied. Ann.* 54, p. 276, 1895.

258. Other Systems of Units. The systems of units that have been explained in this book are those in universal use. The electromagnetic system is the one altogether used in practical measurements, but as we have seen when considering the mutual effects of electrical and magnetic phenomena the Gaussian system is least liable to produce confusion. When only electrostatic phenomena are under consideration the electrostatic system is most convenient.

A change of units has been proposed by Heaviside, who would define the unit of electricity and magnetism in such a way that the flux of force due to unit charge out from a closed surface in air should be unity in value, instead of 4π. This would have the convenient effect of causing the disappearance of the factor 4π from many of our equations, for instance from the equation

$$\mathfrak{F}_{1n_1} + \mathfrak{F}_{2n_2} = 4\pi\sigma,$$

while the energy per unit volume would be

$$\tfrac{1}{2}(X\mathfrak{X} + Y\mathfrak{Y} + Z\mathfrak{Z}),$$
$$\tfrac{1}{2}(L\mathfrak{L} + M\mathfrak{M} + N\mathfrak{N}).$$

A practical advantage would be the disappearance of 4π in the formula connecting current-turns with magnetomotive-force. On the other hand the quantity 4π would be introduced in certain places where it is now absent. For instance the force at a distance r from a charged point m would be

$$\frac{m}{4\pi\epsilon r^2}.$$

It is rather singular that Maxwell adopted this method in his definition of electrical displacement, making the density equal to the divergence of the displacement, but did not do it in the case of the magnetic induction, nor even of the electric force. He was therefore obliged to make the displacement equal to $\frac{\epsilon}{4\pi}$ times the force, and his equations have an unfortunate appearance of dissymmetry. This has been avoided by Hertz, and in the present book, and it therefore seems merely a matter of convenience in writing whether we adopt Heaviside's proposition or not. Heaviside has called the new units *rational*, probably not because they are more reasonable than the old ones, but because of their avoidance in the majority of cases of the irrational number 4π. Of the

convenience of this there can be no question, but the question of units seems now to be beyond the control of theoretical writers*.

A system of units has been proposed by Dr Johnstone Stoney†, who advocates the choice of units so as to make the velocity **v** the unit of velocity. This would make the numerical measure of all quantities the same in both electrostatic and electromagnetic units, which would be convenient, but inasmuch as the velocity **v** is not yet accurately known, the proposal is hardly practical.

A table of dimensions of the principal electric and magnetic quantities is annexed.

TABLE OF DIMENSIONS.

Quantities.	Dimensions.
Fundamental Units.	
Length	L
Mass	M
Time	T
Derived Units.	
Area	L^2
Volume	L^3
Angle	L^0
Solid Angle	L^0
Velocity	LT^{-1}
Angular Velocity	T^{-1}
Acceleration	LT^{-2}
Momentum	MLT^{-1}
Force	MLT^{-2}
Pressure	$ML^{-1}T^{-2}$
Energy	ML^2T^{-2}
Activity or Power	ML^2T^{-3}
Energy-Density	$ML^{-1}T^{-2}$
Energy-Current-Density	MT^{-3}

* Heaviside. "The position of 4π in Electromagnetic Units." *Nature*, July 28, 1892, p. 292; *Papers*, Vol. II. p. 575.

† *B. A. Report*, 1891.

Electrical Units.

	$\epsilon\mu = L^{-2}T^2$	
Charge	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ \mu^{-\frac{1}{2}}$
Volume-Density	$M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{5}{2}}\ \mu^{-\frac{1}{2}}$
Surface-Density	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \mu^{-\frac{1}{2}}$
Moment	$M^{\frac{1}{2}}L^{\frac{5}{2}}\ T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ \mu^{-\frac{1}{2}}$
Polarization	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \mu^{-\frac{1}{2}}$
Potential	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-1}\epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-2}\mu^{\frac{1}{2}}$
Field-strength	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-2}\mu^{\frac{1}{2}}$
Induction	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \mu^{-\frac{1}{2}}$
Induction-Flux	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-1}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ \mu^{-\frac{1}{2}}$
Current	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-2}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-1}\mu^{-\frac{1}{2}}$
Current-Density	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Capacity	$L\ \epsilon$	$L^{-1}T^2\ \mu^{-1}$
Resistance	$L^{-1}T\ \epsilon^{-1}$	$L\ T^{-1}\mu$
Conductance	$L\ T^{-1}\epsilon$	$L^{-1}T\ \mu^{-1}$
Inductivity	ϵ	$L^{-2}T^2\ \mu^{-1}$
Conductivity	$T^{-1}\epsilon$	$L^{-2}T\ \mu^{-1}$
Resistivity	$T\ \epsilon^{-1}$	$L^2\ T^{-1}\mu$

Magnetic Units.

Charge	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-1}\mu^{\frac{1}{2}}$
Volume-Density	$M^{\frac{1}{2}}L^{-\frac{5}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}$
Surface-Density	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Moment	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{5}{2}}\ T^{-1}\mu^{\frac{1}{2}}$
Polarization	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Potential	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-2}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-1}\mu^{-\frac{1}{2}}$
Field-strength	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-2}\epsilon^{\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Induction	$M^{\frac{1}{2}}L^{-\frac{3}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Induction-Flux	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}\ T^{-1}\mu^{\frac{1}{2}}$
Vector-Potential	$M^{\frac{1}{2}}L^{-\frac{1}{2}}\ \epsilon^{-\frac{1}{2}}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}\ T^{-1}\mu^{\frac{1}{2}}$
Inductance	$L^{-1}T^2\ \epsilon^{-1}$	$L\ \mu$
Reluctance	$L\ T^{-2}\epsilon$	$L^{-1}\ \mu^{-1}$
Inductivity	$L^{-2}T^2\ \epsilon^{-1}$	μ
Reluctivity	$L^2\ T^{-2}\epsilon$	μ^{-1}

TABLE OF COMPARATIVE NOTATION.

	This book.	Maxwell.	Kirchhoff.	Helmholtz.*	Heaviside.
Electric Field	F, XYZ	PQR	XYZ	XYZ	**E**
Electric Induction	$\mathfrak{F}$, $\mathfrak{X}\mathfrak{Y}\mathfrak{Z}$	$4\pi f\, 4\pi g\, 4\pi h$		$4\pi\mathfrak{X}\, 4\pi\mathfrak{Y}\, 4\pi\mathfrak{Z}$	4π**D**
Electric Current-density	q, uvw	uvw		uvw	**C**
Magnetic Field	H, LMN	$\alpha\beta\gamma$	XYZ		**H**
Magnetic Induction	$\mathfrak{B}$, $\mathfrak{L}\mathfrak{M}\mathfrak{N}$	abc		$4\pi\mathfrak{L}\, 4\pi\mathfrak{M}\, 4\pi\mathfrak{N}$	**B**
Vector Potential belonging to magnetic induction	FGH	FGH	UVW	UVW $4\pi\mathfrak{U}\, 4\pi\mathfrak{V}\, 4\pi\mathfrak{W}$	**A**
Vector Potential of magnetization	PQR		LMN	LMN	
Polarization	ABC	ABC	$\alpha\beta\gamma$	$\lambda\mu\nu lmn$	**I**
Mechanical Force	ΞHZ	XYZ	XYZ, ABC	XYZ	
Volume Density	ρ	ρ	k	σ, ϵ, τ	ρ
Surface Density	σ	σ	h, e	e	σ
Electrical Potential	V	V, Ψ	V, ϕ	ϕ	P
Magnetic Potential	Ω	Ω	Ω, ϕ	ϕ	Ω
Electric Inductivity	ϵ	K	$1+4\pi\kappa$	$1+4\pi\epsilon$, $4\pi\epsilon$	c
Magnetic Inductivity	μ	μ		$1+4\pi\theta$, $4\pi\mu$	μ
Susceptibility	κ	κ	κ	θ	
Conductance	λ	C	λ		k
Capacity	K				
Resistance	R				
Total Current	I				

* Helmholtz's usage is variable, particularly as to the position of 4π.

APPENDIX.

Note to §§ 199, 200. According to the equations (6) § 199, the force on an element of medium in an electric or magnetic field becomes infinite at the surface between two media of different inductivities, for there μ is discontinuous. The layer in which this takes place is however infinitely thin, so that the total force on the surface is finite. We may find the force, as stated in § 199, by integrating throughout the space included in an infinitely thin layer containing the surface of discontinuity, as in § 85. We may also use the results of § 200, finding the components of the stress in both media by equations (8). The six components X_x, Y_y, Z_z, Y_z, Z_x, X_y, will in general have discontinuities at the surfaces of discontinuity of μ, and the forces on the unit of surface are equal to these discontinuities. For instance let us consider a surface bounding a medium of inductivity μ_2, surrounded by a medium of inductivity μ_1, the surface being such that the lines of force are normal to it in both media. Then taking the direction of the normal to a certain element for that of the X-axis, we have in the medium 1,

(1) $$X_x = \frac{1}{8\pi}\mathfrak{F}_1 F_1, \quad Y_y = Z_z = Y_z = Z_x = X_y = 0,$$

the force being a tension. In the medium 2 we have

(2) $$X_x = \frac{1}{8\pi}\mathfrak{F}_2 F_2.$$

The two tensions being in opposite directions on the two sides of the surface, the resultant force acting on unit of surface is the difference,

(3) $$T = \frac{1}{8\pi}(\mathfrak{F}_1 F_1 - \mathfrak{F}_2 F_2),$$

acting towards the medium 1. But since the induction is continuous, we have $\mathfrak{F}_1 = \mathfrak{F}_2$ and the force becomes,

$$(4) \qquad T = \frac{\mathfrak{F}}{8\pi}(F_1 - F_2) = \frac{\mathfrak{F}^2}{8\pi}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) = \frac{\kappa_{21}\mathfrak{F}^2}{8\pi\mu_2}.$$

We may also obtain the formula (4) in a simple manner by considering the energy-density in the two media. This is in the media 1 and 2

$$\frac{\mathfrak{F}^2}{8\pi\mu_1} \text{ and } \frac{\mathfrak{F}^2}{8\pi\mu_2} \text{ respectively.}$$

If now we consider the surface displaced normally a distance dn toward the medium 1 the prism standing on the element dS exchanges its energy

$$\frac{\mathfrak{F}^2}{8\pi\mu_1}\,dS\,dn,$$

for the amount

$$\frac{\mathfrak{F}^2}{8\pi\mu_2}\,dS\,dn,$$

so that there is a loss of energy

$$\frac{\mathfrak{F}^2}{8\pi}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_2}\right\}dS\,dn,$$

during the motion through the distance dn. Accordingly the force on the element dS is that given by (4) towards the medium 1*. If there is a real charge on the surface, so that the induction is discontinuous, we have

$$(5) \qquad T = \frac{1}{8\pi}(\mathfrak{F}_1 F_1 - \mathfrak{F}_2 F_2),$$

and if $\mu_1 = \mu_2$ this becomes

$$(6) \qquad T = \tfrac{1}{2}\sigma(F_1 + F_2).$$

This is exemplified in § 147.

As a second particular case, let us consider a surface of discontinuity at which the induction is tangential. Then we have in the medium 1 the *pressure*

$$P_1 = \frac{1}{8\pi}\mathfrak{F}_1 F_1,$$

* The deduction given by Maxwell, art. 440, is only approximate, lacking the factor μ_2 in the denominator.

and in the medium 2 the pressure

$$P_2 = \frac{1}{8\pi}\mathfrak{F}_2 F_2,$$

in the opposite direction.

There is accordingly on the element of surface the difference of pressure

(7) $$P = \frac{1}{8\pi}(\mathfrak{F}_2 F_2 - \mathfrak{F}_1 F_1),$$

acting towards the medium 1. If the two media lie side by side between the plates of a plane condenser we have F the same in both media, so that the pressure is greater in the medium for which μ is greater, and the surface is impelled towards the other medium. This has been verified by Quincke*, who blew a large bubble of air into a liquid contained between the plates of a condenser, and observed the additional pressure necessary to be given the air in order to resist the pressure due to the fluid. Quincke also verified the tension in the direction of the lines of force by filling an absolute electrometer with liquid. The stresses in magnetized media have been similarly verified by experiments by Quincke† and Taylor Jones‡.

Upon the principle of the sidewise force Mr A. P. Wills has founded an accurate method for determining κ for substances in which it is extremely small, both for magnetic and diamagnetic substances, by observing the attraction or repulsion on a slab with its edge in a uniform field.

* Quincke, "Electrische Untersuchungen," *Wied. Ann.* XIX. p. 705, 1883.

† *Ibid.* XXIV. p. 347, 1885.

‡ Jones, "On Electromagnetic Stress," *Phil. Mag.* XXXIX. p. 254, 1895.

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